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Research Article

Viscosity Approximation Methods for Equilibrium Problems, Variational Inequality Problems of Infinitely Strict Pseudocontractions in Hilbert Spaces

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We introduce an iterative scheme by the viscosity approximation method for finding a common element of the set of the solutions of the equilibrium problem and the set of fixed points of infinitely strict pseudocontractive mappings. Strong convergence theorems are established in Hilbert spaces. Our results improve and extend the corresponding results announced by many others recently.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty convex subset of H.

A mapping S of C is said to be a κ-strict pseudocontraction if there exists a constant $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \tag{1.1}$$

for all $x, y \in C$; see [1]. We denote the set of fixed points S by $F(S)$ (i.e., $F(S) = \{x \in C : Sx = x\}$).

Note that the class of strict pseudocontraction strictly includes the class of nonexpansive mappings which are mappings S on C such that

$$\|Sx - Sy\| \leq \|x - y\|, \tag{1.2}$$

for all $x, y \in C$. That is, S is nonexpansive if and only if S is a 0-strict pseudocontraction. Let Φ be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\Phi : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $EP(\Phi)$. Given a mapping $B : C \to H$, let $\Phi(x, y) = \langle Bx, y - x \rangle$ for all $x, y \in C$. Then the classical variational inequality problem is to find $x \in C$ such that $\langle Bx, y - x \rangle \geq 0$. We denote the solution of the variational inequality by $\mathrm{VI}(C, B)$; that is

$$\mathrm{VI}(C, B) = \{x \in C : \langle Bx, y - x \rangle \geq 0\}. \tag{1.4}$$

Let A be a strongly positive linear-bounded operator on H if there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.5}$$

A typical problem is to minimize a quadratic function over the set of the fixed points a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in E} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.6}$$

where A is a linear-bounded operator, E is the fixed point set of a nonexpansive mapping S on H, and b is a given point in H. The problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others; see [1–11]. In particular, Combettes and Hirstoaga [4] proposed several methods for solving the equilibrium problem. On the other hand, Mann [6], Shimoji and Takahashi [8] considered iterative schemes for finding a fixed point of a nonexpansive mapping. Further, Acedo and Xu [12] projected new iterative methods for finding a fixed point of strict pseudocontractions.

In 2006, Marino and Xu [7] introduced the general iterative method: for $x_1 = x \in C$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad n \geq 1. \tag{1.7}$$

They proved that the sequence $\{x_n\}$ of parameters satisfies appropriate condition and that the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the variational inequality $\langle (\gamma f - A) q, p - q \rangle \leq 0$, $p \in F(T)$. Recently, Liu [5] considered a general iterative method for equilibrium problems and strict pseudocontractions:

$$\begin{gathered}
\Phi(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = \beta_n u_n + (1 - \beta_n) S u_n, \\
x_{n+1} = \varepsilon_n \gamma f(x_n) + (I - \varepsilon_n A) u_n, \quad \forall n \geq 1,
\end{gathered} \tag{1.8}$$

where S is a k-strict pseudocondition mapping and $\{\varepsilon_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$. They proved that under certain appropriate conditions over $\{\varepsilon_n\}$, $\{\beta_n\}$, and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ both converge strongly to some $q \in F(S)\bigcap EP(\Phi)$, which solves some variational inequality problems. Tian [10] proposed a new general iterative algorithm: for nonexpansive mapping $T : H \to H$ with $F(T) \neq \phi$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \quad \forall n \geq 1, \tag{1.9}$$

where F is a k-Lipschitzian and η-strong monotone operator. He obtained that the sequence x_n generated by (1.9) converges to a point q in $F(T)$, which is the unique solution of the variational inequality $\langle(\gamma f - A)q, p - q\rangle \leq 0,\ p \in F(T)$. Very recently, Wang [13] considered a general composite iterative method for infinite family strict pseudocontractions: for $x_1 = x \in C$,

$$\begin{aligned} y_n &= \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)y_n, \quad \forall n \geq 1, \end{aligned} \tag{1.10}$$

where W_n is a mapping defined by (2.5), F is a k-Lipschitzian, and η-strongly monotone operator. With some appropriate condition, the sequence $\{x_n\}$ generated by (1.10) converges strongly to a common element of the fixed point of an infinite family of λ_i-strictly pseudocontractive mapping, which is a unique solution of the variational inequality $\langle(\gamma f - A)q, p - q\rangle \leq 0,\ p \in F(T)$. Kumam proposed many algorithms for the equilibrium and the fixed point problems with k-strict pseudoconditions; see [14–16]. In particular, in 2011, Kumam and Jaiboon [14] considered a system of mixed equilibrium problems, variational inequality problems, and strict pseudocontractive mappings:

$$\begin{aligned} &x_1 \in E, \quad u_n \in E, \quad v_n \in E, \\ &u_n = T_r^{\phi_1,\varphi} x_n, \\ &v_n = T_s^{\phi_2,\varphi} x_n, \\ &z_n = P_E(u_n - \mu_n C u_n), \\ &y_n = P_E(v_n - \lambda_n C v_n), \\ &k_n = a_n S_k x_n + b_n y_n + c_n z_n, \\ &x_{n+1} = \varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)k_n, \quad \forall n \geq 1, \end{aligned} \tag{1.11}$$

where S is a k-strict pseudocondition mapping. They proved that under certain appropriate conditions over $\{\varepsilon_n\}$, $\{\beta_n\}$, $\{r_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\lambda_n\}$, and $\{\mu_n\}$, the sequence $\{x_n\}$ converges strongly to a point $q \in \Theta$ which is the unique solution of the variational inequality $\langle(A - \gamma f)q, x - q\rangle \geq 0$. Inprasit [17] proposed a viscosity approximation methods to solving the generalized equilibrium and fixed point problems of finite family of nonexpansive mapping in Hilbert spaces.

In this paper, motivated by the above facts, we use the viscosity approximation method to find a common element of the set of solutions of the equilibrium problem $\mathrm{VI}(C,B)$ and the set of fixed points of a infinite family of strict pseudocontractions.

2. Preliminaries

Throughout this paper, we always write $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence. We need some facts and tools in a real Hilbert space H which are listed as below.

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle,\ \forall x, y \in H,$

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,\ \forall t \in [0,1],\ \forall x, y \in H.$

Lemma 2.2 (see [18]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\rho_n)\alpha_n + \sigma_n, \tag{2.1}$$

where $\{\rho_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \rho_n = \infty,$

(ii) $\limsup_{n\to\infty}(\sigma_n/\rho_n) \le 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty.$

Then $\lim_{n\to\infty} a_n = 0.$

Recall that given a nonempty closed convex subset C of a real Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \tag{2.2}$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C. As we all know, $y = P_C x$ if and only if there holds the relation:

$$\langle x - y, y - z\rangle \ge 0 \quad \forall z \in C. \tag{2.3}$$

Lemma 2.3 (see [13]). *Let $A : H \to H$ be an L-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $L > 0$, $\eta > 0$, $0 < \mu < 2\eta/L^2$, and $0 < t < 1$. Then $S = (I - t\mu A) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = (1/2)\mu(2\eta - \mu L^2)$.*

Lemma 2.4 (see [1]). *Let $S : C \to C$ be a κ-strict pseudocontraction. Define $T : C \to C$ by $Tx = \lambda x + (1-\lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.*

Lemma 2.5 (see [10]). *Let H be a Hilbert space and $f : H \to H$ a contraction with coefficient $0 < \alpha < 1$, and $A : H \to H$ an L-Lipschitzian continuous operator and η-strongly monotone with $L > 0$, $\eta > 0$. Then for $0 < \gamma < \mu\eta/\alpha$,*

$$\langle x - y, (\mu A - \gamma f)x - (\mu A - \gamma f)y \rangle \geq (\mu\eta - \gamma\alpha)\|x - y\|^2, \quad x, y \in H. \tag{2.4}$$

That is, $\mu A - \gamma f$ is strongly monotone with coefficient $\mu\eta - \gamma\alpha$.

Let $\{S_n\}$ be a sequence of κ_n-strict pseudo-concontractions. Define $S'_n = \theta_n I + (1 - \theta_n)S_n$, $\theta_n \in [\kappa_n, 1)$. Then, by Lemma 2.4, S'_n is nonexpansive. In this paper, we consider the mapping W_n defined by

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= t_n S'_n U_{n,n+1} + (1 - t_n)I, \\ U_{n,n-1} &= t_{n-1} S'_{n-1} U_{n,n} + (1 - t_{n-1})I, \\ &\cdots, \\ U_{n,i} &= t_i S'_i U_{n,i+1} + (1 - t_i)I, \\ &\cdots, \\ U_{n,2} &= t_2 S'_2 U_{n,3} + (1 - t_2)I, \\ W_n = U_{n,1} &= t_1 S'_1 U_{n,2} + (1 - t_1)I. \end{aligned} \tag{2.5}$$

Lemma 2.6 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E, let $S'_1, S'_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(S'_i) \neq \emptyset$, and let $t_1, t_2, \ldots$ be real numbers such that $0 < t_i \leq b < 1$, for every $i = 1, 2, \ldots$. Then, for any $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.6, one can define the mapping W of C into itself as follows:

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C. \tag{2.6}$$

Lemma 2.7 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $S'_1, S'_2, \ldots$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(S'_i) \neq \emptyset$, and let $t_1, t_2, \ldots$ be real numbers such that $0 < t_i \leq b < 1$, for all $i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0. \tag{2.7}$$

Lemma 2.8 (see [3]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{S'_i : C \to C\}$ be a family of infinite nonexpansive mappings with $\cap_{i=1}^{\infty} F(S'_i) \neq \emptyset$, and let $t_1, t_2, \ldots$ be real numbers such that $0 < t_i \leq b < 1$, for every $i = 1, 2, \ldots$. Then $F(W) = \cap_{i=1}^{\infty} F(S'_i)$.*

For solving the equilibrium problem, let us assume that the bifunction Φ satisfies the following conditions:

(A1) $\Phi(x,x)=0$ for all $x\in C$;

(A2) Φ is monotone; that is $\Phi(x,y)+\Phi(y,x)\leq 0$ for any $x,y\in C$;

(A3) for each $x,y,z\in C$, $\limsup_{t\to 0}\Phi(tz+(1-t)x,y)\leq\Phi(x,y)$;

(A4) $\Phi(x,\cdot)$ is convex and lower semicontinuous for each $x\in C$.

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.9 (see [2]). *Let C be a nonempty closed convex subset of H, let Φ be bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r>0$ and $x\in H$. Then there exists $z\in C$ such that*

$$\Phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \forall y\in C. \tag{2.8}$$

Lemma 2.10 (see [4]). *Let ϕ be a bifunction from $C\times C$ into $\mathbb{R}$ satisfying (A1)–(A4). Then, for any $r>0$ and $x\in H$, there exists $z\in C$ such that*

$$\Phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\quad \forall y\in C. \tag{2.9}$$

Further, if $T_rx=\{z\in C;\Phi(z,y)+(1/r)\langle y-z,z-x\rangle\geq 0,\forall y\in C\}$, then the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive;*

(3) *$F(T_r)=EP(\phi)$;*

(4) *$EP(\phi)$ is closed and convex.*

Lemma 2.11 (see [9]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space, and let $\{\beta_n\}$ be a sequence of real numbers such that $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$ for all $n=0,1,2,\ldots$. Suppose that $x_{n+1}=(1-\beta_n)z_n+\beta_nx_n$ for all $n=0,1,2,\ldots$ and $\limsup_{n\to\infty}\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\leq 0$. Then $\lim_{n\to\infty}\|z_n-x_n\|=0$.*

Lemma 2.12 (see [11]). *Let C,H,F, and T_rx be as in Lemma 2.9. Then the following holds:*

$$\|T_sx-T_tx\|^2\leq\langle T_sx-T_tx,T_sx-x\rangle, \tag{2.10}$$

for all $s,t>0$, and $x\in H$.

Lemma 2.13 (see [13]). *Let H be a Hilbert space, and let C be a nonempty closed convex subset of H, and $T:C\to C$ a nonexpansive mapping with $F(T)\neq\emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x=y$.*

3. Main Results

Now we start and prove our main result of this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let ϕ be a bifunction from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4). Let $S_i : C \rightarrow C$ be a family κ_i-strict pseudocontractions for some $0 \leq \kappa_i < 1$. Assume the set $\Omega = \mathrm{VI}(C, B) \cap_{i=1}^{\infty} F(S_i) \cap EP(\phi) \neq \emptyset$. Let f be a contraction of H into itself with $\alpha \in (0,1)$, and let A be a strongly positive linear bounded operator on H with coefficient $\gamma > 0$ and $0 < \gamma < \overline{\gamma}/\gamma$. Let $B : C \rightarrow H$ be an ξ-inverse strongly monotone mapping. Let W_n be the mapping generated by S_i' and t_i as in* (2.5)*. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}
\phi(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n \rangle &\geq 0, \\
y_n &= P_C(I - \mu_n B) z_n, \\
K_n &= \alpha_n x_n + (1 - \alpha_n) W_n y_n, \\
x_{n+1} &= \varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n) I - \varepsilon_n A) K_n,
\end{aligned} \tag{3.1}$$

where $\{\varepsilon_n\}$, $\{\beta_n\}$, $\{\alpha_n\}$, and $\{\lambda_n\}$ are sequences in $(0,1)$*. Assume that the control sequences satisfy the following restrictions:*

(i) $\lim_{n\to\infty}\varepsilon_n = 0$ *and* $\Sigma_{n=1}^{\infty}\varepsilon_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$*;*

(iii) $0 < \lim_{n\to\infty}(\lambda_n/\lambda_{n+1}) = 1$*;*

(iv) $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$*;*

(v) $0 < \mu_n \leq 2\xi$*;*

(vi) $\lim_{n\to\infty}\alpha_n = a$*.*

Then $\{x_n\}$ converges strongly to $q \in \Omega$ which is the unique solution of the variational inequality

$$\langle (A - \gamma f) q, x - q \rangle \geq 0, \quad \forall x \in \Omega, \tag{3.2}$$

or equivalent $q = P_\Omega(I - A + \gamma f)(q)$, where P is a metric projection mapping form H onto Ω.

Proof. Since $\varepsilon_n \rightarrow 0$, as $n \rightarrow \infty$, we may assume, without loss of generality, that $\varepsilon_n \leq (1 - \beta_n)\|A\|^{-1}$ for all $n \in \mathbb{N}$. By Lemma 2.3, we know that if $0 \leq \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$. We will assume that $\|I - A\| \leq 1 - \overline{\gamma}$. Since A is a strongly positive bounded linear operator on H, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \subseteq H, \|x\| = 1\}. \tag{3.3}$$

Observe that

$$\begin{aligned}
\langle ((1 - \beta_n) I - \varepsilon_n A) x, x \rangle &= (1 - \beta_n)\|x\|^2 - \varepsilon_n \langle Ax, x\rangle \\
&\geq (1 - \beta_n)\|x\|^2 - \varepsilon_n \|A\| \\
&\geq 0.
\end{aligned} \tag{3.4}$$

So this shows that $(1-\beta_n)I-\varepsilon_n A$ is positive. It follows that

$$\begin{aligned}\left\|(1-\beta_n)I-\varepsilon_n A\right\| &= \sup\left\{\left|\left\langle((1-\beta_n)I-\varepsilon_n A)x,x\right\rangle\right| : x\subseteq H, \|x\|=1\right\}\\ &= \sup\left\{1-\beta_n-\varepsilon_n\langle Ax,x\rangle x\subseteq H, \|x\|=1\right\}\le 1-\beta_n-\varepsilon_n\overline{\gamma}.\end{aligned} \tag{3.5}$$

Step 1. We claim that the mapping $P_\Omega(I-A+\gamma f)$ where $\Omega=\bigcap_{i=1}^{\infty}F(S_i)\bigcap EP(\Phi)$ has a unique fixed point. Let f be a contraction of H into itself with $\alpha\in(0,1)$. Then, we have

$$\begin{aligned}\left\|P_C(I-A+\gamma f)(x)-P_C(I-A+\gamma f)(y)\right\| &\le \left\|(I-A+\gamma f)(x)-(\mathrm{I}-A+\gamma f)(y)\right\|\\ &\le \|I-A\|\left\|x-y\right\|+\gamma\left\|f(x)-f(y)\right\|\\ &\le (1-\overline{\gamma})\left\|x-y\right\|+\gamma\alpha\left\|x-y\right\|\\ &= (1-(\overline{\gamma}-\gamma\alpha))\left\|x-y\right\|,\end{aligned} \tag{3.6}$$

for all $x,y\in H$. Since $0<1-(\overline{\gamma}-\gamma\alpha)<1$, it follows that $P_\Omega(I-A+\gamma f)$ is a contraction of H into itself. Therefore the Banach contraction mapping principle implies that there exists a unique element $q\in H$ such that $q=P_\Omega(I-A+\gamma f)(q)$.

Step 2. We shall show that $(I-\mu_n B)$ is nonexpansive. Let $x,y\in C$. Since B is ξ-inverse strongly monotone and $\lambda_n<2\alpha$ for all $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\left\|(I-\mu_n B)x-(I-\mu_n B)y\right\|^2 &= \left\|x-y-\mu_n(Bx-By)\right\|^2\\ &= \left\|x-y\right\|^2-2\mu_n\langle x-y,Bx-By\rangle\\ &\quad+\mu_n^2\left\|Bx-By\right\|^2\\ &\le \left\|x-y\right\|^2-2\xi\mu_n\left\|Bx-By\right\|^2+\mu_n^2\left\|Bx-By\right\|^2\\ &= \left\|x-y\right\|^2+\mu_n(\mu_n-2\xi)\left\|Bx-By\right\|^2\le\left\|x-y\right\|^2,\end{aligned} \tag{3.7}$$

where $\mu_n\le 2\xi$, for all $n\in N$. So we have that the mapping $(I-\lambda_n A)$ is nonexpansive.

Step 3. We claim that $\{x_n\}$ is bounded.

Let $p\in\Omega$; from Lemma 2.10, we have

$$\begin{aligned}p &= P_C(p-\mu_n Bp)=T_{\lambda_n}p,\\ \left\|z_n-p\right\| &= \left\|T_{\lambda_n}x_n-T_{\lambda_n}p\right\|\\ &\le \left\|x_n-p\right\|.\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}
\|y_n - p\| &= \|P_C(I - \mu_n B)z_n - p\| \\
&= \|P_C(I - \mu_n B)z_n - P_C(I - \mu_n B)p\| \\
&\le \|(z_n - \mu_n B z_n) - (p - \mu_n B p)\| \\
&= \|(I - \mu_n B)(z_n - p)\| \\
&\le \|z_n - p\| \\
&\le \|x_n - p\|, \\
\|K_n - p\| &= \|\alpha_n x_n + (1 - \alpha_n) W_n y_n - p\| \\
&= \|\alpha_n (x_n - p) + (1 - \alpha_n)(W_n y_n - p)\| \\
&\le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|W_n y_n - p\| \\
&\le \alpha_n \|x_n - p\| + (1 - \alpha_n)\|y_n - p\| \\
&\le \|x_n - p\|.
\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \varepsilon_n A)K_n - p\| \\
&= \|\varepsilon_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \varepsilon_n A)(K_n - p)\| \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \varepsilon_n \overline{\gamma})\|K_n - p\| \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\| + \beta_n \|x_n - p\| + (1 - \beta_n - \varepsilon_n \overline{\gamma})\|x_n - p\| \\
&\le (1 - \varepsilon_n \overline{\gamma})\|x_n - p\| + \varepsilon_n \gamma \|f(x_n) - f(p)\| + \varepsilon_n \|\gamma f(p) - Ap\| \\
&\le (1 - \varepsilon_n \overline{\gamma})\|x_n - p\| + \varepsilon_n \gamma \alpha \|x_n - p\| + \varepsilon_n \|\gamma f(p) - Ap\| \\
&= (1 - (\overline{\gamma} - \alpha\gamma)\varepsilon_n)\|x_n - p\| + (\overline{\gamma} - \alpha\gamma)\varepsilon_n \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma} \\
&\le \max\left\{ \|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma} \right\}.
\end{aligned} \tag{3.10}$$

By simple induction, we have

$$\|x_n - p\| \le \max\left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\overline{\gamma} - \alpha\gamma} \right\}, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Hence $\{x_n\}$ is bounded. This implies that $\{K_n\}$, $\{f(x_n)\}$ are also bounded.

Step 4. Show that $\limsup_{n\to\infty}\|x_{n+1}-x_n\|=0$.
Observing that $z_n=T_{\lambda_n}x_n$ and $z_{n+1}=T_{\lambda_{n+1}}$, we get

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|T_{\lambda_{n+1}}x_{n+1}-T_{\lambda_n}x_n\| \\ &= \|T_{\lambda_{n+1}}x_{n+1}-T_{\lambda_{n+1}}x_n+T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n\| \\ &\le \|x_{n+1}-x_n\|+\|T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n\|.\end{aligned} \tag{3.12}$$

By Lemma 2.10, we obtain

$$\begin{gathered}\Phi(T_{\lambda_n}x_n,y)+\frac{1}{\lambda_n}\langle y-T_{\lambda_n}x_n,T_{\lambda_n}u_n-x_n\rangle\ge 0,\quad \forall y\in C,\\ \Phi(T_{\lambda_{n+1}}x_n,y)+\frac{1}{\lambda_{n+1}}\langle y-T_{\lambda_{n+1}}x_n,T_{\lambda_{n+1}}x_n-x_n\rangle\ge 0,\quad \forall y\in C.\end{gathered} \tag{3.13}$$

In particular, we have

$$\begin{gathered}\Phi(T_{\lambda_n}x_n,T_{\lambda_{n+1}}x_n)+\frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n,T_{\lambda_n}x_n-x_n\rangle\ge 0,\\ \Phi(T_{\lambda_{n+1}}x_n,T_{\lambda_n}x_n)+\frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}x_n-T_{\lambda_{n+1}}x_n,T_{\lambda_{n+1}}x_n-x_n\rangle\ge 0.\end{gathered} \tag{3.14}$$

Summing up (3.14) and using (A_2), we obtain

$$\frac{1}{\lambda_{n+1}}\langle T_{\lambda_n}x_n-T_{\lambda_{n+1}}x_n,T_{\lambda_{n+1}}x_n-x_n\rangle+\frac{1}{\lambda_n}\langle T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n,T_{\lambda_n}x_n-x_n\rangle\ge 0, \tag{3.15}$$

for all $y\in C$. It follows that

$$\left\langle T_{\lambda_n}x_n-T_{\lambda_{n+1}}x_n,\frac{T_{\lambda_{n+1}}x_n-x_n}{\lambda_{n+1}}-\frac{T_{\lambda_n}x_n-x_n}{\lambda_n}\right\rangle\ge 0. \tag{3.16}$$

This implies

$$\begin{aligned}0&\le\left\langle T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n,T_{\lambda_n}x_n-x_n-\frac{\lambda_n}{\lambda_{n+1}}(T_{\lambda_{n+1}}x_n-x_n)\right\rangle\\ &=\left\langle T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n,T_{\lambda_n}x_n-T_{\lambda_{n+1}}x_n+\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right)(T_{\lambda_{n+1}}x_n-x_n)\right\rangle.\end{aligned} \tag{3.17}$$

It follows that

$$\|T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n\|^2\le\left|1-\frac{\lambda_n}{\lambda_{n+1}}\right|\|T_{\lambda_{n+1}}x_n-T_{\lambda_n}x_n\|(\|T_{\lambda_{n+1}}x_n\|+\|x_n\|). \tag{3.18}$$

Hence, we obtain

$$\|T_{\lambda_{n+1}}x_n - T_{\lambda_n}x_n\|^2 \leq \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L, \tag{3.19}$$

where $L = \sup\{\|x_n\| + \|T_{\lambda_{n+1}}x_n\| : n \in \mathbb{N}\}$. By (3.12) and (3.19), we obtain

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \|u_{n+1} - u_n\| + \|T_{\lambda_{n+1}}u_n - T_{\lambda_n}u_n\| \\
&\leq \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L, \\
\|y_{n+1} - y_n\| &= \|P_C(I - \mu_n B)z_{n+1} - P_C(I - \mu_n B)z_n\| \\
&\leq \|(I - \mu_n B)(z_{n+1} - z_n)\| \\
&\leq \|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L.
\end{aligned} \tag{3.20}$$

From (2.5), we have

$$\begin{aligned}
\|W_{n+1}y_n - W_n y_n\| &= \|t_1 S_1' U_{n+1,2} y_n - t_1 S_1' U_{n,2} y_n\| \\
&\leq t_1 \|U_{n+1,2} y_n - U_{n,2}\| \\
&\leq t_1 \|t_2 S_2' U_{n+1,3} y_n - t_2 S_2' U_{n,3} y_n\| \\
&\leq t_1 t_2 \|U_{n+1,3} y_n - U_{n,3} y_n\| \\
&\leq \cdots \\
&\leq M_1 \prod_{i=1}^{n} t_i,
\end{aligned} \tag{3.21}$$

where $M_1 = \sup_n\{\|U_{n+1,n+1}y_n - U_{n,n+1}y_n\|\}$.

Note that

$$\begin{aligned}
\|K_{n+1} - K_n\| &= \|\alpha_{n+1}x_{n+1} + (1 - \alpha_{n+1})W_{n+1}y_{n+1} - \alpha_n x_n - (1 - \alpha_n)W_n y_n\| \\
&= \|\alpha_{n+1}(x_{n+1} - x_n) + \alpha_{n+1}x_n + (1 - \alpha_{n+1})(W_{n+1}y_{n+1} - W_n y_n) \\
&\quad -\alpha_n x_n + (1 - \alpha_{n+1})W_n y_n - (1 - \alpha_n)W_n y_n\| \\
&= \|\alpha_{n+1}(x_{n+1} - x_n) + (\alpha_{n+1} - \alpha_n)x_n(1 - \alpha_{n+1})(W_{n+1}y_{n+1} - W_n y_n) \\
&\quad +(\alpha_n - \alpha_{n+1})W_n y_n\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_{n+1}\|x_{n+1} - \mathrm{x}_n\| + |\alpha_{n+1} - \alpha_n|\|x_n\| \\
&\quad + (1-\alpha_{n+1})\left\|W_{n+1}y_{n+1} - W_n y_n\right\| + |\alpha_n - \alpha_{n+1}|\left\|W_n y_n\right\| \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|x_n\| \\
&\quad + (1-\alpha_{n+1})\left[\left\|W_{n+1}y_{n+1} - W_{n+1}y_n\right\| + \left\|W_{n+1}y_n - W_n y_n\right\|\right] \\
&\quad + |\alpha_n - \alpha_{n+1}|\left\|W_n y_n\right\| \\
&\leq \alpha_{n+1}\|x_{n+1} - x_n\| + |\alpha_{n+1} - \alpha_n|\|x_n\| \\
&\quad + (1-\alpha_{n+1})\left[\|x_{n+1} - x_n\| + (1-\alpha_{n+1})\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + M_1\prod_{i=1}^{n} t_i\right] \\
&\quad + |\alpha_n - \alpha_{n+1}|\left\|W_n y_n\right\| \\
&\leq \|x_{n+1} - x_n\| + 2|\alpha_{n+1} - \alpha_n| M_2' + (1-\alpha_{n+1})\left(M_1\prod_{i=1}^{n} t_i + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L\right),
\end{aligned}
\tag{3.22}
$$

where $M_2' = \sup\{\|x_n\|, \|W_n y_n\|\}$.

Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n) l_n$, then $l_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n) = (\varepsilon_n \gamma f(x_n) + ((1-\beta_n)I - \varepsilon_n F)K_n)/(1-\beta_n)$.

Hence, we have

$$
\begin{aligned}
l_{n+1} - l_n &= \frac{\varepsilon_{n+1}\gamma f(x_{n+1}) + ((1-\beta_{n+1})I - \varepsilon_n A)K_{n+1}}{1-\beta_{n+1}} - \frac{\varepsilon_n \gamma f(x_n) + ((1-\beta_n)I - \varepsilon_n A)K_n}{1-\beta_n} \\
&= \frac{\varepsilon_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1}) - \frac{\varepsilon_n}{1-\beta_n}\gamma f(x_n) + K_{n+1} - K_n + \frac{\varepsilon_n}{1-\beta_n} A K_n - \frac{\varepsilon_{n+1}}{1-\beta_{n+1}} A K_{n+1} \\
&= \frac{\varepsilon_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1} - A K_{n+1})) + \frac{\varepsilon_n}{1-\beta_n}(A K_n - \gamma f(x_n)) + K_{n+1} - K_n.
\end{aligned}
\tag{3.23}
$$

Then

$$
\begin{aligned}
\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| &\leq \frac{\varepsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|A K_{n+1}\|\right) \\
&\quad + \frac{\varepsilon_n}{1-\beta_n}\left(\|A K_n\| + \left\|\gamma f(x_n)\right\|\right) + \|K_{n+1} - K_n\| - \|x_{n+1} - x_n\| \\
&\leq \frac{\varepsilon_{n+1}}{1-\beta_{n+1}}\left(\left\|\gamma f(x_{n+1})\right\| + \|A K_{n+1}\|\right) + \frac{\varepsilon_n}{1-\beta_n}\left(\|A K_n\| + \left\|\gamma f(x_n)\right\|\right) \\
&\quad + 2|\alpha_{n+1} - \alpha_n| M_2' + (1-\alpha_{n+1})\left(\left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| L + M_1\prod_{i=1}^{n} t_i\right).
\end{aligned}
\tag{3.24}
$$

Combining with (i), (iii), and (iv), we have

$$\limsup_{n\to\infty}(\|l_{n+1}-l_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.25}$$

Hence, by Lemma 2.11, we obtain $\|l_n-x_n\| \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|l_n-x_n\| = 0. \tag{3.26}$$

We also know that

$$\begin{aligned} x_{n+1}-x_n &= \varepsilon_n\gamma f(x_n)+\beta_n x_n+\left[(1-\beta_n)I-\varepsilon_n A\right]K_n-x_n \\ &= \varepsilon_n(\gamma f(x_n)-Ax_n)+\left[(1-\beta_n)I-\varepsilon_n A\right](K_n-x_n). \end{aligned} \tag{3.27}$$

So

$$\lim_{n\to\infty}\|K_n-x_n\| = 0. \tag{3.28}$$

Step 5. We claim that $\|x_n-Wx_n\| \to 0$.
Observe that

$$\begin{aligned} \|x_n-W_ny_n\| &= \|x_n-x_{n+1}+x_{n+1}-y_n+K_n-W_ny_n\| \\ &\le \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|+\alpha_n\|x_n-W_ny_n\|. \end{aligned} \tag{3.29}$$

From (A1), (A3), and (3.28), using step 2, we have

$$\begin{aligned} (1-\alpha_n)\|x_n-W_ny_n\| &\le \|x_n-x_{n+1}\|+\|\varepsilon_n\gamma f(x_n)+\beta_nx_n-\beta_nK_n-\varepsilon_nAK_n\| \\ &\le \|x_n-x_{n+1}\|+\varepsilon_n\|\gamma f(x_n)-AK_n\|+\beta_n\|x_n-K_n\|. \end{aligned} \tag{3.30}$$

This implies that

$$\|x_n-W_ny_n\| \longrightarrow 0 \quad (\text{as } n\longrightarrow\infty). \tag{3.31}$$

Next we want to show $\lim_{n\to\infty}\|x_n-y_n\| = 0$.
Let $p\in\bigcap_{i=1}^{\infty}F(S_i)\bigcap EP(\Phi)$; we have

$$\begin{aligned} \|z_n-p\|^2 &= \|T_{\lambda_n}x_n-T_{\lambda_n}p\|^2 \\ &\le \langle T_{\lambda_n}x_n-T_{\lambda_n}p, x_n-p\rangle \\ &= \langle z_n-p, x_n-p\rangle \\ &= \frac{1}{2}\left(\|z_n-p\|^2+\|x_n-p\|^2-\|x_n-z_n\|^2\right). \end{aligned} \tag{3.32}$$

Therefore

$$\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2. \tag{3.33}$$

Note that

$$\begin{aligned}
\|K_n - p\| &= \|\alpha_n x_n + (1-\alpha_n)W_n y_n - p\|^2 \\
&= \|\alpha_n (x_n - p) + (1-\alpha_n)(W_n y_n - p)\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|z_n - p\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left[\|x_n - p\|^2 - \|x_n - z_n\|^2\right] \\
&= \|x_n - p\|^2 - (1-\alpha_n)\|x_n - z_n\|^2.
\end{aligned} \tag{3.34}$$

From (3.34), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)K_n - p\|^2 \\
&= \|\varepsilon_n (\gamma f(x_n) - Ap)\|^2 + \beta_n \|x_n - p\|^2 + ((1-\beta_n)I - \varepsilon_n A)\|K_n - p\|^2 \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \varepsilon_n \overline{\gamma})\|K_n - p\|^2 \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 \\
&\quad + (1 - \beta_n - \varepsilon_n \overline{\gamma})\left(\|x_n - p\|^2 - (1-\alpha_n)\|x_n - z_n\|^2\right) \\
&= \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + (1 - \varepsilon_n \overline{\gamma})\|x_n - p\|^2 - (1 - \beta_n - \varepsilon_n \overline{\gamma})(1-\alpha_n), \\
\|x_n - z_n\|^2 &\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 + (1 - \beta_n - \varepsilon_n \overline{\gamma})(1-\alpha_n)\|x_n - z_n\|^2.
\end{aligned} \tag{3.35}$$

It follows that

$$\begin{aligned}
(1 - \beta_n - \varepsilon_n \overline{\gamma})(1-\alpha_n)\|x_n - z_n\|^2 &\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&= \varepsilon_n \|\gamma f(x_n) \ - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \\
&\quad \times \|x_n - x_{n+1}\|.
\end{aligned} \tag{3.36}$$

From conditions (i), (vi) and (3.26), we have

$$\|x_n - z_n\| \longrightarrow 0. \tag{3.37}$$

We also compute

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(I-\mu_n B)z_n - P_C(I-\mu_n B)p\| \\
&\le \|(z_n - \mu_n B z_n) - (p - \mu_n B p)\|^2 \\
&= \|z_n - p\|^2 - \ 2\mu_n \langle z_n - p, Bz_n - Bp\rangle + \mu_n^2 \|Bz_n - Bp\|^2 \\
&\le \|x_n - p\|^2 - \mu_n(2\xi - \mu_n)\|Bz_n - Bp\|^2,
\end{aligned} \tag{3.38}$$

$$\begin{aligned}
\|K_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n) W_n y_n - p\|^2 \\
&= \|\alpha_n(x_n - p) - (1-\alpha_n)(W_n y_n - p)\|^2 \\
&\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 \\
&= \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\{\|x_n - p\|^2 - \mu_n(2\xi - \mu_n)\|Bz_n - Bp\|^2\right\} \\
&= \|x_n - p\|^2 - \mu_n(1-\alpha_n)(2\xi - \mu_n)\|Bz_n - Bp\|^2.
\end{aligned} \tag{3.39}$$

So, from (3.39), we get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\varepsilon_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)K_n - p\|^2 \\
&= \|\varepsilon_n(\gamma f(x_n) - Ap)\|^2 + \beta_n(x_n - p) + ((1-\beta_n)I - \varepsilon_n A)\|K_n - p\|^2 \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \varepsilon_n \overline{\gamma})\|K_n - p\|^2 \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \beta_n \|x_n - p\|^2 + (1 - \beta_n - \varepsilon_n \overline{\gamma}) \\
&\quad \times \left\{\|x_n - p\|^2 - \mu_n(1-\alpha_n)(2\xi - \mu_n)\|Bz_n - Bp\|^2\right\} \\
&\le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 \\
&\quad - (1 - \beta_n - \varepsilon_n \overline{\gamma})\mu_n(1-\alpha_n)(2\xi - \mu_n)\|Bz_n - Bp\|^2.
\end{aligned} \tag{3.40}$$

It follows that

$$\begin{aligned}
&(1 - \beta_n - \varepsilon_n \overline{\gamma})\mu_n(1-\alpha_n)(2\xi - \mu_n)\|Bz_n - Bp\|^2 \\
&\quad \le \varepsilon_n \|\gamma f(x_n) - Ap\|^2 + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned} \tag{3.41}$$

So

$$\|Bz_n - Bp\| \longrightarrow 0. \tag{3.42}$$

On the other hand, we also know that

$$\begin{aligned}
\|y_n - p\|^2 &= \|P_C(I-\mu_n B)z_n - P_C(I-\mu_n B)p\|^2 \\
&\leq \langle (I-\mu_n B)z_n - (I-\mu_n B)p, y_n - p\rangle \\
&= \frac{1}{2}\Big\{\|(I-\mu_n B)z_n - (I-\mu_n B)p\|^2 + \|y_n - p\|^2 \\
&\qquad - \|(I-\mu_n B)z_n - (I-\mu_n B)p - (y_n - p)\|^2\Big\} \\
&\leq \frac{1}{2}\Big\{\|x_n - p\|^2 + \|y_n - p\|^2 - \|z_n - y_n\|^2 - \mu_n\|Bz_n - Bp\|^2 \\
&\qquad + 2\mu_n\langle z_n - y_n, Bz_n - Bp\rangle\Big\},
\end{aligned} \tag{3.43}$$

and hence

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - y_n\|^2 + 2\mu_n\|z_n - y_n\|^2\|Bz_n - Bp\|. \tag{3.44}$$

So

$$\begin{aligned}
\|K_n - p\|^2 &= \|\alpha_n x_n + (1-\alpha_n)W_n y_n - p\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\Big\{\|x_n - p\|^2 - \|z_n - y_n\|^2\Big\} \\
&\quad + 2\mu_n\|z_n - y_n\|\|Bz_n - Bp\|
\end{aligned} \tag{3.45}$$

$$\begin{aligned}
&= \|x_n - p\|^2 - (1-\alpha_n)\|z_n - y_n\|^2 + 2(1-\alpha_n)\mu_n\|z_n - y_n\|\|Bz_n - Bp\|, \\
\|x_{n+1} - p\|^2 &= \|\varepsilon_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \varepsilon_n A)K_n - p\|^2 \\
&\leq \varepsilon_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 + (1-\beta_n-\varepsilon_n\overline{\gamma})\|K_n - p\|^2 \\
&\leq \varepsilon_n\|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 - (1-\beta_n-\varepsilon_n\overline{\gamma})(1-\alpha_n)\|z_n - y_n\|^2 \\
&\quad + 2(1-\alpha_n)\mu_n(1-\beta_n-\varepsilon_n\overline{\gamma})\|z_n - y_n\|\|Bz_n - Bp\|.
\end{aligned} \tag{3.46}$$

Hence

$$\begin{aligned}
&(1-\beta_n-\varepsilon_n\overline{\gamma})(1-\alpha_n)\|z_n - y_n\|^2 \\
&\quad\leq \varepsilon_n\|\gamma f(x_n) - Ap\|^2 + 2(1-\alpha_n)\mu_n(1-\beta_n-\varepsilon_n\overline{\gamma})\|z_n - y_n\|\|Bz_n - Bp\| \\
&\qquad + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad\leq \varepsilon_n\|\gamma f(x_n) - Ap\|^2 + 2(1-\alpha_n)\mu_n(1-\beta_n-\varepsilon_n\overline{\gamma})\|z_n - y_n\|\|Bz_n - Bp\| \\
&\qquad + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned} \tag{3.47}$$

From (i), (3.42), and (3.26), we know that

$$\left\|z_n - y_n\right\| \longrightarrow 0. \tag{3.48}$$

From (3.37) and (3.42), we can get

$$\left\|x_n - y_n\right\| \longrightarrow 0. \tag{3.49}$$

On the other hand, we have

$$\begin{aligned}\|x_n - Wx_n\| &\le \left\|x_n - W_n y_n\right\| + \left\|W_n y_n - Wx_n\right\| \\ &\le \left\|x_n - W_n y_n\right\| + \sup\left\|W_n y_n - Wx_n\right\|.\end{aligned} \tag{3.50}$$

By (3.30), (3.49), and using Lemma 2.7, we have

$$\|x_n - Wx_n\| \longrightarrow 0. \tag{3.51}$$

Step 6. We claim that $\limsup_{n\to\infty}\langle(A-\gamma f)q, q-x_n\rangle \le 0$, where $q = P_\Omega(I - A + \gamma f)(q)$ is the unique solution of $\langle(A-\gamma f)q, x-q\rangle \ge 0$, for all $x \in \Omega$.

Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle(A-\gamma f)q, q-x_n\right\rangle = \limsup_{n\to\infty}\left\langle(A-\gamma f)q, q-x_{n_j}\right\rangle. \tag{3.52}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$, which converges weakly to p; without loss of generality, we can assume $x_{n_j} \rightharpoonup p$ and $Wx_{n_j} \rightharpoonup p$, we arrive at

$$\limsup_{n\to\infty}\left\langle(A-\gamma f)q, q-x_{n_j}\right\rangle = \langle(A-\gamma f)q, q-p\rangle \le 0. \tag{3.53}$$

Step 7. We show that $x_n \to q$.

Since

$$\begin{aligned}\left\langle(A-\gamma f)q, q-x_{n+1}\right\rangle &= \left\langle(A-\gamma f)q, x_n - x_{n+1}\right\rangle + \left\langle(A-\gamma f)q, q-x_n\right\rangle \\ &\le \left\|(A-\gamma f)q\right\| \cdot \|x_n - x_{n+1}\| + \left\langle(A-\gamma f)q, q-x_n\right\rangle,\end{aligned} \tag{3.54}$$

so

$$\limsup_{n\to\infty}\left\langle(A-\gamma f)q, q-x_{n+1}\right\rangle \le 0. \tag{3.55}$$

Note that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|\varepsilon_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\varepsilon_n A)K_n-q\|^2\\
&= \|\varepsilon_n(\gamma f(x_n)-Aq)+\beta_n(x_n-q)+((1-\beta_n)I-\varepsilon_n A)(K_n-q)\|^2\\
&\le \left\|(1-\beta_n)\frac{(1-\beta_n)I-\varepsilon_n A}{1-\beta_n}(K_n-q)+\beta_n(x_n-q)\right\|^2\\
&\quad + 2\varepsilon_n\langle (A-\gamma f)q, x_{n+1}-q\rangle\\
&\le (1-\beta_n)\left\|\frac{(1-\beta_n)I-\varepsilon_n A}{1-\beta_n}(K_n-q)\right\|^2+\beta_n\|x_n-q\|^2+2\varepsilon_n\gamma\alpha\\
&\quad \times\langle f(x_n)-f(q), x_{n+1}-q\rangle + 2\varepsilon_n\langle\gamma f(q)-Aq, x_{n+1}-q\rangle\\
&\le (1-\beta_n)\left\|\frac{(1-\beta_n)I-\varepsilon_n A}{1-\beta_n}(K_n-q)\right\|^2+\beta_n\|x_n-q\|^2+2\varepsilon_n\gamma\alpha\\
&\quad \times\|x_n-q\|\cdot\|x_{n+1}-q\| + 2\varepsilon_n\langle\gamma f(q)-Aq, x_{n+1}-q\rangle\\
&\le \frac{\|(1-\beta_n)I-\varepsilon_n A\|^2}{1-\beta_n}\|K_n-q\|^2+\beta_n\|x_n-q\|^2+\varepsilon_n\gamma\alpha\\
&\quad \times\left(\|x_n-q\|^2+\|x_{n+1}-q\|^2\right)+2\varepsilon_n\langle\gamma f(q)-Aq, x_{n+1}-q\rangle\\
&\le \left(\frac{((1-\beta_n)-\overline{\gamma}\varepsilon_n)^2}{1-\beta_n}+\beta_n+\varepsilon_n\gamma\alpha\right)\|x_n-q\|^2+\varepsilon_n\gamma\alpha\|x_{n+1}-q\|^2\\
&\quad + 2\varepsilon_n\langle\gamma f(x_n)-Aq, x_{n+1}-q\rangle,
\end{aligned} \tag{3.56}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\le \left(1-\frac{2(\overline{\gamma}-\alpha\gamma)\varepsilon_n}{1-\alpha\gamma\varepsilon_n}\right)\|x_n-q\|^2\\
&\quad + \frac{\varepsilon_n}{1-\alpha\gamma\varepsilon_n}\left\{\frac{\overline{\gamma}^2\varepsilon_n^2}{1-\beta_n}\|x_n-q\|^2+2\varepsilon_n\langle\gamma f(x_n)-Aq, x_{n+1}-q\rangle\right\}.
\end{aligned} \tag{3.57}$$

Let

$$\begin{aligned}
\sigma_n &= \frac{\varepsilon_n}{1-\alpha\gamma\varepsilon_n}\left\{\frac{\overline{\gamma}^2\varepsilon_n^2}{1-\beta_n}\|x_n-q\|^2+2\varepsilon_n\langle\gamma f(x_n)-Aq, x_{n+1}-q\rangle\right\},\\
\rho_n &= \frac{2(\overline{\gamma}-\alpha\gamma)\varepsilon_n}{1-\alpha\gamma\varepsilon_n},
\end{aligned} \tag{3.58}$$

then we have

$$\limsup_{n\to\infty}\frac{\sigma_n}{\rho_n}\le 0. \tag{3.59}$$

Applying Lemma 2.2, we can conclude that $\{x_n\}$ converges strongly to q in norm. This completes the proof. □

As direct consequences of Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C\to\mathbb{R}$ satisfying (A1)–(A4). Let $S_i : C\to C$ be a family κ_i-strict pseudocontractions for some $0\le\kappa_i<1$. Assume the set $\Omega=\cap_{i=1}^{\infty}F(S_i)\cap EP(F)\neq\emptyset$. Let f be a contraction of H into itself with $\alpha\in(0,1)$ and let A be an α-inverse strongly monotone mapping. Let F be a strongly positive linear-bounded operator on H with coefficient $\gamma>0$ and $0<\gamma<\overline{\gamma}/\alpha$ and $\tau<1$. Let W_n be the mapping generated by S_i' and t_i, where $S_i : C\to C$ is a nonexpansive mapping with a fixed point. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{gathered}F(z_n,y)+\frac{1}{\lambda_n}\langle y-z_n,z_n-x_n\rangle\ge 0,\\ K_n=\alpha_n x_n+(1-\alpha_n)W_n z_n,\\ x_{n+1}=\varepsilon_n\gamma f(x_n)+\beta_n x_n+\left((1-\beta_n)I-\varepsilon_n A\right)K_n,\end{gathered} \tag{3.60}$$

where $\{\varepsilon_n\}$, $\{\beta_n\}$, $\{\alpha_n\}$, and $\{\lambda_n\}$ are sequences in $(0,1)$. Assume that the control sequences satisfy the following restrictions:

(i) $\lim_{n\to\infty}\varepsilon_n=0$ *and* $\Sigma_{n=1}^{\infty}\varepsilon_n=\infty$;

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$;

(iii) $0<\liminf_{n\to\infty}\lambda_n\le\limsup_{n\to\infty}\lambda_n<1$;

(iv) $\lim_{n\to\infty}|\lambda_{n+1}-\lambda_n|=\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n|=0$;

(v) $0<t_n\le b<1$;

(vi) $\lambda_n<2\alpha$.

Then $\{x_n\}$ converges strongly to $w\in\Omega$ where $w=P_\Omega(I-A+\gamma f)w$.

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