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Research Article

Strong Convergence Theorems for a Countable Family of Total Quasi-ϕ-Asymptotically Nonexpansive Nonself Mappings

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The purpose of this paper is to introduce a class of total quasi-ϕ-asymptotically nonexpansive-nonself mappings and to study the strong convergence under a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper extend and improve the corresponding results announced by some authors recently.

1. Introduction

Throughout this paper, we assume that E is a real Banach space, C is a nonempty closed and convex subset of E, E^* is the dual space of E, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \left\{ f^* \in E^*, \langle x, f^* \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad x \in E. \tag{1.1}$$

Recall that a Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U = \{z \in E : \|z\| = 1\}$ with $x \neq y$. E is said to be *uniformly convex*, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|x + y\|/2 < 1 - \delta$ for all $x, y \in U$ with $\|x - y\| \geq \epsilon$. E is said to be *smooth*, if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{**}$$

exists for all $x, y \in U$. And E is said to be *uniformly smooth*, if the above limit is exists uniformly for $x, y \in U$.

In the sequel, we shall denote the fixed point set of a mapping T by $F(T)$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ ($x_n \rightharpoonup x$) will denote strong (weak) convergence of the sequence $\{x_n\}$ to x.

A mapping $T : C \to C$ is said to be *nonexpansive*, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

Recall that a subset C of E is said to be *retract* of E, if there exists a continuous mapping $P : E \to C$ such that $Px = x$, for all $x \in C$.

It is well known that every nonempty closed and convex subset of a uniformly convex Banach space is a retract of E. A mapping $P : E \to C$ is said to be a *retraction*, if $P^2 = P$. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A mapping $P : E \to C$ is said to be a *nonexpansive retraction*, if it is nonexpansive and it is a retraction from E to C.

In the sequel, we assume that E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty closed convex subset of E. Throughout this paper we assume that $\phi : E \times E \to \mathcal{R}^+$ is the Lyapunov function which is defined by

$$\phi(x, y) = \|x\| - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.4}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E, \tag{1.5}$$

$$\phi\Big(x, J^{-1}(\lambda J y + (1 - \lambda) J z)\Big) \leq \lambda \phi(x, y) + (1 - \lambda)\phi(x, z), \quad \forall x, y \in E. \tag{1.6}$$

Following Alber [1], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \arg \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{1.7}$$

Lemma 1.1 (see [1]). *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset of E. Then the following conclusions hold:*

(1) $\phi(x, \Pi_C y) + \phi(\Pi_c y, y) \leq \phi(x, y)$ *for all $x \in C$ and $y \in E$;*

(2) *If $x \in E$ and $z \in C$, then $z = \Pi_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \geq 0$, for all $y \in C$;*

(3) *For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Remark 1.2. If E is a real Hilbert space H, then $\phi(x,y) = \|x-y\|^2$ and $\Pi_C = P_C$ (the metric projection of H onto C).

A mapping $T : C \to C$ is said to be *closed*, if for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

Definition 1.3. Let $P : E \to C$ be the nonexpansive retraction.

(1) $T : C \to E$ is said to be *quasi-ϕ-nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and

$$\phi(u, Tx) \le \phi(u, x), \quad \forall x \in C,\ u \in F(T). \tag{1.8}$$

(2) $T : C \to E$ is said to be *quasi-ϕ-asymptotically nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and there exists a real sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$\phi\left(u, T(PT)^{n-1}x\right) \le k_n \phi(u, x), \quad \forall x \in C,\ u \in F(T),\ n \ge 1. \tag{1.9}$$

(3) $T : C \to E$ is said to be *total quasi-ϕ-asymptotically nonexpansive nonself mapping*, if $F(T) \neq \emptyset$ and there exists nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ with $\rho(0) = 0$ such that for all $x \in C$, $u \in F(T)$

$$\phi\left(u, T(PT)^{n-1}x\right) \le \phi(u, x) + \nu_n \rho(\phi(u, x)) + \mu_n, \quad \forall n \ge 1. \tag{1.10}$$

(4) A countable family of nonself mappings $\{T_i\} : C \to E$ is said to be *uniformly total quasi-ϕ-asymptotically nonexpansive*, if $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and there exists nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ with $\nu_n \to 0$, $\mu_n \to 0$ (as $n \to \infty$) and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ with $\rho(0) = 0$ such that for each $i \ge 1$ and all $x \in C$, $u \in \bigcap_{i=1}^{\infty} F(T_i)$

$$\phi\left(u, T_i(PT_i)^{n-1}x\right) \le \phi(u, x) + \nu_n \rho(\phi(u, x)) + \mu_n, \quad \forall n \ge 1. \tag{1.11}$$

Remark 1.4. From the definitions, it is easy to know that

(1) If T is a quasi-ϕ-nonexpansive nonself mapping, then it must be a quasi-ϕ-asymptotically nonexpansive nonself mapping with $\{k_n = 1\}$.

(2) Taking $\rho(t) = t$, $t > 0$, $\nu_n = (k_n - 1)$ and $\mu_n = 0$, then (1.9) can be rewritten as

$$\phi\left(u, T(PT)^{n-1}x\right) \le \phi(u, x) + \nu_n \rho(\phi(u, x)) + \mu_n, \quad \forall n \ge 1,\ x \in C,\ u \in F(T). \tag{1.12}$$

This implies that each quasi-ϕ-asymptotically nonexpansive nonself mapping must be a total quasi-ϕ-asymptotically nonexpansive nonself mapping, but the converse is not true.

A nonself mapping $T : C \to E$ is said to be *uniformly L-Lipschitz continuous*, if there exists a constant $L > 0$ such that

$$\left\|T(PT)^{n-1}x - T(PT)^{n-1}y\right\| \leq L\|x - y\|, \quad \forall x, y \in C,\ n \geq 1. \tag{1.13}$$

Lemma 1.5 (see [2]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ (as $n \to \infty$) and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$ (as $n \to \infty$).*

Lemma 1.6. *Let E be a smooth, strictly convex, and reflexive Banach space and C be a nonempty closed and convex subset E. Let $T : C \to E$ be a closed and total quasi-ϕ-asymptotically nonexpansive nonself mapping with nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ and $\rho(0) = 0$. Then the fixed point set $F(T)$ is a closed and convex subset of C.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $x_n \to u$ (as $n \to \infty$). Since $Tx_n = x_n \to u$, by the closeness of T, we have $u = Tu$, that is, $u \in F(T)$. This shows that $F(T)$ is a closed set in C.

Next, we prove that $F(T)$ is convex. For any $x, y \in F(T)$, $t \in (0,1)$, putting $q = tx + (1 - t)y$, we prove that $q \in F(T)$. Indeed, let $\{u_n\}$ be a sequence generated by

$$\begin{aligned} &u_1 = Tq, \quad u_2 = TPTq = TPu_1, \quad u_3 = T(PT)^2 q = TPu_2, \ldots, \\ &u_n = T(PT)^{n-1} q = TPu_{n-1}, \ldots, \end{aligned} \tag{1.14}$$

we have

$$\begin{aligned} \phi(q, u_n) &= \|q\|^2 - 2\langle q, Ju_n\rangle + \|u_n\|^2 \\ &= \|q\|^2 - 2t\langle x, Ju_n\rangle - 2(1-t)\langle y, Ju_n\rangle + \|u_n\|^2 \\ &= \|q\|^2 + t\phi(x, u_n) + (1-t)\phi(y, u_n) - t\|x\|^2 - (1-t)\|y\|^2. \end{aligned} \tag{1.15}$$

Since

$$\begin{aligned} &t\phi(x, u_n) + (1-t)\phi(y, u_n) \\ &\quad \leq t(\phi(x, q) + \nu_n\rho(\phi(x, q)) + \mu_n) + (1-t)(\phi(y, q) + \nu_n\rho(\phi(y, q)) + \mu_n) \\ &\quad = t\left(\|x\|^2 - 2\langle x, Jq\rangle + \|q\|^2 + \nu_n\rho(\phi(x, q)) + \mu_n\right) \\ &\qquad + (1-t)\left(\|y\|^2 - 2\langle y, Jq\rangle + \|q\|^2 + \nu_n\rho(\phi(y, q)) + \mu_n\right) \\ &\quad = t\|x\|^2 + (1-t)\|y\|^2 - \|q\|^2 + t\nu_n\rho(\phi(x, q)) + (1-t)\nu_n\rho(\phi(y, q)) + \mu_n. \end{aligned} \tag{1.16}$$

Substituting (1.16) into (1.15), and simplifying we have

$$\phi(q, u_n) \leq t\nu_n\rho(\phi(x, q)) + (1-t)\nu_n\rho(\phi(y, q)) + \mu_n \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{1.17}$$

By Lemma 1.5, we have $u_n \to q$ $(n \to \infty)$. This implies that $u_{n+1} \to q$ $(n \to \infty)$.

Since $u_{n+1} = T(PT)^n q = TPT(PT)^{n-1} q = TPu_n$ and T is closed, we have $q = TPq$. Since $q \in C$, $Pq = q$, thus $q = Tq$. this implies that $F(T)$ is a convex set in C. □

Concerning the strong and weak convergence of asymptotically nonexpansive self or nonself mappings, relatively nonexpansive, quasi-ϕ-nonexpansive and quasi-ϕ-asymptotically nonexpansive self or nonself mappings have been considered extensively by several authors in the setting of Hilbert or Banach spaces (see e.g., [2–19]).

The purpose of this paper is to modify the Halpern and Mann-type iteration algorithm for a family of of total quasi-ϕ-asymptotically nonexpansive nonself mappings and to have the strong convergence under removing $F(T)$ is a convex set of condition and a limit condition only in the framework of Banach spaces. As an application, we utilize our results to study the approximation problem of solution to a system of equilibrium problems. The results presented in the paper extend and improve the corresponding results of Chang et al. [4–7], W. P. Guo and W. Guo [8], Hao et al. [9], Kamimura and Takahashi [10], Kiziltunc and Temir [11], Nilsrakoo and Saejung [2], Pathak et al. [12], Qin et al. [13], Su et al. [14], Thianwan [15], Wang et al. [16], Yıldırım and Özdemir [17], Yang and Xie [18], Zegeye et al. [19], Kanjanasamranwong et al. [20], Saewan and Kumam [21–24] and Wattanawitoon and Kumam [25].

2. Main Results

Theorem 2.1. *Let E be a real uniformly convex and uniformly smooth Banach space, and C be a nonempty closed convex subset E. Let $T_i : C \to E, i = 1,2,\dots$ be a family of closed and uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with nonnegative real sequence $\{\nu_n\}$, $\{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \to \mathcal{R}^+$ such that $\nu_n \to 0$, $\mu_n \to 0$ and $\rho(0) = 0$, and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{\beta_n\}$ be a sequence in $(0,1)$ satisfying the following conditions:*

(a) $\lim_{n\to\infty}\alpha_n = 0$;

(b) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

Let $\{x_n\}$ be a sequence generated by

$$
\begin{gathered}
x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\
y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n)\left(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n\right)\right], \quad i \geq 1, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \theta_n \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1.
\end{gathered}
\tag{2.1}
$$

where $\theta_n = \nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n$, for all $n \geq 1$, $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$. If $\mathcal{F}$ is a nonempty-bounded subset in C, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. We divide the proof of Theorem 2.1 into five steps.

(I) *$\mathcal{F}$ and C_n, $n \geq 1$ are closed and convex subset in C.*

In fact, it follows from Lemma 1.6 that $F(T_i)$, $i \geq 1$ is closed and convex subset of C. Therefore $\mathcal{F}$ is a closed and convex subset in C.

Again by the assumption that $C_1 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \geq 2$. In view of the definition of ϕ we have that

$$\begin{aligned} C_{n+1} &= \left\{ z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \theta_n \right\} \\ &= \bigcap_{i\geq 1} \{ z \in C_n : \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) + \theta_n \} \cap C_n \\ &= \bigcap_{i\geq 1} \Big\{ z \in C_n : 2\alpha_n \langle z, Jx_1\rangle + 2(1-\alpha_n)\langle z, Jx_n\rangle - 2\langle z, Jy_{n,i}\rangle \\ &\qquad \leq \alpha_n \|x_1\|^2 + (1-\alpha_n)\|x_n\|^2 - \|y_{n,i}\|^2 + \theta_n \Big\} \cap C_n. \end{aligned} \tag{2.2}$$

This implies that C_{n+1} is closed and convex. The conclusion is proved.

(II) *Now we prove that* $\mathcal{F} \subset C_n$, $n \geq 1$.

In fact, it is obvious that $\mathcal{F} \subset C_1 = C$. Suppose that $\mathcal{F} \subset C_n$ for some $n \geq 2$. Letting

$$w_{n,i} = J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n\right), \tag{2.3}$$

it follows from (1.6) that for any $u \in \mathcal{F} \subset C_n$ we have

$$\begin{aligned} \phi(u, y_{n,i}) &= \phi\left(u, J^{-1}(\alpha_n J x_1 + (1-\alpha_n) J w_{n,i})\right) \\ &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, w_{n,i}), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \phi(u, w_{n,i}) &= \phi\left(u, J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_i (PT_i)^{n-1} x_n\right)\right) \\ &\leq \beta_n \phi(u, x_n) + (1-\beta_n)\phi\left(u, T_i (PT_i)^{n-1} x_n\right) \\ &\leq \beta_n \phi(u, x_n) + (1-\beta_n)\{\phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n\} \\ &= \phi(u, x_n) + (1-\beta_n)(\nu_n \rho(\phi(u, x_n)) + \mu_n) \\ &\leq \phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n, \end{aligned} \tag{2.5}$$

therefore we have

$$\begin{aligned} \sup_{i\geq 1}\phi(u, y_{n,i}) &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\{\phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n\} \\ &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n \\ &\leq \alpha_n \phi(u, x_1) + (1-\alpha_n)\phi(u, x_n) + \theta_n, \end{aligned} \tag{2.6}$$

where $\theta_n = \nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n$. This shows that $u \in C_{n+1}$, and so $\mathcal{F} \subset C_{n+1}$. The conclusion is proved.

(III) *Next we prove that* $\{x_n\}$ *is a Cauchy sequence in* C.

In fact, since $x_n = \Pi_{C_n} x_1$, from Lemma 1.1(2) we have

$$\langle x_n - y, Jx_1 - Jx_n \rangle \geq 0, \quad \forall y \in C_n. \tag{2.7}$$

Again since $\mathcal{F} \subset C_n$, for all $n \geq 1$, we have

$$\langle x_n - u, Jx_1 - Jx_n \rangle \geq 0, \quad \forall u \in \mathcal{F}. \tag{2.8}$$

It follows from Lemma 1.1(1) that for each $u \in \mathcal{F}$ and for each $n \geq 1$

$$\phi(x_n, x_1) = \phi(\Pi_{C_n} x_1, x_1) \leq \phi(u, x_1) - \phi(u, x_n) \leq \phi(u, x_1). \tag{2.9}$$

Therefore $\{\phi(x_n, x_1)\}$ is bounded. By virtue of (1.5), $\{x_n\}$ is also bounded.

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we have $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$, for all $n \geq 1$. This implies that $\{\phi(x_n, x_1)\}$ is nondecreasing. Hence the limit $\lim_{n\to\infty} \phi(x_n, x_1)$ exists. By the construction of C_n, for any positive integer $m \geq n$, we have $C_m \subset C_n$ and $x_m = \Pi_{C_m} x_1 \in C_n$. This shows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \longrightarrow 0, \quad \text{as } n, m \longrightarrow \infty. \tag{2.10}$$

It follows from Lemma 1.5 that $\lim_{m,n\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a nonempty closed subset of Banach space E, it is complete, without loss of generality, we can assume that $x_n \to x^*$ $(n \to \infty)$.

By the assumption, it is easy to see that

$$\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \left(\nu_n \sup_{u\in\mathcal{F}} \rho(\phi(u, x_n)) + \mu_n \right) = 0. \tag{2.11}$$

(IV) *Now we prove that* $x^* \in \mathcal{F}$.

In fact, since $x_{n+1} \in C_{n+1}$ and $\alpha_n \to 0$, it follows from (2.1) and (2.11) that

$$\sup_{i\geq 1} \phi(x_{n+1}, y_{n,i}) \leq \alpha_n \phi(x_{n+1}, x_1) + (1 - \alpha_n)\phi(x_{n+1}, x_n) + \theta_n \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty). \tag{2.12}$$

Since $x_n \to x^*$, by virtue of Lemma 1.5 for each $i \geq 1$,we have

$$\lim_{n\to\infty} y_{n,i} = x^*. \tag{2.13}$$

Since $\{x_n\}$ is bounded, $\{T_i\}_{i=1}^{\infty}$ is uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with nonnegative real sequence $\{\nu_n\}, \{\mu_n\}$ and a strictly increasing continuous function $\rho : \mathcal{R}^+ \rightarrow \mathcal{R}^+$ such that $\nu_n \rightarrow 0$, $\mu_n \rightarrow 0$, and $\rho(0) = 0$, for any given $u \in \mathcal{F}$, we have

$$\phi\left(u, T_i(PT_i)^{n-1}x_n\right) \leq \phi(u, x_n) + \nu_n \rho(\phi(u, x_n)) + \mu_n. \tag{2.14}$$

This implies that $\{T_i(PT_i)^{n-1}x_n\}$ is uniformly bounded. Since

$$\begin{aligned} \|w_{n,i}\| &= \left\| J^{-1}\left(\beta_n J x_n + (1-\beta_n) J T_i(PT_i)^{n-1}x_n\right)\right\| \\ &\leq \beta_n \|x_n\| + (1-\beta_n)\left\| T_i \|PT_i\|^{n-1} x_n \right\| \\ &\leq \|x_n\| + \left\| T_i(PT_i)^{n-1}x_n \right\|. \end{aligned} \tag{2.15}$$

This implies that $\{w_{n,i}\}$ is also uniformly bounded.

Since $\alpha_n \rightarrow 0$, from (2.1), for each $i \geq 1$ we have

$$\lim_{n\to\infty} \left\| J y_{n,i} - J w_{n,i} \right\| = \lim_{n\to\infty} \alpha_n \|J x_1 - J w_{n,i}\| = 0. \tag{2.16}$$

Since J^{-1} is uniformly continuous on each bounded subset of E^*, it follows from (2.13) and (2.16) that

$$\lim_{n\to\infty} w_{n,i} = x^* \quad \text{for each } i \geq 1. \tag{2.17}$$

Since J is uniformly continuous on each bounded subset of E, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|J w_{n,i} - J x^*\| \\ &= \lim_{n\to\infty} \left\| \beta_n J x_n + (1-\beta_n) J T_i(PT_i)^{n-1}x_n - Jx^* \right\| \\ &= \lim_{n\to\infty} \left\| \beta_n (J x_n - J x^*) + (1-\beta_n)\left(J T_i(PT_i)^{n-1}x_n - Jx^* \right) \right\| \\ &= \lim_{n\to\infty} (1-\beta_n) \left\| J T_i(PT_i)^{n-1}x_n - Jx^* \right\|. \end{aligned} \tag{2.18}$$

By condition (b), we have that

$$\lim_{n\to\infty} \left\| J T_i(PT_i)^{n-1}x_n - Jx^* \right\| = 0. \tag{2.19}$$

Since J is uniformly continuous, this shows that $\lim_{n\to\infty} T_i(PT_i)^{n-1}x_n = x^*$ uniformly in $i \geq 1$.

Again by the assumptions that for each $i \geq 1$, T_i is uniformly L_i-Lipschitz continuous, thus we have

$$\begin{aligned}
&\left\| T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n \right\| \\
&\quad \leq \left\| T_i(PT_i)^n x_n - T_i(PT_i)^n x_{n+1} \right\| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| \\
&\qquad + \| x_{n+1} - x_n \| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\| \\
&\quad \leq (L_i + 1)\| x_n - x_{n+1} \| + \left\| T_i(PT_i)^n x_{n+1} - x_{n+1} \right\| + \left\| x_n - T_i(PT_i)^{n-1} x_n \right\|.
\end{aligned} \tag{2.20}$$

Since $\lim_{n\to\infty} T_i(PT_i)^{n-1} x_n = x^*$ and $x_n \to x^*$, these together with (2.20) imply that $\lim_{n\to\infty} \| T_i(PT_i)^n x_n - T_i(PT_i)^{n-1} x_n \| = 0$ and $\lim_{n\to\infty} T_i(PT_i)^n x_n = x^*$, that is,

$$\lim_{n\to\infty} T_i P (PT_i)^{n-1} x_n = x^*. \tag{2.21}$$

In view continuity of $T_i P$, it yields that $T_i P x^* = x^*$. Since $x^* \in C$, $Px^* = x^*$. This shows that $Tx^* = x^*$. By the arbitrariness of $i \geq 1$, we have $x^* \in \mathcal{F}$.

(V) *Finally we prove that* $x_n \to x^* = \Pi_{\mathcal{F}} x_1$.

Let $w = \Pi_{\mathcal{F}} x_1$. Since $w \in \mathcal{F} \subset C_n$ and $x_n = \Pi_{C_n} x_1$, we have $\phi(x_n, x_1) \leq \phi(w, x_1)$, for all $n \geq 1$. This implies that

$$\phi(x^*, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w, x_1). \tag{2.22}$$

In view of the definition of $\Pi_{\mathcal{F}} x_1$, from (2.22) we have $x^* = w$. Therefore $x_n \to x^* = \Pi_{\mathcal{F}} x_1$. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.1. Let $T_i : C \to E$, $i = 1, 2, \ldots$ be a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$, and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered}
x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\
y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1 - \alpha_n)\left(\beta_n J x_n + (1 - \beta_n) J T_i (PT_i)^{n-1} x_n\right)\right], \quad i \geq 1, \\
C_{n+1} = \left\{ z \in C_n : \sup_{i\geq 1} \phi(z, y_{n,i}) \leq \alpha_n \phi(z, x_1) + (1 - \alpha_n)\phi(z, x_n) + \theta_n \right\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1,
\end{gathered} \tag{2.23}$$

where $\theta_n = (k_n - 1)\sup_{u\in\mathcal{F}} \phi(u, x_n)$, $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i)$. If $\mathcal{F}$ is a nonempty bounded subset in C, then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. By Remark 1.4 $T_i : C \to E$, $i = 1, 2, \ldots$ be a family of closed and uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings that it is a family of closed and uniformly total quasi-ϕ-asymptotically nonexpansive nonself mappings with taking $\rho(t) = t$, $t > 0$, $\nu_n = (k_n - 1)$ and $\mu_n = 0$. Therefore all conditions in Theorem 2.1 are satisfied. By the similar methods as given in the proof of Theorem 2.1, we can prove that the sequence $\{x_n\}$ defined by (2.23) converges strongly to $\Pi_{\mathcal{F}} x_1$. □

Theorem 2.3. *Let E, C, $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.2. Let $T_i : C \to E$, $i = 1, 2, \ldots$ be a family of quasi-ϕ-nonexpansive nonself mappings such that $\mathcal{F} := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ and for each $i \geq 1$, T_i be uniformly L_i-Lipschitz continuous. Let $\{x_n\}$ be a sequence generated by*

$$\begin{gathered} x_1 \in E \text{ chosen arbitrarily; } C_1 = C, \\ y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n)\left(\beta_n J x_n + (1-\beta_n) J T_i x_n\right)\right], \quad i \geq 1, \\ C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \phi\left(z, y_{n,i}\right) \leq \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{gathered} \tag{2.24}$$

Then $\{x_n\}$ converges strongly to $\Pi_{\mathcal{F}} x_1$.

Proof. By Remark 1.4 $T_i : C \to E$, $i = 1, 2, \ldots$ be a family of quasi-ϕ-nonexpansive nonself mappings that it is a family of uniformly quasi-ϕ-asymptotically nonexpansive nonself mappings with sequence $\{k_n\} = \{1\}$. Hence $\theta_n = (k_n - 1)\sup_{u \in \mathcal{F}} \phi(u, x_n) = 0$ Therefore all conditions in Theorem 2.2 are satisfied. By the similar methods, we can prove that the sequence $\{x_n\}$ defined by (2.24) converges strongly to $\Pi_{\mathcal{F}} x_1$. □

3. Application and Example

In this section we utilize the results presented in Section 2 to prove a strong convergence theorem concerning maximal monotone operators in Hilbert spaces.

Let E be a real Hilbert space and let A be a maximal monotone operator from E to E. For each $r > 0$, we can define a single valued mapping $J_r^A : E \to E$ by $J_r^A = (I + rA)^{-1}$ and such a mapping J_r^A is called the *resolvent of* A. It is easy to prove that J_r^A is a nonexpansive mapping and $A^{-1}(0) = F(J_r^A)$ for all $r > 0$. Therefore it is a uniformly 1-Lipschitz continuous and quasi-ϕ-nonexpansive mapping. Hence for each $p \in F(J_r^A)$ and $w \in E$, we have

$$\phi\left(p, J_r^A w\right) \leq \phi(p, w), \tag{3.1}$$

and $F(J_r^A) = A^{-1}(0)$. These show that all conditions in Theorem 2.3 are satisfied. Hence from Theorem 2.3 we have the following.

Theorem 3.1. *Let E be a real Hilbert space. Let A_1, A_2 be two maximal monotone operators from E to E such that $\mathcal{F} = A_1^{-1}(0) \cap A_2^{-1}(0) \neq \emptyset$. Let $J_r^{A_1}$ and $J_r^{A_2}$ be the resolvent of A_1 and A_2, respectively, where $r > 0$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be the same as in Theorem 2.3 and $\{x_n\}$ be the sequence defined by*

$$\begin{aligned} & x_1 \in E \text{ chosen arbitrarily; } C_1 = E, \\ & y_{n,i} = J^{-1}\left[\alpha_n J x_1 + (1-\alpha_n)\left(\beta_n J x_n + (1-\beta_n) J J_r^{A_i} x_n\right)\right], \quad i = 1,2, \\ & C_{n+1} = \left\{z \in C_n : \max_{i=1,2} \phi(z, y_{n,i}) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\right\}, \\ & x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{aligned} \tag{3.2}$$

where P_C is the metric projection from H onto the subset $C \subset H$. Then the sequence $\{x_n\}$ defined by (3.2) *converges strongly to $P_{\mathcal{F}} x_1$.*

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