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Research Article

Constructions of Vector-Valued Filters and Vector-Valued Wavelets

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Let $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ be an m-dimensional vector. Then, it can be identified with an $m \times m$ circulant matrix. By using the theory of matrix-valued wavelet analysis (Walden and Serroukh, 2002), we discuss the vector-valued multiresolution analysis. Also, we derive several different designs of finite length of vector-valued filters. The corresponding scaling functions and wavelet functions are given. Specially, we deal with the construction of filters on symmetric matrix-valued functions space.

1. Introduction

Wavelet analysis has been investigated extensively due to its wide applications in pure and applied sciences. Many interesting books and papers on this topic have been published (see [1–7]). The construction of filter banks is very important in applied aspects. The analogous theory can be extended to the cases of vector-valued and matrix-valued function spaces (see [8–11]). For example, Xia and Suter in [11] proposed vector-valued wavelets and vector filter banks and established a sufficient condition on the matrix-valued filters such that the solution of the corresponding two-scale dilation equation is a matrix-valued scaling function for a matrix-valued multiresolution analysis. But they did not give any example of finite length matrix-valued filter. As for this reason, Walden and Serroukh in [9] studied the wavelet analysis of matrix-valued time series and gave the construction of several different finite length matrix-valued (2×2) filters; naturally, how to construct the filters for symmetric matrix case. Possible practical application of this scheme in signal and image processing is numerous. In voice privacy systems, a number of signals may need to be transmitted from one place to another, intermixing of the signals before transmission via the matrix-valued filters, combined with perfect reconstruction, adds greatly to the likelihood of secure communications. In a scalable coding application, the high-quality lower-resolution approximations produced may be transmitted via slower communication channels, while the original can be reproduced using the perfect reconstruction filter banks. Other application areas are in progressive coding scheme, multisatellite measurements of electromagnetic wave fields, analysis of climate-related time series, and analysis of space weather effects, and so on. Here, we shall mention the theory of continuous wavelet transforms for quaternion-valued functions (see [12, 13]). Applying the theory of matrix-valued wavelet analysis, The authors in [8] gave the construction of scaling functions and wavelet functions by identifying the quaternion-valued functions with the complex duplex matrix-valued functions. Also, Bahri in [14] discussed the construction of filter banks of quaternion-valued functions. On the other hand, a quaternion $a + bi + cj + dk$ $(a, b, c, d \in \mathbb{R})$ can be identified with a matrix $\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}$. Recently, a new work is to construct the filter banks of quaternion-valued functions by using method in [9] with this identification. It is well known that every m-dimensional vector corresponds to an $m \times m$ matrix, which is called the circulant matrix. Our purpose of the present paper is to study wavelet analysis of vector-valued time series directly by identifying the vector-valued functions with matrix-valued functions and derive several different designs of finite length vector-valued filters. In order to get more length filter banks, we need to improve the value of parameter ϵ in [9]. Also, the corresponding scaling function and wavelet function are given in the paper. Since the scaling and wavelet functions are connected with the vanishing moments and regularity, we will consider this problem in late publication.

This paper is organized as follows. In the remainder of this section, we state some preliminaries. Section 2 will introduce some important results of multiresolution analysis theory in the matrix-valued function cases. In Section 3, we give the construction of finite length vector-valued filters. Followed by several different filter designs, we gain the scaling functions and wavelet functions, respectively. In the last section, we deal with the same problem for the symmetric matrix-valued function cases.

Throughout this paper, the black characters are representation of vectors. Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$. Then, $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{C}^m$ denotes an m-dimensional vector. The mapping $\mathcal{M}$ from $\mathbb{C}^m$ to $\mathbb{C}^{m\times m}$ is defined by

$$\mathcal{M} : \mathbf{a} = (a_1, a_2, \ldots, a_m) \longmapsto \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_m \\ a_m & a_1 & a_2 & \cdots & a_{m-1} \\ a_{m-1} & a_m & a_1 & \cdots & a_{m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix} = \mathcal{M}(\mathbf{a}). \tag{1.1}$$

Clearly, if $m = 2$, $\mathcal{M}(\mathbf{a}) = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$ is a symmetric matrix, whose diagonal has the same number. If $m \geq 3$, $\mathcal{M}(\mathbf{a})$ is not symmetric. For example, let $m = 3$, we get $\mathcal{M}(\mathbf{a}) = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$, and $\mathcal{M}(\mathbf{a})$ is called the circulant matrix. It is a very important class of Toeplitz matrices (see [15, page 201]). And we can verify that $\mathcal{M}(\mathbf{a})\mathcal{M}(\mathbf{b}) = \mathcal{M}(\mathbf{c})$, where

$$\begin{aligned}\mathbf{c} = (c_1, c_2, \ldots, c_m) = (&a_1b_1 + a_2b_m + a_3b_{m-1} + \cdots + a_mb_2, a_1b_2 \\ &+ a_2b_1 + a_3b_m + \cdots + a_mb_3, \ldots, a_1b_m \\ &+ a_2b_{m-1} + a_3b_{m-2} + \cdots + a_mb_1).\end{aligned} \tag{1.2}$$

This is to say that for any $\mathbf{a}, \mathbf{b}$, $\mathcal{M}(\mathbf{a})\mathcal{M}(\mathbf{b})$ is closed under the matrix multiplication. An $m \times m$ complex matrix B is said to be normal if $B^H B = BB^H$, where B^H denotes the complex-conjugate transpose of B. Thus, we can see that, for every $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathcal{M}(\mathbf{a})$ is normal. Let

$$\mathbf{M} = \{\mathcal{M}(\mathbf{a}) : \mathbf{a} \in \mathbb{C}^m\}, \qquad \mathbf{M}^* = \{\mathcal{M}(\mathbf{a}) : \mathbf{a} \in \mathbb{C}^m, \det \mathcal{M}(\mathbf{a}) \neq 0\}. \tag{1.3}$$

Then, we have the following.

Theorem 1.1. $\mathbf{M}^*$ *is a subgroup of* $\mathbf{GL}(m, \mathbb{C})$ *in the sense of matrix multiplication, where* $\mathbf{GL}(m, \mathbb{C})$ *is the set of all nonsingular linear transforms on* $\mathbb{C}^m$.

2. Multiresolution Analysis on $L^2_M(\mathbb{R}, \mathbb{C}^{m\times m})$

Firstly, we introduce the basic knowledge of vector-valued functions which can be found in [10, 11]. Let

$$L^2(\mathbb{R}, \mathbb{C}^{m\times m}) = \left\{\mathbf{F}(t) = \left(F_{l,j}(t)\right)_{m\times m} : t \in \mathbb{R},\ F_{l,j}(t) \in L^2(\mathbb{R}, \mathbb{C}),\ 1 \le l,\ j \le m\right\} \tag{2.1}$$

denote the space of matrix-valued functions defined on $\mathbb{R}$ with value in $C^{m\times m}$. The Frobenius norm on $L^2(\mathbb{R}, \mathbb{C}^{m\times m})$ is defined by

$$\|\mathbf{F}(t)\| = \left(\sum_{l,j} \int_{\mathbb{R}} \left|F_{l,j}(t)\right|^2 dt\right)^{1/2}. \tag{2.2}$$

For $\mathbf{F}$, $\mathbf{G} \in L^2(\mathbb{R}, \mathbb{C}^{m\times m})$, the integral of matrix product $\mathbf{F}(t)\mathbf{G}^H(t)$ is denoted by

$$\langle \mathbf{F}, \mathbf{G} \rangle = \int_{\mathbb{R}} \mathbf{F}(t)\mathbf{G}^H(t)dt. \tag{2.3}$$

The above operation is not inner product in the common sense; however, it has the linear and commutative properties:

$$\langle \mathbf{F}, \alpha\mathbf{G}_1 + \beta\mathbf{G}_2 \rangle = \alpha^H \langle \mathbf{F}, \mathbf{G}_1 \rangle + \beta^H \langle \mathbf{F}, \mathbf{G}_2 \rangle, \qquad \langle \mathbf{F}, \mathbf{G} \rangle = \langle \mathbf{G}, \mathbf{F} \rangle^H. \tag{2.4}$$

For convenience, we also call the operator in (2.3) the "inner product." The concept of orthogonality on $L^2(\mathbb{R},\mathbb{C}^{m\times m})$ can be given in natural way: for $\mathbf{F}_j,\mathbf{F}_k \in L^2(\mathbb{R},\mathbb{C}^{m\times m})$, $\mathbf{F}_j,\mathbf{F}_k$ is called orthogonal if $\langle \mathbf{F}_j,\mathbf{F}_k\rangle = \mathbf{I}_m\delta_{j,k}$, where $\delta_{j,k}$ is the Kronecker delta. Let $L^2_M(\mathbb{R},\mathbb{C}^{m\times m})$ be a subspace of $L^2(\mathbb{R},\mathbb{C}^{m\times m})$ which is defined by

$$L^2_M(\mathbb{R},\mathbb{C}^{m\times m}) = \left\{ \mathbf{x}(t) = \begin{pmatrix} x_1(t) & x_2(t) & x_3(t) & \cdots & x_m(t) \\ x_m(t) & x_1(t) & x_2(t) & \cdots & x_{m-1}(t) \\ x_{m-1}(t) & x_m(t) & x_1(t) & \cdots & x_{m-2}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2(t) & x_3(t) & x_4(t) & \cdots & x_1(t) \end{pmatrix} : x_1(t), x_2(t), \ldots, x_m(t) \in L^2(\mathbb{R}) \right\}. \tag{2.5}$$

Let $\mathbf{Z}$ be a set of all integers. A sequence $\{\mathbf{\Phi}_k(t)\}_{k\in\mathbf{Z}}$ in $L^2_M(\mathbb{R},\mathbb{C}^{m\times m})$ is an orthogonal basis if it is orthogonal, and, for all $\mathbf{\Phi}(t) \in L^2_M(\mathbb{R},\mathbb{C}^{m\times m})$, there is a constant sequence $\{\mathbf{A}_k\}_{k\in\mathbf{Z}}$ in $\mathbf{M}$ such that

$$\mathbf{\Phi}(t) = \sum_{k\in\mathbf{Z}} \mathbf{A}_k \mathbf{\Phi}_k(t). \tag{2.6}$$

It is obvious that $\mathbf{A}_k = \langle \mathbf{\Phi}, \mathbf{\Phi}_k\rangle$. Let $x(t)$ be a function defined on $\mathbb{R}$. The Fourier transform is

$$\widehat{x}(f) = \int_{\mathbb{R}} x(t) e^{-2i\pi t f}\, dt. \tag{2.7}$$

Suppose that

$$\mathbf{\Phi}(t) = \begin{pmatrix} \phi_1(t) & \phi_2(t) & \phi_3(t) & \cdots & \phi_m(t) \\ \phi_m(t) & \phi_1(t) & \phi_2(t) & \cdots & \phi_{m-1}(t) \\ \phi_{m-1}(t) & \phi_m(t) & \phi_1(t) & \cdots & \phi_{m-2}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_2(t) & \phi_3(t) & \phi_4(t) & \cdots & \phi_1(t) \end{pmatrix}. \tag{2.8}$$

We say that $\mathbf{\Phi}(t)$ generates a multiresolution analysis for $L^2_M(\mathbb{R},\mathbb{C}^{m\times m})$ if the sequence of closed subspaces

$$V_j = \operatorname{span}\left\{ 2^{-j/2}\mathbf{\Phi}\left(\frac{t}{2^j} - k\right) : k \in \mathbf{Z} \right\} \tag{2.9}$$

is nested, such that

(1) $\cdots \subset V_3 \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \cdots$,

(2) $\overline{\bigcup_{j\in\mathbf{Z}} V_j} = L^2_M(\mathbb{R},\mathbb{C}^{m\times m})$, and $\bigcap_{j\in\mathbf{Z}} V_j = \{\mathbf{0}_m\}$, where $\mathbf{0}_m$ is an $m\times m$ matrix of zeros,

(3) $\mathbf{\Phi}(t) \in V_0$ if and only if $\mathbf{\Phi}(t-k) \in V_0$ for all $k \in \mathbf{Z}$,

(4) $\mathbf{\Phi}(t) \in V_j$ if and only if $(1/\sqrt{2})\mathbf{\Phi}(t/2) \in V_{j+1}$,

(5) $\{\mathbf{\Phi}(t-k) : k \in \mathbf{Z}\}$ is an orthonormal basis for V_0.

In this case, $\mathbf{\Phi}$ is called a scaling function. Let $\mathbf{\Phi} \in L_M^2(\mathbb{R}, \mathbb{C}^{m\times m})$. Then, the Fourier transform of $\mathbf{\Phi}$ is given by

$$\widehat{\mathbf{\Phi}}(f) = \begin{pmatrix} \widehat{\phi}_1(f) & \widehat{\phi}_2(f) & \widehat{\phi}_3(f) & \cdots & \widehat{\phi}_m(f) \\ \widehat{\phi}_m(f) & \widehat{\phi}_1(f) & \widehat{\phi}_2(f) & \cdots & \widehat{\phi}_{m-1}(f) \\ \widehat{\phi}_{m-1}(f) & \widehat{\phi}_m(f) & \widehat{\phi}_1(f) & \cdots & \widehat{\phi}_{m-2}(f) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \widehat{\phi}_2(f) & \widehat{\phi}_3(f) & \widehat{\phi}_4(f) & \cdots & \widehat{\phi}_1(f) \end{pmatrix}. \tag{2.10}$$

Evidently, $\widehat{\mathbf{\Phi}} \in L_M^2(\mathbb{R}, \mathbb{C}^{m\times m})$ and the Fourier transform of $\mathbf{\Phi}(2t-k)$ with respect to the variable t is

$$\frac{1}{2}\widehat{\mathbf{\Phi}}\left(\frac{f}{2}\right)e^{-i\pi k f}. \tag{2.11}$$

Notice that $\mathbf{\Phi} \in V_0 \subset V_{-1}$, $\{\mathbf{\Phi}(2t-k) : k \in \mathbf{Z}\}$ is an orthonormal basis for V_{-1}, it follows that there exist constant matrices $\mathbf{G}_k \in \mathbf{M}$, such that two-scale dilation equation holds:

$$\mathbf{\Phi}(t) = \sqrt{2}\sum_{k\in\mathbf{Z}}\mathbf{G}_k\mathbf{\Phi}(2t-k). \tag{2.12}$$

Let $\mathbf{G}(f) = \sum_{k\in\mathbf{Z}}\mathbf{G}_k e^{-i2\pi k f}$, then we have

$$\widehat{\mathbf{\Phi}}(f) = \frac{1}{\sqrt{2}}\mathbf{G}\left(\frac{f}{2}\right)\widehat{\mathbf{\Phi}}\left(\frac{f}{2}\right). \tag{2.13}$$

By the orthogonality,

$$\int_{\mathbf{R}}\mathbf{\Phi}(t)\mathbf{\Phi}^H(t-k)dt = \mathbf{I}_m\delta_{k0}, \quad k \in \mathbf{Z}. \tag{2.14}$$

We know that $\sum_l \mathbf{G}_l\mathbf{G}_{2k+l}^H = I_m\delta_{k0}$, $k \in \mathbf{Z}$. This implies that

$$\mathbf{G}(f)\mathbf{G}^H(f) + \mathbf{G}\left(f+\frac{1}{2}\right)\mathbf{G}^H\left(f+\frac{1}{2}\right) = 2\mathbf{I}_m. \tag{2.15}$$

Setting $f = 0$, we have

$$\mathbf{G}(0) = \sum_{k\in\mathbf{Z}}\mathbf{G}_k = \sqrt{2}\mathbf{I}_m, \qquad \mathbf{G}\left(\frac{1}{2}\right) = \mathbf{0}_m. \tag{2.16}$$

For $f \in \mathbb{R}$, we let

$$\mathbf{H}(f) = \sum_{k \in \mathbf{Z}} \mathbf{H}_k e^{-i2\pi k f} \tag{2.17}$$

satisfy

$$\begin{aligned} &\mathbf{G}(f)\mathbf{H}^H(f) + \mathbf{G}\left(f + \frac{1}{2}\right)\mathbf{H}^H\left(f + \frac{1}{2}\right) = \mathbf{0}_m, \\ &\mathbf{H}(f)\mathbf{H}^H(f) + \mathbf{H}\left(f + \frac{1}{2}\right)\mathbf{H}^H\left(f + \frac{1}{2}\right) = 2\mathbf{I}_m. \end{aligned} \tag{2.18}$$

Analogous to the proof of in [11, Proposition 1], we can get the following.

Theorem 2.1. *Suppose that $\widehat{\mathbf{\Psi}}(f) = (1/\sqrt{2})\mathbf{H}(f/2)\widehat{\mathbf{\Phi}}(f/2)$, then $\mathbf{\Psi}_k(t) = \mathbf{\Psi}(t-k)$, $k \in \mathbf{Z}$ constitute an orthonormal basis for $V_{-1} = W_0 \bigoplus V_0$, where $\mathbf{\Psi}$ is called a wavelet function.*

The matrix filters $\mathbf{G}(f)$ and $\mathbf{H}(f)$ are called matrix quadrature mirror filters (MQMF). Since $\mathbf{G}(f)$ is normal, by the spectral theorem of normal matrices in [16], we can obtain that $\mathbf{G}(f)$ is unitarily equivalent to a diagonal matrix, namely,

$$\mathbf{G}(f) = U \operatorname{diag}(\lambda_1(\mathbf{G}(f)), \lambda_2(\mathbf{G}(f)), \ldots, \lambda_m(\mathbf{G}(f)))U^H, \tag{2.19}$$

where $U \in \mathbf{U}(m)$, $\mathbf{U}(m)$ denotes the unitary matrix group of order m, $\lambda_q(\mathbf{G}(f))$ ($q = 1, 2, \ldots, m$) are the eigenvalues of $\mathbf{G}(f)$, and "diag" means the diagonal matrix. Generally, even if $m = 2$, it is possible that $\lambda_1(f) \neq \lambda_2(f)$. Also, it seems to be true that U in (2.19) should belong to $\mathbf{M}$. However, it is not the case; we shall display this fact. For simplicity, we assume that $m = 2$. Let $U = \left(\begin{smallmatrix} \alpha & \beta \\ \beta & \alpha \end{smallmatrix}\right) \in \mathbf{U}(2) \cap \mathbf{M}$ satisfy the relation

$$\mathbf{G}(f) = U \operatorname{diag}(\lambda_1(\mathbf{G}(f)), \lambda_2(\mathbf{G}(f)))U^H, \tag{2.20}$$

where $\lambda_1(\mathbf{G}(f)) \neq \lambda_2(\mathbf{G}(f))$ are nonzero eigenvalues. Write $\lambda_j = \lambda_j(\mathbf{G}(f))$. Since $\mathbf{G}(f) \in L^2_M(\mathbb{R}, \mathbb{C}^{2\times 2})$, it follows that $|\alpha|^2\lambda_1 + |\beta|^2\lambda_2 = |\alpha|^2\lambda_2 + |\beta|^2\lambda_1$. But $|\alpha|^2 + |\beta|^2 = 1$. Therefore, we have $|\alpha| = |\beta| = 1/\sqrt{2}$. On the other hand, from the equality $\overline{\alpha}\beta\lambda_1 + \overline{\beta}\alpha\lambda_2 = \overline{\alpha}\beta\lambda_2 + \overline{\beta}\alpha\lambda_1$, we have $(\overline{\alpha}\beta - \overline{\beta}\alpha)\lambda_1 = (\overline{\alpha}\beta - \overline{\beta}\alpha)\lambda_2$. But $(\overline{\alpha}\beta + \overline{\beta}\alpha) = 0$, which implies that $\overline{\alpha}\beta\lambda_1 = \overline{\alpha}\beta\lambda_2$. This is a contradiction.

It is natural to ask what is the form of 2×2 unitary matrices in $\mathbf{M}$? The following theorem will give the answer.

Theorem 2.2. *Let U be a 2×2 unitary matrix in $\mathbf{M}$. Then, $U = \left(\begin{smallmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{smallmatrix}\right)$ or $U = \left(\begin{smallmatrix} i\cos\theta & \sin\theta \\ \sin\theta & i\cos\theta \end{smallmatrix}\right)$.*

From the discussion for [11, Proposition 2, page 513], we have the following theorem.

Theorem 2.3. *If* $\inf_{|f|\le 1/4}|\lambda_q(\mathbf{G}(f))| > 0$ *for any* $1 \le q \le m$*, then the solution* $\mathbf{\Phi}$ *of the two-scale dilation equation* (2.13) *is a scaling function for* $L^2_M(\mathbb{R}, \mathbb{C}^{m\times m})$*, and*

$$\left\{\mathbf{\Psi}_{j,k} = 2^{-j/2}\mathbf{\Psi}\left(\frac{t}{2^j} - k\right) : j, k \in \mathbf{Z}\right\} \tag{2.21}$$

constitutes an orthonormal basis for the space $L^2_M(\mathbb{R}, \mathbb{C}^{m\times m})$.

From the construction of quaternion-valued filters in [8], we have seen that the estimation of the eigenvalues $\lambda_q(\mathbf{G}(f))$ can be transferred to calculate the value of $\det \mathbf{G}(f)$, while the latter is easy to be checked in practice. But in the present case, the situation is different; we have to involve the computation for all eigenvalues of $\mathbf{G}(f)$.

3. Construction of Filters

Let $\mathbf{G}(f)$ be a finite polynomial matrix in $e^{-i2\pi f}$, that is,

$$\mathbf{G}(f) = \sum_{l=0}^{L-1}\mathbf{G}_l e^{-il2\pi f}. \tag{3.1}$$

Suppose that $\mathbf{G}(0) = \sqrt{2}\mathbf{I}_m$ and satisfies

$$\mathbf{G}(f)\mathbf{G}^H(f) + \mathbf{G}\left(f + \frac{1}{2}\right)\mathbf{G}^H\left(f + \frac{1}{2}\right) = 2\mathbf{I}_m. \tag{3.2}$$

If $\inf_{|f|\le 1/4}|\lambda_q(\mathbf{G}(f))| > 0$ for all $1 \le q \le m$, then the solution of the two-scale dilation equation (2.13) is a scaling function in $L^2_M(\mathbb{R}, \mathbb{C}^{m\times m})$, and

$$\left\{\mathbf{\Psi}_{j,k}(t) = 2^{-j/2}\mathbf{\Psi}\left(\frac{t}{2^j} - k\right) : j, k \in \mathbf{Z}\right\} \tag{3.3}$$

is an orthonormal basis in $L^2_M(\mathbb{R}, \mathbb{C}^{m\times m})$. In order to get the designs of the vector-valued filters, we need to deduce that $\mathbf{G}(f)$ and $\mathbf{H}(f)$ should satisfy the necessary condition. Firstly, we consider that $\mathbf{G}(f)$ has the form

$$\mathbf{G}(f) = \frac{e^{i2\pi f\gamma}}{\sqrt{2}}\left(\mathbf{I}_m + e^{\epsilon i2\pi f}\mathbf{R}(2f)\right), \quad \epsilon \in \{-1, 1\}, \tag{3.4}$$

where γ is an integer, $\mathbf{R}(f) \in L_M^2(\mathbb{R}, \mathbb{C}^{m\times m})$ is a paraunitary matrix with unit periodicity, namely, $\mathbf{R}(f)\mathbf{R}^H(f) = \mathbf{I}_m$, $\mathbf{R}(f+1) = \mathbf{R}(f)$, $\mathbf{R}(0) = \mathbf{I}_m$. Actually, if ϵ in (3.4) is taken to be 0 and $\mathbf{R}(f)$ satisfies $\mathbf{R}(f)\mathbf{R}^H(f) = \mathbf{I}_m$, $\mathbf{R}(f+1) = -\mathbf{R}(f)$, $\mathbf{R}(0) = \mathbf{I}_m$, then we still have

$$\begin{aligned}
&\mathbf{G}(f)\mathbf{G}^H(f) + \mathbf{G}\left(f+\frac{1}{2}\right)\mathbf{G}^H\left(f+\frac{1}{2}\right) \\
&\quad = \frac{1}{2}\left[(\mathbf{I}_m + \mathbf{R}(2f))\left(\mathbf{I}_m + \mathbf{R}^H(2f)\right) + (\mathbf{I}_m + \mathbf{R}(2f+1))\left(\mathbf{I}_m + \mathbf{R}^H(2f+1)\right)\right] \\
&\quad = \frac{1}{2}\left[(\mathbf{I}_m + \mathbf{R}(2f))\left(\mathbf{I}_m + \mathbf{R}^H(2f)\right) + (\mathbf{I}_m - \mathbf{R}(2f))\left(\mathbf{I}_m - \mathbf{R}^H(2f)\right)\right] \\
&\quad = 2\mathbf{I}_m.
\end{aligned} \tag{3.5}$$

Namely, (2.15) and (2.16) are still true. Thus,

$$\lambda_q(\mathbf{G}(f)) = \frac{e^{i2\pi f\gamma}}{\sqrt{2}}\left\{1 + e^{\epsilon i2\pi f}\lambda_q(\mathbf{R}(2f))\right\}. \tag{3.6}$$

We are now in a position to define $\mathbf{H}(f)$. As shown in [9], the matrix $\mathbf{H}(f)$ can be given in the following:

$$\mathbf{H}(f) = e^{-2i\pi f(S-1+\delta)}\mathbf{G}\left(f+\frac{1}{2}\right), \tag{3.7}$$

where L is the design length of the filter $\mathbf{G}_l$, and $\delta \in \{0,1\}$, such that $L-1+\delta$ is odd. Actually, it is deduced that

$$\mathbf{H}(f) = \sum_{m=\delta}^{L-1+\delta} (-1)^{L-1+\delta-m}\mathbf{G}^H_{L-1+\delta-m}e^{-i2\pi fm}. \tag{3.8}$$

In particular, if L is even, then we take $\delta = 0$. Thus,

$$\mathbf{H}(f) = \sum_{l=0}^{L-1}\mathbf{H}_l e^{-i2\pi fl}, \tag{3.9}$$

where $\mathbf{H}_l = (-1)^{l+1}\mathbf{G}^H_{L-l-1}$, $l = 0, 1, \ldots, L-1$. If L is odd, then take $\delta = 1$ and let $\mathbf{G}_L = \mathbf{0}_m$, we add the filter length of $\mathbf{G}(f)$ to $L+1$. Suppose that

$$\mathbf{G}(f) = \sum_{L+1}^{l=0}\mathbf{G}_l e^{-il2\pi f}. \tag{3.10}$$

The similar design can be realized.

Design 1

Let $\mathbf{G}(f) = (1/\sqrt{2})(\mathbf{I}_2 + e^{i2\pi f}\operatorname{diag}(e^{-i4\pi f}, e^{-i4\pi f}))$, $\mathbf{G}(f)$ has the same eigenvalues $\lambda(\mathbf{G}(f)) = (1/\sqrt{2})(1+e^{-i2\pi f})$. It is obvious that $\inf_{|f|\le 1/4}|\lambda(\mathbf{G}(f))| = \sqrt{1+\cos 2\pi f} > 0$. At the same time, $\mathbf{G}(0) = \sqrt{2}\mathbf{I}_2$, $\mathbf{G}(1/2) = \mathbf{0}_2$, $\mathbf{G}(f+1) = \mathbf{G}(f)$. Consequently, we have

$$\begin{gathered}\mathbf{G}_0 = \mathbf{G}_1 = \frac{1}{\sqrt{2}}\mathbf{I}_2,\\ \mathbf{H}_0 = -\frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{H}_1 = \frac{1}{\sqrt{2}}\mathbf{I}_2.\end{gathered} \tag{3.11}$$

If $\mathbf{G}(f)$ is defined in the form $\mathbf{G}(f) = (1/\sqrt{2})(\mathbf{I}_2 + e^{-i2\pi f}\operatorname{diag}(e^{i4\pi f}, e^{i4\pi f}))$, we have

$$\begin{gathered}\mathbf{G}_0 = \mathbf{G}_{-1} = \frac{1}{\sqrt{2}}\mathbf{I}_2,\\ \mathbf{H}_0 = -\frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{H}_{-1} = \frac{1}{\sqrt{2}}\mathbf{I}_2.\end{gathered} \tag{3.12}$$

This is a simple case. In the following, we shall deal with the case that $\mathbf{G}(f)$ has different eigenvalues. For a diagonal matrix $\operatorname{diag}(\lambda_1, \lambda_2)$, if $\lambda_1 \neq \lambda_2$, $\rho_\theta = \left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$ is a rotation transform on $\mathbb{R}^2$, when $\theta = \pi/4$, $\rho_\theta \operatorname{diag}(\lambda_1, \lambda_2)\rho_\theta^H \in \mathbf{M}$. We state it as follows.

Theorem 3.1. *Let* $\operatorname{diag}(\lambda_1, \lambda_2)$ *be an arbitrary diagonal matrix. Then,* $\left(\begin{smallmatrix}\sqrt{2}/2 & \sqrt{2}/2\\ -\sqrt{2}/2 & \sqrt{2}/2\end{smallmatrix}\right)\operatorname{diag}(\lambda_1, \lambda_2)\left(\begin{smallmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{smallmatrix}\right) \in \mathbf{M}$.

Depending on this, we give the following designs.

Design 2

Let $\rho_\theta = \left(\begin{smallmatrix}\sqrt{2}/2 & \sqrt{2}/2\\ -\sqrt{2}/2 & \sqrt{2}/2\end{smallmatrix}\right)$. Then,

$$\begin{aligned}\mathbf{R}(2f) &= \begin{pmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}\operatorname{diag}\left(1, e^{i4\pi f}\right)\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{2}+\frac{1}{2}e^{i4\pi f} & -\frac{1}{2}+\frac{1}{2}e^{i4\pi f}\\ -\frac{1}{2}+\frac{1}{2}e^{i4\pi f} & \frac{1}{2}+\frac{1}{2}e^{i4\pi f}\end{pmatrix}\\ &= \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}e^{i4\pi f} + \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}.\end{aligned} \tag{3.13}$$

Clearly, $\mathbf{R}(f)\mathbf{R}^H(f) = \mathbf{I}_2$, $\mathbf{R}(f+1) = \mathbf{R}(f)$, $\mathbf{R}(0) = \mathbf{I}_2$. Taking $\gamma = -1$, $\epsilon = -1$ in (3.4), we obtain

$$\mathbf{G}(f) = \frac{e^{-i2\pi f}}{\sqrt{2}}\left(\mathbf{I}_2 + \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} e^{i2\pi f} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} e^{-i2\pi f}\right). \tag{3.14}$$

Consequently, we have

$$\lambda_1(\mathbf{G}(f)) = \frac{e^{-i2\pi f}}{\sqrt{2}}\left(1 + e^{i2\pi f}\right), \qquad \lambda_2(\mathbf{G}(f)) = \frac{e^{-i2\pi f}}{\sqrt{2}}\left(1 + e^{-i2\pi f}\right). \tag{3.15}$$

It is easy to verify that

$$\inf_{|f|\le 1/4}\left|\lambda_q(\mathbf{G}(f))\right| = \inf_{|f|\le 1/4}\sqrt{1+\cos(2\pi f)} > 0 \tag{3.16}$$

for $q = 1, 2$. Therefore, we get

$$\begin{gathered}\mathbf{G}(f) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \mathbf{I}_2 e^{-i2\pi f} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} e^{-i4\pi f}\right], \\ \mathbf{G}_0 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{G}_1 = \frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{G}_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.\end{gathered} \tag{3.17}$$

Let $\mathbf{G}_3 = 0_2$ and $L = 4$, from the relation (3.9), we then find that

$$\mathbf{H}_0 = \mathbf{0}_2, \qquad \mathbf{H}_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad \mathbf{H}_2 = -\frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{H}_3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.18}$$

To change the coefficients of filter bank $\mathbf{G}(f)$, it is valid to produce a type of unitary matrices U and U^H in Theorem 2.2 on left and right sides of $\mathbf{R}(2f)$.

In the following, we shall give a construction of more length filter banks.

Design 3

Let $\rho_\theta = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$. Setting $\alpha = (e^{2i\pi f} + e^{-2i\pi f})$, $\beta = (e^{6i\pi f} + e^{2i\pi f} + e^{-2i\pi f} + e^{-i6\pi f})$. Then,

$$\begin{aligned}
\mathbf{R}(2f) &= \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \operatorname{diag}\left(\frac{1}{2}\alpha, \frac{1}{4}\beta\right) \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \frac{1}{2}\alpha + \frac{1}{4}\beta & -\frac{1}{2}\alpha + \frac{1}{4}\beta \\ -\frac{1}{2}\alpha + \frac{1}{4}\beta & \frac{1}{2}\alpha + \frac{1}{4}\beta \end{pmatrix} \\
&= \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^{i6\pi f} + \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} e^{i2\pi f} + \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} e^{-i2\pi f} + \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^{-i6\pi f}.
\end{aligned} \tag{3.19}$$

Taking $\gamma = -3$, $\epsilon = 0$ in (3.4), we obtain $\mathbf{R}(f)\mathbf{R}^H(f) = \mathbf{I}_m$, $\mathbf{R}(f+1) = -\mathbf{R}(f)$, $\mathbf{R}(0) = \mathbf{I}_m$, and

$$\begin{aligned}
\inf_{|f|\le 1/4} \left|\lambda_1(\mathbf{G}(f))\right| &= \frac{1}{\sqrt{2}} \inf_{|f|\le 1/4} \sqrt{1+\cos(2\pi f)} > 0, \\
\inf_{|f|\le 1/4} \left|\lambda_2(\mathbf{G}(f))\right| &= \frac{1}{\sqrt{2}} \inf_{|f|\le 1/4} \sqrt{1+\frac{1}{2}(\cos(2\pi f)+\cos(6\pi f))} > 0.
\end{aligned} \tag{3.20}$$

Therefore, we get

$$\begin{aligned}
\mathbf{G}(f) = {} & \frac{1}{8\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{8\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} e^{-i4\pi f} + \frac{1}{\sqrt{2}}\mathbf{I}_2 e^{-6i\pi f} \\
& + \frac{1}{8\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} e^{-i8\pi f} + \frac{1}{8\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} e^{-i12\pi f}.
\end{aligned} \tag{3.21}$$

Consequently,

$$\mathbf{G}_0 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{G}_1 = \mathbf{0}_2, \qquad \mathbf{G}_2 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \tag{3.22}$$

$$\mathbf{G}_3 = \frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{G}_4 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \qquad \mathbf{G}_5 = \mathbf{0}_2, \qquad \mathbf{G}_6 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.23}$$

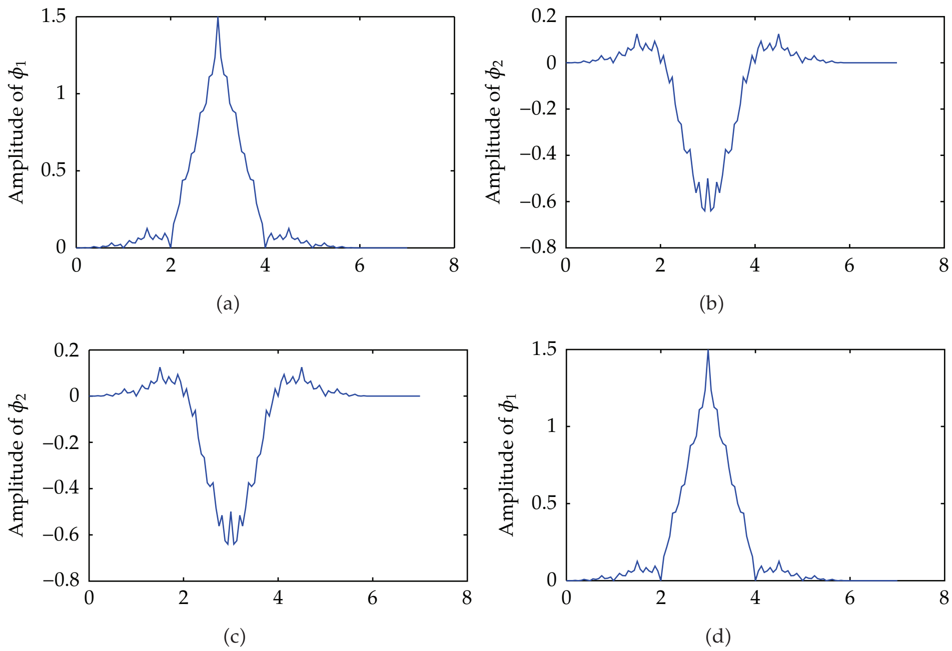

Figure 1: The graph of the scale function $\mathbf{\Phi}$ for Design 3.

Here, $L = 7$ is odd, so we let $\mathbf{G}_7 = \mathbf{0}_2$. Also, from (3.4), we have

$$\mathbf{H}_0 = \mathbf{0}_2, \quad \mathbf{H}_1 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{H}_2 = \mathbf{0}_2, \quad \mathbf{H}_3 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix},$$
$$\mathbf{H}_4 = -\frac{1}{\sqrt{2}}\mathbf{I}_2, \quad \mathbf{H}_5 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad \mathbf{H}_6 = \mathbf{0}_2, \quad \mathbf{H}_7 = \frac{1}{8\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3.24}$$

The corresponding 2×2 matrix functions $\mathbf{\Phi}$ and $\mathbf{\Psi}$ of this design are plotted in Figures 1 and 2, respectively. Notice that we take the values of $\phi_1(3)$ and $\phi_2(3)$ as $3/2$ and $-1/2$, respectively, in the figure plotting. However, this design does not determine the values of $\phi_1(3)$ and $\phi_2(3)$. Thus, in practical application, we have the freedom to choose the values of $\phi_1(3)$ and $\phi_2(3)$.

Next, we are going to construct the filters for the case of $m \geq 3$. We need to use the methods of discrete Fourier transform matrix (DFT). Let $\omega_m = e^{-2i\pi/m} = \cos(2\pi/m) - i\sin(2\pi/m)$. The parameter ω_m is an mth root of unity due to $\omega_m^m = 1$. Write $\beta_{jk} = \omega_m^{(j-1)(k-1)}$. Then, $\mathbf{F}_m = (\beta_{jk})_{m\times m}$ is called the discrete Fourier transform matrix of order m. For example, if $m = 2$, the discrete Fourier transform matrix $\mathbf{F}_2$ of order 2 is $\left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$,

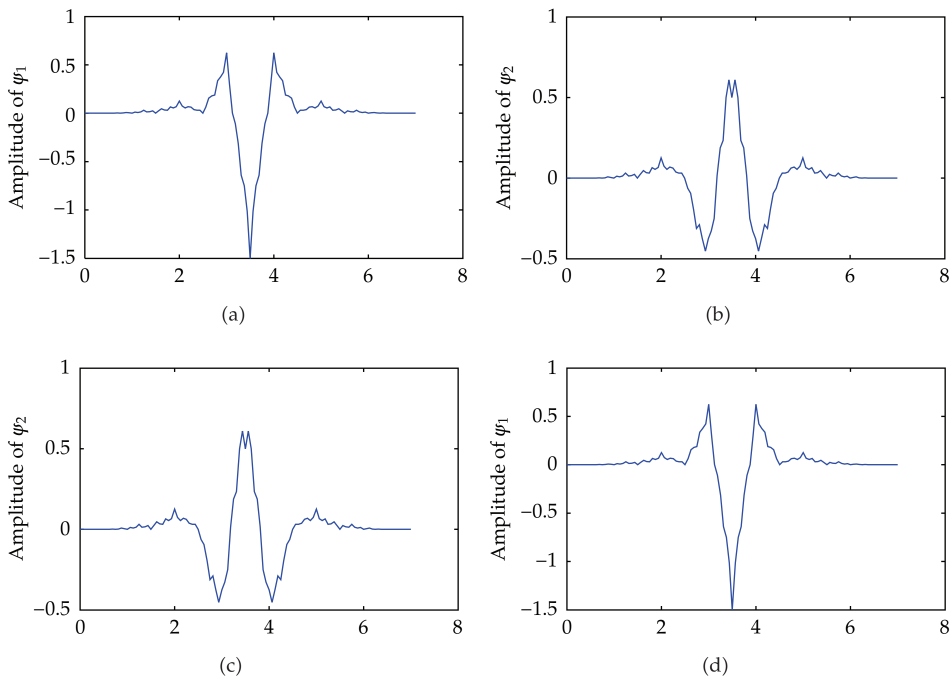

Figure 2: The graph of the wavelet function $\boldsymbol{\Psi}$ for Design 3.

$\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)\operatorname{diag}(\lambda_1(\mathbf{G}(f)), \lambda_2(\mathbf{G}(f)))\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right) \in \mathbf{M}$. We can construct the filters by substituting $\left(\begin{smallmatrix}1 & 1\\ 1 & -1\end{smallmatrix}\right)$ for $(1/\sqrt{2})\left(\begin{smallmatrix}1 & 1\\ -1 & 1\end{smallmatrix}\right)$. Now, we describe another case. Let $m = 4$, the discrete Fourier transform matrix $\mathbf{F}_4$ of order m is given by

$$\mathbf{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}, \tag{3.25}$$

and the inverse matrix of $\mathbf{F}_4$ is

$$\mathbf{F}_4^{-1} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}. \tag{3.26}$$

We have known that the discrete Fourier transform is an extremely important tool in applied mathematics and engineering, specially in signal processing. Let us define the downshift permutation $\mathbf{S}_4$ by

$$\mathbf{S}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{3.27}$$

and a vector $\mathbf{b} = (a_1, a_2, a_3, a_4)^T$, where T means the transpose. Then,

$$\mathcal{M}\left(\mathbf{b}^T\right) = \left(\mathbf{b}, \mathbf{S}_4\mathbf{b}, \mathbf{S}_4^2\mathbf{b}, \mathbf{S}_4^3\mathbf{b}\right) = \mathbf{F}_4^{-1}\operatorname{diag}(\mathbf{F}_4\mathbf{b})\mathbf{F}_4. \tag{3.28}$$

Actually, we can verify that $\mathbf{F}_4^{-1}\operatorname{diag}(\lambda_1(\mathbf{G}(f)), \lambda_2(\mathbf{G}(f)), \lambda_3(\mathbf{G}(f)) = \lambda_4(\mathbf{G}(f)))\mathbf{F}_4 \in \mathbf{M}$. According to this theory, we can do some designs of the filters for the case $m = 4$. Other cases can be done analogously.

4. Symmetric Matrix Case

In the above examples, we deal with the constructions of filter banks for $L_M^2(\mathbb{R}, \mathbb{C}^{2\times 2})$. As an application of this theory, we will discuss the same problem on the space

$$L_S^2\left(\mathbb{R}, \mathbb{C}^{2\times 2}\right) = \left\{ \begin{pmatrix} x_1(t) & x_2(t) \\ x_2(t) & x_3(t) \end{pmatrix} : x_1(t), x_2(t), x_3(t) \in L^2(\mathbb{R}) \right\}. \tag{4.1}$$

Obviously, $L_M^2(\mathbb{R}, \mathbb{C}^{2\times 2}) \subset L_S^2(\mathbb{R}, \mathbb{C}^{2\times 2})$. Since

$$\begin{aligned} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} &= \begin{pmatrix} a + 2b\cos\theta\sin\theta & b\left(\cos^2\theta - \sin^2\theta\right) \\ b\left(\cos^2\theta - \sin^2\theta\right) & a - 2b\cos\theta\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} a + b\sin 2\theta & b\cos 2\theta \\ b\cos 2\theta & a - b\sin 2\theta \end{pmatrix}, \end{aligned} \tag{4.2}$$

we can see that $\rho_\theta\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)\rho_\theta^H \notin \mathbf{M}$; however, $\rho_\theta\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)\rho_\theta^H$ is symmetric with different elements in diagonal if $\theta \neq 0, \pi/2, \pi, 3/2\pi, 2\pi$. From this, we can give some designs of filter banks in $L_S^2(\mathbb{R}, \mathbb{C}^{2\times 2})$.

Design 4

Multiplying matrices $\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$ on the left and right sides of $\mathbf{G}(f)$, respectively, in Design 3, we get the coefficients of the filter banks:

$$\mathbf{G}_0 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 2-\sqrt{3} & -1 \\ -1 & 2+\sqrt{3} \end{pmatrix}, \qquad \mathbf{G}_1 = \mathbf{0}_2, \qquad \mathbf{G}_2 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 6+\sqrt{3} & 1 \\ 1 & 6-\sqrt{3} \end{pmatrix},$$
$$\mathbf{G}_3 = \frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{G}_4 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 6+\sqrt{3} & 1 \\ 1 & 6-\sqrt{3} \end{pmatrix},$$
$$\mathbf{G}_5 = \mathbf{0}_2, \qquad \mathbf{G}_6 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 2-\sqrt{3} & -1 \\ -1 & 2+\sqrt{3} \end{pmatrix}, \tag{4.3}$$
$$\mathbf{H}_0 = \mathbf{0}_2, \qquad \mathbf{H}_1 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 2-\sqrt{3} & -1 \\ -1 & 2+\sqrt{3} \end{pmatrix},$$
$$\mathbf{H}_2 = \mathbf{0}_2, \qquad \mathbf{H}_3 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 6+\sqrt{3} & 1 \\ 1 & 6-\sqrt{3} \end{pmatrix},$$
$$\mathbf{H}_4 = -\frac{1}{\sqrt{2}}\mathbf{I}_2, \qquad \mathbf{H}_5 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 6+\sqrt{3} & 1 \\ 1 & 6-\sqrt{3} \end{pmatrix},$$
$$\mathbf{H}_6 = \mathbf{0}_2, \qquad \mathbf{H}_7 = \frac{1}{16\sqrt{2}}\begin{pmatrix} 2-\sqrt{3} & -1 \\ -1 & 2+\sqrt{3} \end{pmatrix}. \tag{4.4}$$

Notice that in this design the scaling function $\mathbf{\Lambda}$ and the wavelet function $\mathbf{\Upsilon}$ are

$$\mathbf{\Lambda} = \begin{pmatrix} \phi_1 - \frac{\sqrt{3}}{2}\phi_2 & -\frac{1}{2}\phi_2 \\ -\frac{1}{2}\phi_2 & \phi_1 + \frac{\sqrt{3}}{2}\phi_2 \end{pmatrix}, \qquad \mathbf{\Upsilon} = \begin{pmatrix} \psi_1 - \frac{\sqrt{3}}{2}\psi_2 & -\frac{1}{2}\psi_2 \\ -\frac{1}{2}\psi_2 & \psi_1 + \frac{\sqrt{3}}{2}\psi_2 \end{pmatrix}. \tag{4.5}$$

The corresponding 2×2 matrix functions $\mathbf{\Lambda}$ and $\mathbf{\Upsilon}$ of this design are plotted in Figures 3 and 4, respectively.

It is a well-known fact that every rotation operator $\rho_{\alpha,\beta,\gamma}$ on $\mathbb{R}^3$ can be described by three Euler's angles which are given by

$$\rho_{\alpha,\beta,\gamma} = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 \\ -\sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.6}$$

where $\alpha, \gamma \in [0, 2\pi]$, $\beta \in [0, \pi]$. Obviously, $\rho_{\alpha,\beta,\gamma}\,\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)\rho^H_{\alpha,\beta,\gamma}$ is also symmetric. The description of the rotation operator on $\mathbb{R}^n$ is analogous. Thus, we can easily construct the filter banks on $L^2_S(\mathbb{R}, \mathbb{C}^{n\times n})$.

As a final example of this paper, we give a construction in case $m = 4$ in space $L^2_S(\mathbb{R}, \mathbb{C}^{4\times 4})$.

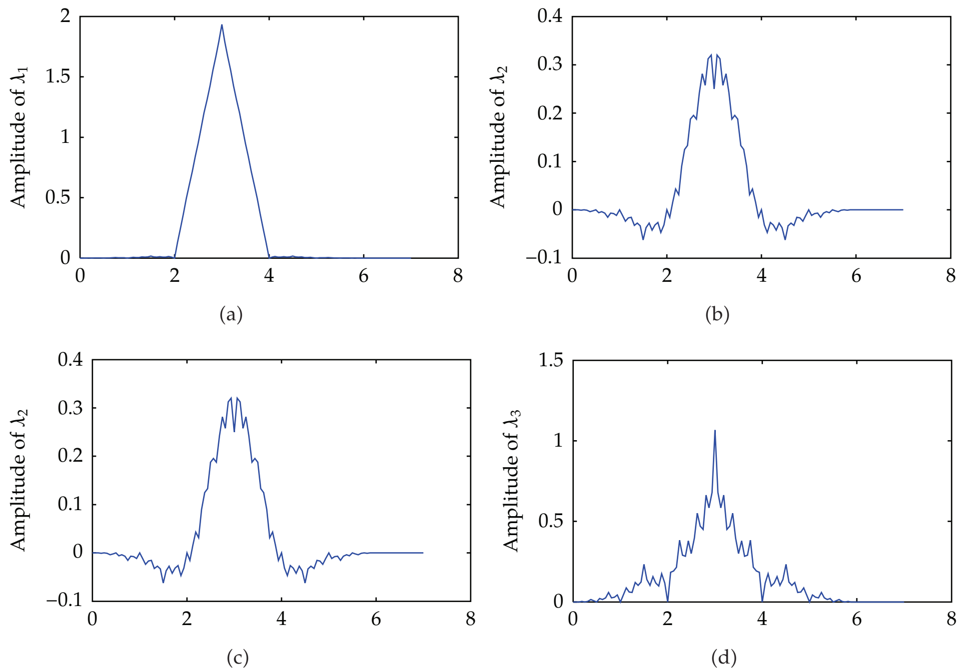

Figure 3: The graph of the scaling function $\boldsymbol{\Lambda}$ for Design 4.

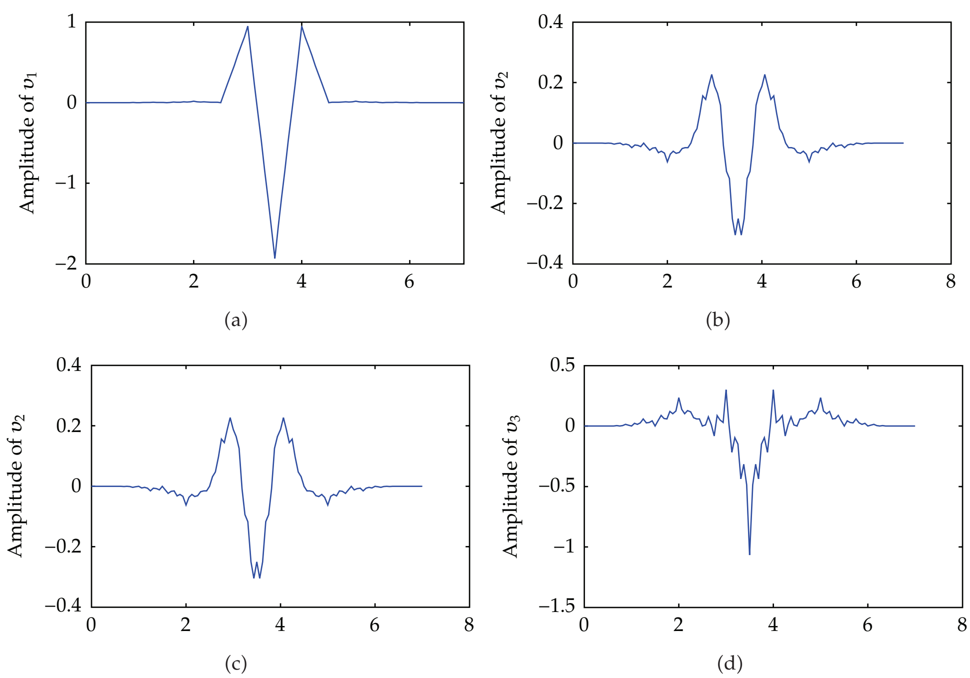

Figure 4: The graph of the wavelet function $\boldsymbol{\Upsilon}$ for Design 4.

Design 5

Let $U = (1/2)\left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{smallmatrix}\right)$. Then,

$$\begin{aligned} \mathbf{R}(2f) &= \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \operatorname{diag}\left(1, 1, e^{-i4\pi f}, e^{-i4\pi f}\right) \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \\ &= \frac{1}{4}\left(\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix} e^{-i4\pi f}\right). \end{aligned} \tag{4.7}$$

Taking $\gamma = -1, \epsilon = 1$ in (3.4), we obtain

$$\inf_{|f| \le 1/4} \left|\lambda_q(\mathbf{G}(f))\right| = \inf_{|f| \le 1/4} \sqrt{1 + \cos(2\pi f)} > 0 \tag{4.8}$$

for $q = 1, 2$, and

$$\begin{aligned} \mathbf{G}(f) &= \frac{e^{-i2\pi f}}{\sqrt{2}}\left(\mathbf{I}_4 + \frac{1}{4}\left[\begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} e^{i2\pi f} + \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -2 & 2 & 0 \\ -2 & 0 & 0 & 2 \end{pmatrix} e^{-2i\pi f}\right]\right) \\ &= \left(\frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{2}}\mathbf{I}_4 e^{-i2\pi f} + \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} e^{-4i\pi f}\right). \end{aligned} \tag{4.9}$$

Thus, we get

$$\mathbf{G}_0 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{G}_1 = \frac{1}{\sqrt{2}}\mathbf{I}_4, \qquad \mathbf{G}_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{4.10}$$

Let $\mathbf{G}_3 = \mathbf{0}_2$ and $L = 4$, from the relation (3.9), we then obtain

$$\mathbf{H}_0 = \mathbf{0}_4, \qquad \mathbf{H}_1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{H}_2 = -\frac{1}{\sqrt{2}}\mathbf{I}_4, \qquad \mathbf{H}_3 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}. \tag{4.11}$$

5. Conclusion

In this work, we discuss the problem of construction for vector-valued filters. By using the theory of matrix-valued wavelet analysis and technique of discrete Fourier transform matrix, we get some designs of vector-valued filters (circulant matrix space). And the corresponding scaling functions of multiresolution analysis and wavelet functions are obtained. Also, the analogous problem on the symmetric matrix space can be solved thoroughly.

Acknowledgments

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