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Research Article

Sufficient and Necessary Conditions of Complete Convergence for Weighted Sums of PNQD Random Variables

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The complete convergence for pairwise negative quadrant dependent (PNQD) random variables is studied. So far there has not been the general moment inequality for PNQD sequence, and therefore the study of the limit theory for PNQD sequence is very difficult and challenging. We establish a collection that contains relationship to overcome the difficulties that there is no general moment inequality. Sufficient and necessary conditions of complete convergence for weighted sums of PNQD random variables are obtained. Our results generalize and improve those on complete convergence theorems previously obtained by Baum and Katz (1965) and Wu (2002).

1. Introduction and Lemmas

Random variables X and Y are said to be negative quadrant dependent (NQD) if

$$P(X \le x, Y \le y) \le P(X \le x)P(Y \le y), \tag{1.1}$$

for all $x, y \in R$. A sequence of random variables $\{X_n; n \ge 1\}$ is said to be pairwise negative quadrant dependent (PNQD) if every pair of random variables in the sequence is NQD. This definition was introduced by Lehmann [1]. Obviously, PNQD sequence includes many negatively associated sequences, and pairwise independent random sequence is the most special case.

In many mathematics and mechanic models, a PNQD assumption among the random variables in the models is more reasonable than an independence assumption. PNQD series have received more and more attention recently because of their wide applications in mathematics and mechanic models, percolation theory, and reliability theory. Many statisticians have investigated PNQD series with interest and have established a series of useful results. For example, Matula [2], Li and Yang [3], and Wu and Jiang [4] obtained the strong law of large numbers, Wang et al. [5] obtained the Marcinkiewicz's weak law of large numbers, Wu [6] obtained the strong convergence properties of Jamison weighted sums, the three-series theorem, and complete convergence theorem, and Li and Wang [7] obtained the central limit theorem. It is interesting for us to extend the limit theorems to the case of PNQD series. However, so far there has not been the general moment inequality for PNQD sequence, and therefore the study of the limit theory for PNQD sequence is very difficult and challenging. In the above-mentioned conclusions, only the Kolmogorov-type strong law of large numbers obtained by Matula [2, Theorem 1] and Baum and Katz-type complete convergence theorem obtained by Wu [6, Theorem 4] achieve the corresponding conclusions of independent cases, and the rest did not achieve the optimal results of independent cases.

Complete convergence is one of the most important problems in probability theory. Recent results of the complete convergence can be found in Wu [6], Chen and Wang [8], and Li et al. [9]. In this paper, we establish a collection that contains relationship to overcome the difficulties that there is no the general moment inequality and obtain the complete convergence theorem for weighted sums of PNQD sequence, which extend and improve the corresponding results of Baum and Katz [10] and Wu [6].

Lemma 1.1 (see [1]). *Let X and Y be NQD random variables. Then*

(i) $\operatorname{cov}(X, Y) \le 0$,

(ii) $P(X > x, Y > y) \le P(X > x)P(Y > y)$, *for all* $x, y \in R$,

(iii) *if f and g are Borel functions, both of which being monotone increasing (or both are monotone decreasing), then $f(X)$ and $g(Y)$ are NQD.*

Lemma 1.2 (see [6, Lemma 2]). *Let $\{X_n; n \ge 1\}$ be a sequence of PNQD random variables with $EX_n = 0$, $EX_n^2 < \infty$, $T_j(k) \,\hat{=}\, \sum_{i=j+1}^{j+k} X_i$, $j \ge 0$. Then*

$$\begin{aligned} E\left(T_j(k)\right)^2 &\le \sum_{i=j+1}^{j+k} EX_i^2, \\ E\max_{1\le k\le n}\left(T_j(k)\right)^2 &\le \frac{4\log^2 n}{\log^2 2}\sum_{i=j+1}^{j+n} EX_i^2. \end{aligned} \tag{1.2}$$

Lemma 1.3 (see [2, Lemma 1]). (i) *If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n; i.o.) = 0$.*

(ii) *if $P(A_k A_m) \le P(A_k)P(A_m)$, $k \ne m$, and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(A_n; i.o.) = 1$.*

Lemma 1.4. *Let $\{X_n; n \ge 1\}$ be a sequence of PNQD random variables. Then for any $x \ge 0$, there exists a positive constant c such that for all $n \ge 1$,*

$$\left(1 - P\left(\max_{1\le k\le n}|X_k| > x\right)\right)^2 \sum_{k=1}^{n} P(|X_k| > x) \le cP\left(\max_{1\le k\le n}|X_k| > x\right). \tag{1.3}$$

Proof. We can prove the Lemma by Lemma A.6 of Zhang and Wen [11]. □

2. Main Results and the Proof

In the following, the symbol c stands for a generic positive constant which may differ from one place to another. Let $a_n \ll b_n$ ($a_n \gg b_n$) denote that there exists a constant $c > 0$ such that $a_n \le cb_n$ ($a_n \ge cb_n$) for all sufficiently large n, and let $X_i \prec X$ ($X_i \succ X$) denote that there exists a constant $c > 0$ such that $P(|X_i| > x) \le cP(|X| > x)$ ($P(|X_i| > x) \ge cP(|X| > x)$) for all $i \ge 1$ and $x > 0$.

Theorem 2.1. *Let $\{X_n; n \ge 1\}$ be a sequence of PNQD random variables with $X_i \prec X$. Let $\{a_{nk}; k \le n, n \ge 1\}$ be a sequence of real numbers such that*

$$|a_{nk}| \ll n^{-\alpha}, \quad \forall k \le n,\ n \ge 1. \tag{2.1}$$

Let for $\alpha p > 1,\ 0 < p < 2,\ \alpha > 0$, and $EX_i = 0$, for $\alpha \le 1$. If

$$E|X|^p < \infty, \tag{2.2}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} |S_{nk}| > \varepsilon\right) < \infty, \quad \forall \varepsilon > 0, \tag{2.3}$$

where $S_{nk} = \sum_{i=1}^{k} a_{ni}X_i$.

Theorem 2.2. *Let $\{X_n; n \ge 1\}$ be a sequence of PNQD random variables with $X_i \succ X$. Let $\{a_{nk}; k \le n,\ n \ge 1\}$ be a sequence of real numbers such that $|a_{nk}| \gg n^{-\alpha}$, for all $k \le n,\ n \ge 1$. Let for $\alpha > 0,\ \alpha p > 1,\ 0 < p < 2$. If (2.3) holds, then (2.2) holds.*

Remark 2.3. Taking $a_{ni} = n^{-\alpha}$, for all $i \le n,\ n \ge 1$ in Theorem 2.1, then

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} |S_{nk}| > \varepsilon\right) &= \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n} n^{-\alpha}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon\right) \\ &= \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| > \varepsilon n^{\alpha}\right).\end{aligned} \tag{2.4}$$

Hence, Theorem 4 in Wu [6] is a particular case of our Theorem 2.1.

Remark 2.4. When $\{X_n; n \ge 1\}$ is i.i.d. and $a_{ni} = n^{-\alpha}$, for all $i \le n,\ n \ge 1$, then Theorems 2.1 and 2.2 become Baum and Katz [10] complete convergence theorem. Hence, our Theorems 2.1 and 2.2 improve and extend the well-known Baum and Katz theorem.

Proof of Theorem 2.1. Without loss of generality, assume that $a_{nk} > 0$ for $k \le n,\ n \ge 1$. Let $q > 0$ such that $(1 + (1/\alpha p))/2 < q < 1$. For all $i \le n$, let

$$\begin{gathered}
Y_{ni} = -a_{ni}^{-1} n^{\alpha(q-1)} I\left(a_{ni}X_i < -n^{\alpha(q-1)}\right) + X_i I\left(a_{ni}|X_i| \le n^{\alpha(q-1)}\right) \\
+ a_{ni}^{-1} n^{\alpha(q-1)} I\left(a_{ni}X_i > n^{\alpha(q-1)}\right), \\
U_{nk} = \sum_{i=1}^{k} a_{ni} Y_{ni}.
\end{gathered} \tag{2.5}$$

Write

$$\begin{gathered}
A_n = \bigcup_{j=1}^{n} \left(\left|a_{nj}X_j\right| \ge \varepsilon\right), \\
B_n = \bigcup_{1\le i<j\le n} \left(\left(a_{ni}X_i > n^{\alpha(q-1)}, a_{nj}X_j > n^{\alpha(q-1)}\right) \bigcup \left(a_{ni}X_i < -n^{\alpha(q-1)}, a_{nj}X_j < -n^{\alpha(q-1)}\right)\right).
\end{gathered} \tag{2.6}$$

Firstly, we prove that

$$\begin{aligned}
\left(\max_{1\le k\le n}|S_{nk}| < 6\varepsilon\right) \supseteq{}& A_n^c \bigcap \left(\max_{1\le k\le n}|U_{nk}| < 2\varepsilon\right) \bigcap B_n^c \\
={}& \bigcap_{j=1}^{n} \left(\left|a_{nj}X_j\right| < \varepsilon\right) \bigcap \left(\max_{1\le k\le n}|U_{nk}| < 2\varepsilon\right) \bigcap \\
& \bigcap_{1\le i<j\le n} \left[\left(\left(a_{ni}X_i \le n^{\alpha(q-1)}\right) \bigcup \left(a_{nj}X_j \le n^{\alpha(q-1)}\right)\right)\right. \\
& \qquad \left.\bigcap \left(\left(a_{ni}X_i \ge -n^{\alpha(q-1)}\right) \bigcup \left(a_{nj}X_j \ge -n^{\alpha(q-1)}\right)\right)\right] \\
\widehat{=}{}& D_n.
\end{aligned} \tag{2.7}$$

For any $\omega \in D_n$, we have

$$\left|a_{nj}X_j\right| < \varepsilon, \quad \left|a_{nj}Y_{nj}\right| \le \left|a_{nj}X_j\right| < \varepsilon, \quad \forall 1 \le j \le n, \qquad \max_{1\le k\le n}|U_{nk}| < 2\varepsilon, \tag{2.8}$$

and for any $1 \le i < j \le n$,

$$\begin{gathered}
a_{ni}X_i \le n^{\alpha(q-1)}, \quad \text{or} \quad a_{nj}X_j \le n^{\alpha(q-1)}, \\
a_{ni}X_i \ge -n^{\alpha(q-1)}, \quad \text{or} \quad a_{nj}X_j \ge -n^{\alpha(q-1)}.
\end{gathered} \tag{2.9}$$

Hence

$$a \hat{=} \sharp\left\{i; 1 \leq i \leq n, a_{ni}X_i(\omega) > n^{\alpha(q-1)}\right\} \leq 1,$$
$$b \hat{=} \sharp\left\{i; 1 \leq i \leq n, a_{ni}X_i(\omega) < -n^{\alpha(q-1)}\right\} \leq 1, \tag{2.10}$$

where the symbol $\sharp A$ denotes the number of elements in the set A.

When $a = b = 0$, then $|a_{ni}X_i(\omega)| \leq n^{\alpha(q-1)}$ for any $1 \leq i \leq n$; thus, $Y_{ni}(\omega) = X_i(\omega)$, and therefore by (2.8),

$$\max_{1\leq k\leq n}|S_{nk}| = \max_{1\leq k\leq n}|U_{nk}| < 2\varepsilon < 6\varepsilon. \tag{2.11}$$

When $a = 1,\ b = 0$ (or $a = 0,\ b = 1$), then there exists only an i_0: $1 \leq i_0 \leq n$ such that $a_{ni_0}X_{i_0}(\omega) > n^{\alpha(q-1)}$ (or $a_{ni_0}X_{i_0}(\omega) < -n^{\alpha(q-1)}$), the remaining j, $|a_{nj}X_{nj}(\omega)| \leq n^{\alpha(q-1)}$; thus, $X_j(\omega) = Y_{nj}(\omega)$. If $1 \leq k \leq i_0 - 1$, then $S_{nk}(\omega) = U_{nk}(\omega)$. If $i_0 \leq k \leq n$, then by (2.8),

$$\begin{aligned}
\max_{1\leq k\leq n}|S_{nk}(\omega)| &= \max_{1\leq k\leq n}\left|\sum_{1\leq i\leq k, i\neq i_0} a_{ni}X_i(\omega) + a_{ni_0}X_{i_0}(\omega)\right| \\
&= \max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}Y_{ni}(\omega) - a_{n_0}Y_{ni_0}(\omega) + a_{ni_0}X_{i_0}(\omega)\right| \\
&\leq \max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}Y_{ni}(\omega)\right| + |a_{ni_0}Y_{ni_0}(\omega)| + |a_{ni_0}X_{i_0}(\omega)| \\
&< 2\varepsilon + \varepsilon + \varepsilon < 6\varepsilon.
\end{aligned} \tag{2.12}$$

When $a = b = 1$, then there exist $1 \leq i_1,\ i_2 \leq n$ such that $a_{ni_1}X_{i_1}(\omega) > n^{\alpha(q-1)}$, $a_{ni_2}X_{i_2}(\omega) < -n^{\alpha(q-1)}$, the remaining j, $|a_{nj}X_j(\omega)| \leq n^{\alpha(q-1)}$; thus, $X_j(\omega) = Y_{nj}(\omega)$. Without loss of generality, assume that $i_1 \leq i_2$. If $1 \leq k \leq i_1 - 1$, then $S_{nk}(\omega) = U_{nk}(\omega)$; if $i_1 \leq k < i_2$, then by (2.8),

$$\begin{aligned}
\max_{1\leq k\leq n}|S_{nk}(\omega)| &\leq \max_{1\leq k\leq n}|U_{nk}(\omega)| + |a_{ni_1}Y_{ni_1}(\omega)| + |a_{ni_1}X_{i_1}(\omega)| \\
&< 2\varepsilon + \varepsilon + \varepsilon < 6\varepsilon.
\end{aligned} \tag{2.13}$$

If $k \geq i_2$, then by (2.8),

$$\begin{aligned}
\max_{1\leq k\leq n}|S_{nk}(\omega)| &= \max_{1\leq k\leq n}\left|\sum_{1\leq i\leq k, i\neq i_1, i_2} a_{ni}X_i(\omega) + a_{ni_1}X_{i_1}(\omega) + a_{ni_2}X_{i_2}(\omega)\right| \\
&\leq \max_{1\leq k\leq n}|U_{nk}(\omega)| + |a_{ni_1}Y_{ni_1}(\omega)| + |a_{ni_2}Y_{ni_2}(\omega)|
\end{aligned}$$

$$
\begin{aligned}
&+ |a_{ni_1} X_{i_1}(\omega)| + |a_{ni_2} X_{i_2}(\omega)| \\
&< 6\varepsilon.
\end{aligned}
\tag{2.14}
$$

Hence, (2.7) holds, that is:

$$
\left(\max_{1\le k\le n}|S_{nk}| \ge 6\varepsilon\right) \subseteq A_n \bigcup \left(\max_{1\le k\le n}|U_{nk}| \ge 2\varepsilon\right) \bigcup B_n. \tag{2.15}
$$

Therefore, in order to prove (2.3), we only need to prove that

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} P(A_n) < \infty, \tag{2.16}
$$

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} P(B_n) < \infty, \tag{2.17}
$$

$$
\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}|U_{nk}| \ge 2\varepsilon\right) < \infty, \quad \forall \varepsilon > 0. \tag{2.18}
$$

By (2.1), (2.2), $X_i \prec X$, and $\alpha p > 1$,

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2} P(A_n) &\le \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{j=1}^{n} P\left(\left|a_{nj} X_j\right| \ge \varepsilon\right) \\
&\le \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{j=1}^{n} P\left(\left|X_j\right| \ge \varepsilon a_{nj}^{-1} \ge \varepsilon c n^{\alpha}\right) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p-1} P(|X| \ge \varepsilon c n^{\alpha}) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-1} \sum_{j=n}^{\infty} P\left(\varepsilon c j^{\alpha} \le |X| < \varepsilon c (j+1)^{\alpha}\right) \\
&= \sum_{j=1}^{\infty} \sum_{n=1}^{j} n^{\alpha p-1} P\left(\varepsilon c j^{\alpha} \le |X| < \varepsilon c (j+1)^{\alpha}\right) \\
&\le \sum_{j=1}^{\infty} j^{\alpha p} P\left(\varepsilon c j^{\alpha} \le |X| < \varepsilon c (j+1)^{\alpha}\right) \\
&\ll E|X|^p < \infty.
\end{aligned}
\tag{2.19}
$$

That is, (2.16) holds.

By Lemma 1.1(ii), $X_i \prec X$, and the definition of q, $\alpha p(1-2q) < -1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\alpha p-2} P(B_n) &\le \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{1\le i<j\le n} \Big(P\Big(a_{ni}X_i > n^{\alpha(q-1)}\Big) P\Big(a_{nj}X_j > n^{\alpha(q-1)}\Big) \\
&\qquad + P\Big(a_{ni}X_i < -n^{\alpha(q-1)}\Big) P\Big(a_{nj}X_j < -n^{\alpha(q-1)}\Big)\Big) \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p} P^2(|X| > cn^{\alpha q}) \le \sum_{n=1}^{\infty} n^{\alpha p} n^{-2\alpha pq} \left(E|X|^p\right)^2 \\
&\ll \sum_{n=1}^{\infty} n^{\alpha p(1-2q)} < \infty.
\end{aligned} \tag{2.20}$$

That is, (2.17) holds.

In order to prove (2.18), firstly, we prove that

$$\max_{1\le k\le n} |EU_{nk}| = \max_{1\le k\le n} \left| E\sum_{i=1}^{k} a_{ni} Y_{ni} \right| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.21}$$

(i) When $\alpha \le 1$, then $p > 1/\alpha \ge 1$; from $EX_i = 0$ and the definition of q, we have $q < 1$, $\alpha pq > \alpha p + 1 - \alpha pq = 1 + \alpha p(1-q) > 1$:

$$\begin{aligned}
&\max_{1\le k\le n} \left| E\sum_{i=1}^{k} a_{ni} Y_{ni} \right| \\
&\quad\le \sum_{i=1}^{n} a_{ni} |EY_{ni}| = \sum_{i=1}^{n} a_{ni} |E(X_i - Y_{ni})| \\
&\quad\le \sum_{i=1}^{n} a_{ni} \left\{ E\left| X_i + a_{ni}^{-1} n^{\alpha(q-1)} \right| I_{(a_{ni}X_i < -n^{\alpha(q-1)})} + E\left| X_i - a_{ni}^{-1} n^{\alpha(q-1)} \right| I_{(a_{ni}X_i > n^{\alpha(q-1)})} \right\} \\
&\quad\ll \sum_{i=1}^{n} a_{ni} E|X_i| I_{(|a_{ni}X_i| > n^{\alpha(q-1)})} \le \sum_{i=1}^{n} a_{ni} E|X_i| \left(\frac{a_{ni}|X_i|}{n^{\alpha(q-1)}} \right)^{p-1} \\
&\quad= \sum_{i=1}^{n} a_{ni}^{p} E|X_i|^p n^{\alpha(1-q)(p-1)} \\
&\quad\ll n^{-\alpha p + 1 + \alpha p - \alpha - \alpha pq + \alpha q} = n^{-(\alpha pq - 1) - \alpha(1-q)} \\
&\quad\longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{2.22}$$

(ii) When $\alpha > 1$, and $p \ge 1$, then $E|X| < \infty$ from (2.2), thus,

$$\max_{1\le k\le n} \left| E\sum_{i=1}^{k} a_{ni} Y_{ni} \right| \le \sum_{i=1}^{n} a_{ni} E|X| \ll n^{-\alpha+1} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{2.23}$$

(iii) When $\alpha > 1$, and $p < 1$, by $-(\alpha p-1)-\alpha(1-q)(1-p) < 0$, and $-\alpha(1-q)-(\alpha pq-1) < 0$, we get

$$\begin{aligned}
\max_{1\le k\le n}\left|E\sum_{i=1}^{k}a_{ni}Y_{ni}\right| &\le \sum_{i=1}^{n}a_{ni}\left(E|X_i|I_{(a_{ni}|X_i|\le n^{\alpha(q-1)})}+a_{ni}^{-1}n^{\alpha(q-1)}P\left(|a_{ni}X_i|>n^{\alpha(q-1)}\right)\right)\\
&\le \sum_{i=1}^{n}a_{ni}^{p}E|X_i|^{p}|a_{ni}X_i|^{1-p}I_{(a_{ni}|X_i|\le n^{\alpha(q-1)})}+\sum_{i=1}^{n}n^{\alpha(q-1)}a_{ni}^{p}n^{-\alpha p(q-1)}E|X_i|^{p}\\
&\ll n^{-(\alpha p-1)-\alpha(1-p)(1-q)}+n^{-\alpha(1-q)-(\alpha pq-1)}\longrightarrow 0.
\end{aligned}\tag{2.24}$$

Hence, (2.21) holds; that is, for any $\varepsilon > 0$, we have $\max_{1\le k\le n}|EU_{nk}| < \varepsilon$ for all sufficiently large n. Thus,

$$P\left(\max_{1\le j\le n}|U_{nj}|\ge 2\varepsilon\right)\le P\left(\max_{1\le j\le n}|U_{nj}-EU_{nj}|>\varepsilon\right).\tag{2.25}$$

Let $\widetilde{Y}_{ni} = Y_{ni} - EY_{ni}$. Obviously, Y_{ni} is monotonic on X_i. By Lemma 1.1(iii), $\{Y_{ni}; n \ge 1, i \le n\}$ is also a sequence of PNQD random variables with $E\widetilde{Y}_{ni} = 0$, by Lemma 1.2 and $-1-\alpha(1-q)(2-p) < -1$:

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha p-2}P\left(\max_{1\le j\le n}|U_{nj}-EU_{nj}|>\varepsilon\right)\\
&\quad\ll \sum_{n=1}^{\infty}n^{\alpha p-2}\log^2 n\sum_{j=1}^{n}Ea_{nj}^2Y_{nj}^2\\
&\quad\ll \sum_{n=1}^{\infty}n^{\alpha p-2}\log^2 n\sum_{j=1}^{n}\left(Ea_{nj}^2X_j^2I_{(a_{nj}|X_j|\le n^{\alpha(q-1)})}+n^{2\alpha(q-1)}P\left(a_{nj}|X_j|>n^{\alpha(q-1)}\right)\right)\\
&\quad\le \sum_{n=1}^{\infty}n^{\alpha p-2}\log^2 n\sum_{j=1}^{n}\left(E|a_{nj}X_j|^{p}n^{\alpha(q-1)(2-p)}+n^{2\alpha(q-1)-\alpha p(q-1)}E|a_{nj}X_j|^{p}\right)\\
&\quad\ll \sum_{n=1}^{\infty}\left(n^{\alpha p-1-\alpha p+\alpha(q-1)(2-p)}+n^{-1+\alpha p-\alpha pq+2\alpha q-2\alpha}\right)\log^2 n\\
&\quad= 2\sum_{n=1}^{\infty}n^{-1-\alpha(1-q)(2-p)}\log^2 n\\
&\quad< \infty.
\end{aligned}\tag{2.26}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Noting that $\max_{1\le k\le n}|a_{nk}X_k| \le 2\max_{1\le k\le n}|S_{nk}|$ and $|a_{nk}| \gg n^{-\alpha}$, from (2.3),

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\le k\le n}|X_k| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0. \tag{2.27}$$

Thus, by $\alpha p - 2 > -1$, we get

$$\sum_{j=1}^{\infty} P\left(\max_{1\le k\le 2^j}|X_k| > \varepsilon 2^{\alpha(j+1)}\right) \ll \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le k\le n}|X_k| > \varepsilon n^{\alpha}\right) < \infty, \quad \forall \varepsilon > 0. \tag{2.28}$$

This implies that

$$\max_{2^{m-1}\le n\le 2^m} P\left(\max_{1\le j\le n}\left|X_j\right| > \varepsilon 2^{2\alpha} n^{\alpha}\right) \le P\left(\max_{1\le j\le 2^m}\left|X_j\right| > \varepsilon 2^{\alpha(m+1)}\right) \longrightarrow 0. \tag{2.29}$$

Hence, for all sufficiently large n,

$$P\left(\max_{1\le j\le n}\left|X_j\right| > \varepsilon 2^{2\alpha} n^{\alpha}\right) < \frac{1}{2}. \tag{2.30}$$

By Lemma 1.4,

$$\sum_{k=1}^{n} P(|X_k| > \varepsilon n^{\alpha}) \le 4cP\left(\max_{1\le k\le n}|X_k| > \varepsilon n^{\alpha}\right), \quad \forall \varepsilon > 0, \tag{2.31}$$

which together with (2.27),

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|X_k| > \varepsilon n^{\alpha}) < \infty, \quad \forall \varepsilon > 0. \tag{2.32}$$

By $X_k \succ X$, we obtain

$$E|X|^p \ll \sum_{n=1}^{\infty} n^{\alpha p-1} P(|X| > \varepsilon n^{\alpha}) < \infty. \tag{2.33}$$

This completes the proof of Theorem 2.2. □

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