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Letter to the Editor

Variational Iteration Method for q-Difference Equations of Second Order

Guo-Cheng Wu

Key Laboratory of Numerical Simulation of Sichuan Province, Neijiang Normal University, Sichuan 641112, China

Correspondence should be addressed to Guo-Cheng Wu, wuguocheng2002@yahoo.com.cn

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Recently, Liu extended He's variational iteration method to strongly nonlinear q-difference equations Liu (2010). In this study, the iteration formula and the Lagrange multiplier are given in a more accurate way. The q-oscillation equation of second order is approximately solved to show the new Lagrange multiplier's validness.

1. Introduction

Generally, applying the variational iteration method (VIM) [1, 2] in differential equations follows the three steps:

(a) establishing the correction functional;

(b) identifying the Lagrange multipliers;

(c) determining the initial iteration.

Obviously, the step (b) is crucial and critical in the method.

For the strongly nonlinear q-difference equation,

$$\frac{d_q^2}{d_q t^2}x + (2+\varepsilon x)\frac{d_q}{d_q t}x + \Omega^2 x + x^2 = 0, \tag{1.1}$$

where $d_q/d_q t$ is the q-derivative [3], Liu [4] used the Lagrange multiplier

$$\lambda(t,s) = s - t, \tag{1.2}$$

which results in the iteration formula (see [4, (4.10) and (4.11)]):

$$x_{n+1} = x_n + \int_0^t (s-t)\left(\frac{d_q^2}{d_q s^2} x_n + (2 + \varepsilon x_n) \frac{d_q}{d_q s} x_n + \Omega^2 x_n + x_n^2 \right) d_q s. \tag{1.3}$$

In this paper, it is pointed out that the iteration formula (1.3) can be given in a more accurate way and a new Lagrange multiplier is explicitly identified.

2. Properties of q-Calculus

2.1. q-Calculus

Let $f(x)$ be a real continuous function. The q-derivative is defined as

$$\frac{d_q}{d_q x} f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0,\ 0 < q < 1, \tag{2.1}$$

and $(d_q/d_q x) f(x)|_{x=0} = \lim_{n\to\infty}((f(q^n) - f(0))/q^n)$.

The partial q-derivative with respect to x is

$$\frac{\partial_q}{\partial_q x} f(x; y; \ldots) = \frac{f(qx; y; \ldots) - f(x; y; \ldots)}{(q-1)x}. \tag{2.2}$$

The corresponding q-integral [5] is

$$\int_0^x f(t) d_q t = (1-q) x \sum_{n=0}^{\infty} q^n f(q^n x). \tag{2.3}$$

2.2. q-Leibniz Product Law

One has

$$\frac{d_q}{d_q x}[g(x) f(x)] = g(qx) \frac{d_q}{d_q x}[f(x)] + f(x) \frac{d_q}{d_q x}[g(x)]. \tag{2.4}$$

2.3. q-Integration by Parts

One has

$$\int_a^b g(qt) \frac{d_q}{d_q t} f(t) d_q t = f(t) g(t)\big|_a^b - \int_a^b f(t) \frac{d_q}{d_q t} g(t) d_q t. \tag{2.5}$$

The properties above are needed in the construction of the correction functional for q-difference equations. For more results and properties in q-calculus, readers are referred to the recent monographs [5–8].

3. A q-Analogue of Lagrange Multiplier

In order to identify the Lagrange multipliers of the q-difference equations, we first establish the correctional functional for (1.1) as

$$x_{n+1} = x_n + \int_0^t \lambda\left(t, q^2 s\right)\left(\frac{d_q^2}{d_q s^2}x_n + (2 + \varepsilon x_n)\frac{d_q}{d_q s}x_n + \Omega^2 x_n + x_n^2\right)d_q s. \tag{3.1}$$

The correction functional here is different from the one in ordinary calculus since the parameter q "disappears" after the integration by parts (2.5) each time. As a result, we use $\lambda(t, q^2 s)$ in the above functional.

We only need to consider the leading term $(d_q^2/d_q t^2)x$ when other terms are restricted variations in (1.1)

$$x_{n+1} = x_n + \int_0^t \lambda\left(t, q^2 s\right)\left(\frac{d_q^2}{d_q s^2}x_n + (2 + \varepsilon x_n)\frac{d_q}{d_q s}x_n + \Omega^2 x_n + x_n^2\right)d_q s. \tag{3.2}$$

Through the integration by parts (2.5), we can have

$$\delta x_{n+1} = \left(1 - q\frac{\partial_q}{\partial_q s}\lambda(t,s)\Big|_{s=t}\right)\delta x_n + \lambda(t,qs)\big|_{s=t}\delta x_n' - q\int_0^t \frac{\partial_q^2}{\partial_q s^2}\lambda(t,s)\delta x_n d_q s, \tag{3.3}$$

where δ is the variation operator and "$'$" denotes the q-derivative with respect to t. As a result, the system of the Lagrange multiplier can be obtained:

the coefficient of $\delta x_n : 1 - q(\partial_q/\partial_q s)\lambda(t,s)|_{s=t} = 0$,

the coefficient of $\delta x_n' : \lambda(t,qs)|_{s=t} = 0$,

the coefficient of δx_n in the q-integral: $q(\partial_q^2/\partial_q s^2)\lambda(t,s) = 0$,

from which we can get

$$\lambda(t,s) = q^{-1}(s - tq), \tag{3.4}$$

instead of $\lambda(t,s) = s - t$ in [4]. More introductions to the identification of various Lagrange multipliers of the VIM can be found in [9, 10].

We also can show the above q-analogue of Lagrange multiplier's validness. For $0 < q < 1$, let T_q be the time scale: $T_q = \{q^n : n \in Z\} \cup \{0\}$, where Z is the set of positive integers. For the real continuous function $u(t) : T_q \to R$, a q-oscillator equation of second order is

$$\frac{d_q^2}{d_q t^2}u - u = 0, \qquad u(0) = 1, \qquad \frac{d_q}{d_q t}u\Big|_{t=0} = 1. \tag{3.5}$$

From (3.4), the iteration formula can be given as

$$u_{n+1} = u_n + \int_0^t q^{-1}\left(q^2 s - tq\right)\left[\frac{d_q^2}{d_q s^2} u_n(s) - u_n(s)\right] d_q s. \tag{3.6}$$

Starting from the initial iteration $u_0 = 1 + t/[1]_q!$, the successive approximate solutions can be obtained as

$$\begin{aligned} u_0 &= 1 + \frac{t}{[1]_q!}, \\ u_1 &= 1 + \frac{t}{[1]_q!} + \frac{t^2}{[2]_q!} + \frac{t^3}{[3]_q!}, \\ &\vdots \\ u_n &= \sum_{k=0}^{2n+1} \frac{t^k}{[k]_q!}. \end{aligned} \tag{3.7}$$

The limit $u = \lim_{n\to\infty} u_n = e_q(t)$ is an exact solution of (3.5). Here $e_q(t)$ is one of the q-exponential functions.

4. Conclusions

In the past ten years, the VIM has been one of the often used nonlinear methods. The q-derivative is a deformation of the classical derivative and it has played a crucial role in quantum mechanics and quantum calculus. In this study, the method is successfully extended to q difference equations of second order. A q-analogue of Lagrange multiplier is presented. Readers who feel interested in the initial value problems of the q difference equations are referred to [11–17].

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