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Research Article

Existence of Positive Solutions for m-Point Boundary Value Problem for Nonlinear Fractional Differential Equation

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We investigate an m-point boundary value problem for nonlinear fractional differential equations. The associated Green function for the boundary value problem is given at first, and some useful properties of the Green function are obtained. By using the fixed point theorems of cone expansion and compression of norm type and Leggett-Williams fixed point theorem, the existence of multiple positive solutions is obtained.

1. Introduction

In recent years, the existence of positive solutions multipoint boundary value problems of fractional order differential equations has been studied by many authors using various methods (see [1–7]).

The study of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [8, 9].

Since then, nonlinear multipoint boundary value problems have been studied by several authors (see [10–14]). Recently, in [15], the authors have studied the existence of at least one positive solution for the following nth-order three-point boundary value problem:

$$\begin{gathered} u^{(n)}(t) + h(t) f(t, u(t)) = 0, \quad t \in [a, b], \\ u(a) = \alpha u(\eta), \qquad u'(a) = u''(a) = \cdots = u^{(n-2)}(a) = 0, \qquad u(b) = \beta u(\eta), \end{gathered} \tag{1.1}$$

where $a < \eta < b$, $0 \leq \alpha < 1$, $f \in C([a,b] \times [0,\infty), [0,\infty))$ and $h \in C([a,b] \times [0,\infty))$ may be singular at $t = a$ and $t = b$.

Goodrich [16] considered the BVP for thehigher-dimensional fractional differential equation as follows:

$$\begin{gathered}
-D_{0^+}^{\nu} y(t) = f\left(t, y(t)\right), \quad 0 < t < 1,\ n-1 < \nu \leq n, \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq n-2, \\
\left[D_{0^+}^{\alpha} y(t)\right]_{t=1} = 0, \quad 1 \leq \alpha \leq n-2,
\end{gathered} \tag{1.2}$$

and a Harnack-like inequality associated with the Green function related to the above problem is obtained improving the results in [17].

Motivated by the aforementioned results and techniques in coping with those boundary value problems of fractional differential equations, we then turn to investigate the existence and multiplicity of positive solutions for the following BVP:

$$^{C}D_{a^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad a \leq t \leq b,\ n-1 \leq \alpha < n,\ n > 2, \tag{1.3}$$

$$u'(a) = \sum_{i=1}^{m-2} \beta_i u'(\eta_i), \qquad u''(a) = u'''(a) = \cdots = u^{(n-1)}(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \gamma_i u(\eta_i), \tag{1.4}$$

where $a < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < b$, $\sum_{i=1}^{m-2} \beta_i < 1$, $\sum_{i=1}^{m-2} \gamma_i < 1$ and $^{C}D_{a^+}^{\alpha}$ are the Caputo fractional derivative.

In this paper, we study the existence of at least one positive solution, existence of two positive solutions associated with the BVP (1.3)-(1.4) by applying the fixed point theorems of cone expansion and compression of norm type, and the existence of at least three positive solutions for BVP (1.3)-(1.4) by using Leggett-Williams fixed point theorem.

The rest of the paper is organized as follows. In Section 2, we introduce some basic definitions and preliminaries used later. In Section 3, the existence of multipoint boundary value problem (1.3)-(1.4) will be discussed.

2. Preliminaries

In this section, we introduce definitions and preliminary facts which are used throughout this paper.

Definition 2.1 (see [18]). For a function $y : (a,\infty) \to R$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$^{C}D_{a^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds, \quad n-1 < \alpha \leq n. \tag{2.1}$$

Definition 2.2 (see [18]). The standard Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (a, \infty) \to R$ is given by

$$D_{a^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} y(s) ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided that the integral on the right-hand side converges.

Definition 2.3 (see [18]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (a, \infty) \to R$ is given by

$$I_{a^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} y(s) ds \tag{2.3}$$

provided that the integral on the right-hand side converges.

Definition 2.4 (see [19]). Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called cone of E if it satisfies the following conditions:

(1) $x \in K,\ \sigma \geq 0$ implies $\sigma x \in K$;

(2) $x \in K,\ -x \in K$ implies $x = 0$.

Definition 2.5. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem 2.6 (see [20]). *Let E be a Banach space and $K \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be completely continuous operator. In addition, suppose either*

(i) $\|Tu\| \leq \|u\|$, *for all* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, *for all* $u \in K \cap \partial\Omega_2$ *or*

(ii) $\|Tu\| \leq \|u\|$, *for all* $u \in K \cap \partial\Omega_2$, *and* $\|Tu\| \geq \|u\|$, *for all* $u \in K \cap \partial\Omega_1$

holds. Then, T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.7. *For $\alpha > 0$, the general solution of the fractional differential equation ${}^C D_{a^+}^{\alpha} u(t) = 0$ is given by*

$$u(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}, \tag{2.4}$$

where $c_i \in R,\ i = 0, 1, 2, \ldots, n-1$.

Remark 2.8 (see [18]). In view of Lemma 2.7, it follows that

$$I_{a^+}^{\alpha}\,{}^{C}D_{a^+}^{\alpha}u(t) = u(t) + c_0 + c_1(t-a) + c_2(t-a)^2 + \cdots + c_{n-1}(t-a)^{n-1}, \tag{2.5}$$

for some $c_i \in R$, $i = 0, 1, 2, \ldots, n-1$.

Definition 2.9. The map θ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\theta : P \to [0,\infty)$ is continuous and

$$\theta(\lambda x + (1-\lambda)y) \geq \lambda\theta(x) + (1-\lambda)\theta(y), \tag{2.6}$$

for all $x, y \in P$, $0 \leq \lambda \leq 1$.

Lemma 2.10 (see [21]). *Let P be a cone in a real Banach space E, $P_c = \{x \in P : \|x\| < c\}$, θ is a nonnegative continuous concave functional on P such that $\theta(x) \leq \|x\|$, for all $x \in P_c$, and $P(\theta, b, d) = \{x \in P : b \leq \theta(x), x \leq d\}$. Suppose that $T : P_c \to P_c$ is completely continuous and there exist positive constants* $0 < a < b < d \leq c$ *such that*

(C1) $\{x \in P(\theta, b, d) : \theta(x) > b\} \neq \phi$ *and* $\theta(x) > b$ *for* $x \in P(\theta, b, d)$,

(C2) $\|Tx\| < a$ *for* $x \in P_a$,

(C3) $\theta(Tx) > b$ *for* $x \in P(\theta, b, d)$ *with* $\|Tx\| > d$,

then T has at least three fixed points x_1, x_2, and x_3 with

$$\|x_1\| < a, \quad b < \theta(x_2), \quad a < \|x_3\| \quad \text{with } \theta(x_3) < b. \tag{2.7}$$

Lemma 2.11. *For a given $y(t) \in C[a,b]$ and $n-1 \leq \alpha < n$, the unique solution of the boundary value problem*

$${}^{C}D_{a^+}^{\alpha}u(t) + y(t) = 0, \quad a \leq t \leq b, \; n-1 \leq \alpha < n, \; n > 2, \; n \in N, \tag{2.8}$$

$$u'(a) = \sum_{i=1}^{m-2}\beta_i u'(\eta_i), \qquad u''(a) = u'''(a) = \cdots = u^{(n-1)}(a) = 0, \qquad u(b) = \sum_{i=1}^{m-2}\gamma_i u(\eta_i), \tag{2.9}$$

is given by

$$u(t) = \int_a^b G(t,s)y(s)ds + \int_a^b H(t,s;\eta_1,\ldots,\eta_{m-2})y(s)ds, \tag{2.10}$$

where

$$
\begin{gathered}
G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}(b-s)^{\alpha-1}-(t-s)^{\alpha-1}, & a\le s\le t\le b,\\ (b-s)^{\alpha-1}, & a\le t\le s\le b,\end{cases}\\
H(t,s;\eta_1,\ldots,\eta_{m-2})=\begin{cases}\dfrac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)}\\ \quad+\dfrac{\mu(t)\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha-1)}, & a\le s\le\eta_i,\ i=1,2,\ldots,m-2,\\ \dfrac{\sum_{i=1}^{m-2}\gamma_i(b-s)^{\alpha-1}}{\delta_2\Gamma(\alpha)}, \quad \eta_i\le s\le b,\ i=1,2,\ldots,m-2,\end{cases}\\
\delta_1=1-\sum_{i=1}^{m-2}\beta_i,\qquad \delta_2=1-\sum_{i=1}^{m-2}\gamma_i,\qquad \mu(t)=\left(b-\sum_{i=1}^{m-2}\gamma_i\eta_i\right)-\delta_2 t.
\end{gathered}
\tag{2.11}
$$

Proof. Using Remark 2.8, for arbitrary constants $c_i\in R,\ i=0,1,2,\ldots,n-1$, we have

$$
\begin{aligned}
u(t)&=\frac{-1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}y(s)ds+c_0+c_1(t-a)+c_2(t-a)^2+\cdots+c_{n-1}(t-a)^{n-1}\\
&=-I_{a^+}^{\alpha}y(t)+c_0+c_1(t-a)+c_2(t-a)^2+\cdots+c_{n-1}(t-a)^{n-1}.
\end{aligned}
\tag{2.12}
$$

In view of the relations ${}^{C}D_{a^+}^{\alpha}I_{a^+}^{\alpha}u(t)=u(t)$ and $I_{a^+}^{\alpha}I_{a^+}^{\beta}u(t)=I_{a^+}^{\alpha+\beta}u(t)$ for $\alpha,\beta>0$, we obtain

$$
\begin{aligned}
u'(t)&=-I_{a^+}^{\alpha-1}y(t)+c_1+2c_2(t-a)+\cdots+(n-1)c_{n-1}(t-a)^{n-2},\\
u''(t)&=-I_{a^+}^{\alpha-2}y(t)+2c_2+\cdots+(n-1)(n-2)c_{n-1}(t-a)^{n-3},\\
&\ \vdots\\
u^{(n-1)}(t)&=-I_{a^+}^{\alpha-n+1}y(t)+(n-1)!c_{n-1}.
\end{aligned}
\tag{2.13}
$$

Applying the boundary conditions (2.9), we find that

$$
\begin{gathered}
c_2=c_3=\cdots=c_{n-1}=0,\\
\left(1-\sum_{i=1}^{m-2}\beta_i\right)c_1=-\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i),
\end{gathered}
\tag{2.14}
$$

then $c_1 = -\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)/\delta_1$, and

$$\begin{aligned}
c_0 &= \frac{I_{a^+}^{\alpha}y(b)}{\left(1-\sum_{i=1}^{m-2}\gamma_i\right)} - \frac{\sum_{i=1}^{m-2}\gamma_i I_{a^+}^{\alpha}y(\eta_i)}{\left(1-\sum_{i=1}^{m-2}\gamma_i\right)} + \frac{\left[(b-a)-\sum_{i=1}^{m-2}\gamma_i(\eta_i-a)\right]\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)}{\left(1-\sum_{i=1}^{m-2}\beta_i\right)\left(1-\sum_{i=1}^{m-2}\gamma_i\right)}\\
&= \frac{I_{a^+}^{\alpha}y(b)}{\delta_2} - \frac{\sum_{i=1}^{m-2}\gamma_i I_{a^+}^{\alpha}y(\eta_i)}{\delta_2} + \frac{\left[(b-a)-\sum_{i=1}^{m-2}\gamma_i(\eta_i-a)\right]\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)}{\delta_1\delta_2}.
\end{aligned} \tag{2.15}$$

Substituting the values of the constants $c_i,\ i=0,1,2,\dots,n-1$, in (2.12), we obtain

$$\begin{aligned}
u(t) &= -I_{a^+}^{\alpha}y(t) + \frac{I_{a^+}^{\alpha}y(b)}{\delta_2} - \frac{\sum_{i=1}^{m-2}\gamma_i I_{a^+}^{\alpha}y(\eta_i)}{\delta_2} + \frac{\left[(b-a)-\sum_{i=1}^{m-2}\gamma_i(\eta_i-a)\right]\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)}{\delta_1\delta_2}\\
&\quad - \frac{\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)}{\delta_1}(t-a)\\
&= -I_{a^+}^{\alpha}y(t) + \frac{I_{a^+}^{\alpha}y(b)}{\delta_2} - \frac{\sum_{i=1}^{m-2}\gamma_i I_{a^+}^{\alpha}y(\eta_i)}{\delta_2} + \frac{\mu(t)}{\delta_1\delta_2}\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)\\
&= -I_{a^+}^{\alpha}y(t) + I_{a^+}^{\alpha}y(b) + \frac{(1-\delta_2)}{\delta_2}I_{a^+}^{\alpha}y(b) - \frac{\sum_{i=1}^{m-2}\gamma_i I_{a^+}^{\alpha}y(\eta_i)}{\delta_2} + \frac{\mu(t)}{\delta_1\delta_2}\sum_{i=1}^{m-2}\beta_i I_{a^+}^{\alpha-1}y(\eta_i)\\
&= -\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \frac{\sum_{i=1}^{m-2}\gamma_i}{\delta_2}\int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\
&\quad - \frac{\sum_{i=1}^{m-2}\gamma_i}{\delta_2}\int_a^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \frac{\mu(t)}{\delta_1\delta_2}\sum_{i=1}^{m-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds\\
&= -\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_a^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\\
&\quad + \frac{\sum_{i=1}^{m-2}\gamma_i}{\delta_2}\int_a^{\eta_i}\left[\frac{(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}\right]y(s)ds\\
&\quad + \frac{\sum_{i=1}^{m-2}\gamma_i}{\delta_2}\int_{\eta_i}^b \frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \frac{\mu(t)}{\delta_1\delta_2}\sum_{i=1}^{m-2}\beta_i\int_a^{\eta_i}\frac{(\eta_i-s)^{\alpha-2}}{\Gamma(\alpha-1)}y(s)ds.
\end{aligned} \tag{2.16}$$

□

Lemma 2.12. *$\mu(t) = (b-\sum_{i=1}^{m-2}\gamma_i\eta_i) - \delta_2 t \geq 0$, for $t,\eta_i \in [a,b],\ i=1,2,\dots,m-2$.*

Proof. We have

$$\begin{aligned}\mu &= \left(b - \sum_{i=1}^{m-2}\gamma_i\eta_i\right) - \delta_2 t \\ &\geq \left(b - \sum_{i=1}^{m-2}\gamma_i\eta_i\right) - \left(1 - \sum_{i=1}^{m-2}\gamma_i\right)b \\ &\geq \sum_{i=1}^{m-2}\gamma_i(b-\eta_i) > 0.\end{aligned} \tag{2.17}$$

□

Lemma 2.13. *The functions $G(t,s)$, $H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})$ defined by* (2.11) *satisfy*

(i) $G(t,s) \geq 0$, $H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) \geq 0$, *for all* $t,s \in [a,b]$,

(ii) $\min_{\tau_1\leq t\leq\tau_2} G(t,s) \geq \tau_0 \max_{a\leq t\leq b} G(t,s) = \tau_0 G(s,s)$, *for all* $t,s \in (a,b)$, $a < \tau_1 < \tau_2 < b$, $\tau_0 = \min_{\tau_1\leq t\leq\tau_2}\varphi(t) = (b-\tau_2)/(b-a)$,

(iii) $N_2 q(s) \leq H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) \leq N_1 q(s)$, *where*

$$\begin{gathered} q(s) = \frac{(b-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha)}, \qquad N_1 = (\alpha-1)\left[\delta_1(b-a)\sum_{i=1}^{m-2}\gamma_i + b\sum_{i=1}^{m-2}\beta_i\right], \\ N_2 = \min\left\{\frac{\delta_1\sum_{i=1}^{m-2}\gamma_i}{(\alpha-1)}, \delta_1\sum_{i=1}^{m-2}\gamma_i(b-\eta_i)\right\}, \end{gathered} \tag{2.18}$$

(iv) $\min_{\tau_1\leq t\leq\tau_2} H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) \geq \tau^* \max_{a\leq t\leq b} H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})$, $s \in (a,b)$, *where*

$$\tau^* = \frac{\left[\left(b - \sum_{i=1}^{m-2}\gamma_i\eta_i\right) - \delta_2\tau_2\right]}{\left[\left(b - \sum_{i=1}^{m-2}\gamma_i\eta_i\right) - \delta_2 a\right]} < 1, \quad a < \tau_1 < \tau_2 < b. \tag{2.19}$$

Proof. It is clear that (i) holds. So, we prove that (ii) is true.

(ii) For $\alpha > 1$, in view of the expression for $G(t,s)$, it follows that $G(t,s) \leq G(s,s)$ for all $s,t \in [a,b]$, where $G(s,s) = (b-s)^{\alpha-1}/\Gamma(\alpha)$.

If $a \le s \le t \le b$, we have

$$\begin{aligned}
\frac{G(t,s)}{G(s,s)} &= \frac{\left[(b-s)^{\alpha-1}-(t-s)^{\alpha-1}\right]}{(b-s)^{\alpha-1}} \\
&= \frac{\left[(b-s)^{\alpha-2}(b-s)-(t-s)^{\alpha-2}(t-s)\right]}{(b-s)^{\alpha-1}} \\
&\ge \frac{(b-s)^{\alpha-2}[(b-s)-(t-s)]}{(b-s)^{\alpha-1}} \\
&= \frac{(b-t)}{(b-a)} := \varphi(t).
\end{aligned} \tag{2.20}$$

If $a \le t \le s \le b$, then we have

$$\frac{G(t,s)}{G(s,s)} = 1 \ge \frac{(b-t)}{(b-a)} := \varphi(t). \tag{2.21}$$

Thus

$$\max_{a\le t\le b} G(t,s) = G(s,s), \quad \varphi(t)G(s,s) \le G(t,s) \le G(s,s), \quad \forall t,s \in (a,b). \tag{2.22}$$

Therefore,

$$\min_{\tau_1\le t\le \tau_2} G(t,s) \ge \tau_0 \max_{a\le t\le b} G(t,s) = \tau_0 G(s,s), \quad \forall t,s \in (a,b),\ a<\tau_1<\tau_2<b. \tag{2.23}$$

(iii) If $a \le s \le \eta_i,\ i = 1,2,\dots,m-2$, then

$$\begin{aligned}
H(t,s;\eta_1,\eta_2,\dots,\eta_{m-2}) &= \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)} + \frac{\mu(t)\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha-1)} \\
&\le \frac{\sum_{i=1}^{m-2}\gamma_i(b-s)^{\alpha-1}}{\delta_2\Gamma(\alpha)} + \frac{(\alpha-1)b\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha)} \\
&= \frac{1}{\delta_1\delta_2\Gamma(\alpha)}\left[\delta_1\sum_{i=1}^{m-2}\gamma_i(b-s)^{\alpha-1}+(\alpha-1)b\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}\right] \\
&\le \frac{(\alpha-1)(b-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha)}\left[\delta_1\sum_{i=1}^{m-2}\gamma_i(b-s)+b\sum_{i=1}^{m-2}\beta_i\right] \\
&\le \frac{(\alpha-1)(b-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha)}\left[\delta_1(b-a)\sum_{i=1}^{m-2}\gamma_i+b\sum_{i=1}^{m-2}\beta_i\right] := N_1 q(s),
\end{aligned}$$

$$H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) = \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)} + \frac{\mu(t)\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha-1)}$$
$$\geq \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)}$$
$$= \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-2}(b-s)-(\eta_i-s)^{\alpha-2}(\eta_i-s)\right]}{\delta_2\Gamma(\alpha)}$$
$$= \frac{(b-s)^{\alpha-2}}{\delta_2\Gamma(\alpha)}\sum_{i=1}^{m-2}\gamma_i(b-\eta_i) \geq N_2 q(s). \tag{2.24}$$

If $\eta_i \leq s \leq b,\ i = 1,2,\ldots,m-2$, then we have

$$\begin{aligned} H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) &= \frac{\sum_{i=1}^{m-2}\gamma_i(b-s)^{\alpha-1}}{\delta_2\Gamma(\alpha)} \\ &= \delta_1\sum_{i=1}^{m-2}\gamma_i(b-s)\frac{(b-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha)} \\ &\leq \delta_1\sum_{i=1}^{m-2}\gamma_i(b-a)q(s) \\ &< N_1 q(s), \\ H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) &= \frac{\sum_{i=1}^{m-2}\gamma_i(\alpha-1)(b-s)^{\alpha-1}}{(\alpha-1)\delta_2\Gamma(\alpha)} \\ &\geq \frac{\sum_{i=1}^{m-2}\gamma_i(b-s)^{\alpha-2}}{(\alpha-1)\delta_2\Gamma(\alpha)} \\ &\geq \frac{\delta_1\sum_{i=1}^{m-2}\gamma_i}{(\alpha-1)}q(s) \geq N_2 q(s). \end{aligned} \tag{2.25}$$

(iv) Since $\partial H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})/\partial t = -\left(\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}/\delta_1\Gamma(\alpha-1)\right) \leq 0$, then $H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})$ is nonincreasing in t, so

$$\begin{aligned} \max_{a\leq t\leq b} H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) &= H(a,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) \\ &= \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)} \\ &\quad + \frac{\left[\left(b-\sum_{i=1}^{m-2}\gamma_i\eta_i\right)-\delta_2 a\right]\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha-1)}, \end{aligned}$$

$$\begin{aligned}
\min_{\tau_1\le t\le\tau_2} H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2}) &= \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)} \\
&\quad+\frac{\left[\left(b-\sum_{i=1}^{m-2}\gamma_i\eta_i\right)-\delta_2\tau_2\right]\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha-1)} \\
&= \frac{\sum_{i=1}^{m-2}\gamma_i\left[(b-s)^{\alpha-1}-(\eta_i-s)^{\alpha-1}\right]}{\delta_2\Gamma(\alpha)} \\
&\quad+\tau^*\frac{\left[\left(b-\sum_{i=1}^{m-2}\gamma_i\eta_i\right)-\delta_2 a\right]\sum_{i=1}^{m-2}\beta_i(\eta_i-s)^{\alpha-2}}{\delta_1\delta_2\Gamma(\alpha-1)} \\
&> \tau^*\max_{a\le t\le b} H(t;\eta_1,\eta_2,\ldots,\eta_{m-2},s).
\end{aligned}\tag{2.26}$$

□

3. Main Results

Let us denote by $E = C[a,b]$ the Banach space of all continuous real functions on $[a,b]$ endowed with the norm $\|u\| = \max_{a\le t\le b}|u(t)|$ and P the cone

$$P = \left\{u \in E : u \ge 0,\ \min_{\tau_1\le t\le\tau_2} u(t) \ge \tau\|u\|,\ t\in[a,b]\right\},\tag{3.1}$$

where $\tau = \min\{\tau_0,\tau^*\}$, since τ_0, τ^* are constants do not depend on t.

Let the nonnegative continuous concave functional θ on the cone P be defined by $\theta(u) = \min_{\tau_1\le t\le\tau_2} u(t)$.

Set $T : P \to E$ by

$$Tu(t) = \int_a^b \left[G(t,s) + H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})\right] f(s,u(s))ds, \quad a \le t \le b,\tag{3.2}$$

where $G(t,s)$, $H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})$ are defined as in Lemma 2.11.

From (3.2) and Lemma 2.13, we have

$$\begin{aligned}
\min_{\tau_1\le t\le\tau_2}(Tu(t)) &\ge \int_a^b\left[\tau_0 G(s,s) + \tau^*\max_{a\le t\le b} H(t,s;\eta_1,\ldots,\eta_{m-2})\right] f(s,u(s))ds \\
&\ge \tau\|Tu\|.
\end{aligned}\tag{3.3}$$

Hence, we have $T(P) \subset P$.

By standard argument, one can prove that $T : P \to P$ is a completely continuous operator.

The Existence of One Positive Solution

We introduce the following definitions:

$$\begin{gathered}
\overline{f}(u) := \sup_{t\in[a,b]} f(t,u), \qquad \underline{f}(u) := \inf_{t\in[a,b]} f(t,u), \\
f^0 = \limsup_{u\to 0^+} \frac{\overline{f}(u)}{u}, \qquad f_0 = \liminf_{u\to 0^+} \frac{\underline{f}(u)}{u}, \\
f^\infty = \limsup_{u\to \infty} \frac{\overline{f}(u)}{u}, \qquad f_\infty = \liminf_{u\to \infty} \frac{\underline{f}(u)}{u}, \\
M = \left(\int_a^b \left[G(s,s) + N_1 q(s)\right] ds\right)^{-1}, \qquad N = \left(\int_{\tau_1}^{\tau_2} \tau \left[G(s,s) + N_2 q(s)\right] ds\right)^{-1}.
\end{gathered} \tag{3.4}$$

Theorem 3.1. *Let $f(t,u)$ be continuous on $[a,b]\times[0,\infty) \rightarrow [0,\infty)$. If there exist two positive constants $r_2 > r_1 > 0$ such that*

(H_1) $f(t,u) \le Mr_2$, *for* $(t,u) \in [a,b]\times[0,r_2]$,

(H_2) $f(t,u) \ge Nr_1$, *for* $(t,u) \in [a,b]\times[0,r_1]$,

then the BVP (1.3)-(1.4) *has at least a positive solution.*

Proof. We know that the operator $T : P \rightarrow P$ defined by (3.2) is completely continuous.

(a) Let $\Omega_2 = \{u \in E : \|u\| < r_2\}$. For any $u \in P \cap \partial\Omega_2$, we have $\|u\| = r_2$ which implies that $0 \le u(t) \le r_2$ for every $t \in [a,b]$:

$$\begin{aligned}
Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\ldots,\eta_{m-2})\right] f(s,u(s)) ds \\
&\le Mr_2 \int_a^b \left[G(s,s) + N_1 q(s)\right] ds \\
&\le r_2 = \|u\|,
\end{aligned} \tag{3.5}$$

which implies that

$$\|Tu\| \le \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.6}$$

(b) Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$. For any $t \in [\tau_1, \tau_2],\ u \in P \cap \partial\Omega_1$. We have

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\ldots,\eta_{m-2})\right] f(s,u(s))ds \\ &\geq \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] f(s,u(s))ds \\ &\geq N r_1 \int_{\tau_1}^{\tau_2} \tau\left[G(s,s) + N_2 q(s)\right] ds \\ &= r_1 = \|u\|, \end{aligned} \tag{3.7}$$

which implies that

$$\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.8}$$

In view of Theorem 2.6, T has a fixed point $u_0 \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ which is a solution of the BVP (1.3)-(1.4). □

The Existence of Two Positive Solutions

Theorem 3.2. *Assume that all assumptions of Theorem 3.1, hold. Moreover, one assumes that $f(t,u)$ also satisfies*

(H_3) $f_\infty = \infty$.

Then, the BVP (1.3)-(1.4) *has at least two positive solutions.*

Proof. At first, it follows from condition (H_1) that

$$\|Tu\| \leq \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.9}$$

Further, it follows from condition (H_2) that

$$\|Tu\| \geq \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.10}$$

Finally, since $f_\infty = \infty$, there exists $\psi > \left(\tau^2 \int_{\tau_1}^{\tau_2}[G(s,s) + N_2 q(s)]ds\right)^{-1}$ and $r_3 > r_2$ such that

$$f(t,u) \geq \psi u(t), \quad t \in [a,b],\ u \geq r_3. \tag{3.11}$$

Let $r^* = \max\{2r_2, \tau^{-1}r_3\}$ and set $\Omega_3 = \{u \in E : \|u\| < r^*\}$, then $u \in P \cap \partial\Omega_3$implies $\min_{\tau_1 \le t \le \tau_2} u(t) \ge \tau\|u\| \ge \tau r^* \ge r_3$,

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})\right] f(s,u(s))ds \\ &\ge \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] f(s,u(s))ds \\ &\ge \psi \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] u(s)ds \\ &\ge r^*\psi \int_{\tau_1}^{\tau_2} \tau^2 \left[G(s,s) + N_2 q(s)\right] ds > r^* = \|u\|. \end{aligned} \tag{3.12}$$

Therefore, we have

$$\|Tu\| \ge \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{3.13}$$

Thus, from (3.6), (3.8), (3.13), and Theorem 2.6, T has a fixed point u_1, in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ and a fixed point u_2, in $P \cap (\overline{\Omega}_3 \setminus \Omega_2)$. Both are positive solutions of BVP (1.3)-(1.4) and satisfy

$$0 < \|u_1\| < r_2 < \|u_2\|. \tag{3.14}$$

□

Theorem 3.3. *Assume that $f(t,u)$ be continuous on $[a,b] \times [0,\infty) \to [0,\infty)$. If the following assumptions hold:*

(H_1) $f_0 > \psi$,

(H_2) $f_\infty > \psi$,

(H_3) *there exists a constant $\rho > 0$ such that $f(t,u) \le \rho M$, $(t,u) \in [a,b] \times [0,\rho]$,*

then the BVP (1.3)-(1.4) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho < \|u_2\|. \tag{3.15}$$

Proof. At first, it follows from condition (H_1) that we may choose $\rho_1 \in (0,\rho)$ such that

$$f(t,u) > \psi u, \quad 0 < u \le \rho_1, \tag{3.16}$$

where ψ is defined as in Theorem 3.2. Set $\Omega_1 = \{u \in E : \|u\| < \rho_1\}$, and $u \in P \cap \partial\Omega_1$; from (3.2) and Lemma 2.13, for $a \le t \le b$, we have

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\eta_2,\dots,\eta_{m-2})\right] f(s,u(s))ds \\ &\ge \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] f(s,u(s))ds \\ &\ge \psi \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] u(s)ds \\ &\ge \rho_1 \psi \int_{\tau_1}^{\tau_2} \tau^2 \left[G(s,s) + N_2 q(s)\right] ds \\ &> \rho_1 = \|u\|. \end{aligned} \tag{3.17}$$

Therefore, we have

$$\|Tu\| \ge \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{3.18}$$

Further, it follows from condition (H_2) that there exists $\rho_2 > \rho$ such that

$$f(t,u) > \psi u(t), \quad u \ge \rho_2. \tag{3.19}$$

Let $\rho^* = \max\{2\rho, \tau^{-1}\rho_2\}$, set $\Omega_2 = \{u \in E : \|u\| < \rho^*\}$, then $u \in P \cap \partial\Omega_2$implies $\min_{\tau_1 \le t \le \tau_2} u(t) \ge \tau\|u\| \ge \tau\rho^* \ge \rho_2$,

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\eta_2,\dots,\eta_{m-2})\right] f(s,u(s))ds \\ &\ge \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] f(s,u(s))ds \\ &\ge \psi \int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right] u(s)ds \\ &\ge \rho^* \psi \int_{\tau_1}^{\tau_2} \tau^2 \left[G(s,s) + N_2 q(s)\right] ds > \rho^* = \|u\|. \end{aligned} \tag{3.20}$$

Therefore, we have

$$\|Tu\| \ge \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3.21}$$

Finally, let $\Omega_3 = \{u \in E : \|u\| < \rho\}$ and $u \in P \cap \partial\Omega_3$. By condition (H_3), we have

$$\begin{aligned} Tu(t) &\leq \int_a^b [G(s,s) + N_1 q(s)] f(s,u(s)) ds \\ &\leq M\rho \int_a^b [G(s,s) + N_1 q(s)] ds \\ &= \rho = \|u\|, \end{aligned} \tag{3.22}$$

which implies

$$\|Tu\| \leq \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{3.23}$$

Thus, from (3.18), (3.21), (3.23), and Theorem 2.6, T has a fixed point u_1 in $P \cap (\overline{\Omega}_3 \setminus \Omega_1)$ and a fixed point u_2, in $P \cap (\overline{\Omega}_2 \setminus \Omega_3)$. Both are positive solutions of BVP (1.3)-(1.4) and satisfy

$$0 < \|u_1\| < \rho < \|u_2\|. \tag{3.24}$$

□

Theorem 3.4. *Assume that $f(t,u)$ be continuous on $[a,b] \times [0,\infty) \rightarrow [0,\infty)$. If the following assumptions hold:*

(H_1') $f_0 = \infty$,

(H_2') $f_\infty = \infty$,

(H_3') *there exists a constant $\rho' > 0$ such that $f(t,u) \leq \rho' M$, $(t,u) \in [a,b] \times [0,\rho']$,*

then the BVP (1.3)-(1.4) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho' < \|u_2\|. \tag{3.25}$$

The proof of Theorem 3.4 is very similar to that of Theorem 3.3 and therefore is omitted.

Theorem 3.5. *Assume that $f(t,u)$ be continuous on $[a,b] \times [0,\infty) \rightarrow [0,\infty)$. If the following assumptions hold:*

(H_1) $f^0 < M$,

(H_2) $f^\infty < M$,

(H_3) *there exists a constant $l > 0$ such that $f(t,u) \geq Nl$, $(t,u) \in [a,b] \times [\tau l, l]$,*

then the BVP (1.3)-(1.4) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < l < \|u_2\|. \tag{3.26}$$

Proof. It follows from condition (H_1) that we may choose $\rho_3 \in (0,l)$ such that

$$f(t,u) < Mu, \quad 0 < u \leq \rho_3. \tag{3.27}$$

Set $\Omega_4 = \{u \in E : \|u\| < \rho_3\}$, and $u \in P \cap \partial\Omega_4$; from (3.2) and Lemma 2.13, for $a \le t \le b$, we have

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H\left(t,s;\eta_1,\eta_2,\dots,\eta_{m-2}\right)\right] f(s,u(s))ds \\ &< M\int_a^b \left[G(s,s) + N_1 q(s)\right] ds \cdot \|u\| = M \cdot M^{-1}\|u\| = \|u\|. \end{aligned} \tag{3.28}$$

Therefore, we have

$$\|Tu\| \le \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{3.29}$$

It follows from condition (H_2) that there exists $\rho_4 > l$ such that

$$f(t,u) < Mu, \quad u \ge \rho_4, \tag{3.30}$$

and we consider two cases.

Case 1. Suppose that f is unbounded, there exists $l^* > \rho_4$ such that $f(t,u) \le f(t,l^*)$ for $0 < u \le l^*$.

Then, for $u \in P$ and $\|u\| = l^*$, we have

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H\left(t,s;\eta_1,\eta_2,\dots,\eta_{m-2}\right)\right] f(s,u(s))ds \\ &\le \int_a^b \left[G(s,s) + N_1 q(s)\right] f(s,l^*)ds \\ &< Ml^* \int_a^b \left[G(s,s) + N_1 q(s)\right] ds = l^* = \|u\|. \end{aligned} \tag{3.31}$$

Case 2. If f is bounded, that is, $f(t,u) \le k$ for all $u \in [0,\infty)$, taking $l^* \ge \max\{2l, kM^{-1}\}$, for $u \in P$ and $\|u\| = l^*$, then we have

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H\left(t,s;\eta_1,\eta_2,\dots,\eta_{m-2}\right)\right] f(s,u(s))ds \\ &\le k\int_a^b \left[G(s,s) + N_1 q(s)\right] ds = kM^{-1} \le l^* = \|u\|. \end{aligned} \tag{3.32}$$

Hence, in either case, we always may set $\Omega_5 = \{u \in E : \|u\| < l^*\}$ such that

$$\|Tu\| \le \|u\|, \quad u \in P \cap \partial\Omega_5. \tag{3.33}$$

Finally, set $\Omega_6 = \{u \in E : \|u\| < l\}$, then $u \in P \cap \partial\Omega_6$ and

$$\min_{\tau_1 \le t \le \tau_2} u(t) \ge \tau\|u\| = \tau l, \tag{3.34}$$

and by condition (H_3) and (3.2), we have

$$\begin{aligned}
Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\eta_2,\ldots,\eta_{m-2})\right] f(s,u(s))ds \\
&\ge Nl\int_a^b \varphi(t)\left[G(s,s) + N_2 q(s)\right]ds \\
&\ge Nl\int_{\tau_1}^{\tau_2} \tau\left[G(s,s) + N_2 q(s)\right]ds = l = \|u\|.
\end{aligned} \tag{3.35}$$

Hence, we have

$$\|Tu\| \ge \|u\|, \quad u \in P \cap \partial\Omega_6. \tag{3.36}$$

Thus, from (3.29), (3.33), (3.36) and Theorem 2.6, T has a fixed point u_1 in $P \cap (\overline{\Omega}_6 \setminus \Omega_4)$ and a fixed point u_2 in $P \cap (\overline{\Omega}_5 \setminus \Omega_6)$. Both are positive solutions of BVP (1.3)-(1.4) and satisfy

$$0 < \|u_1\| < l < \|u_2\|. \tag{3.37}$$

□

Theorem 3.6. *Assume that $f(t,u)$ be continuous on $[a,b] \times [0,\infty) \to [0,\infty)$. If the following assumptions hold:*

(H_1') $f^0 = 0$,

(H_2') $f^\infty = 0$,

(H_3') *there exists a constant $\rho'' > 0$ such that $f(t,u) \ge N\rho''$, $(t,u) \in [a,b] \times [\tau\rho'', \rho'']$,*

then the BVP (1.3)-(1.4) *has at least two positive solutions u_1 and u_2 such that*

$$0 < \|u_1\| < \rho'' < \|u_2\|. \tag{3.38}$$

The proof of Theorem 3.6 is very similar to that of Theorem 3.5 and therefore omitted.

The Existence of Three Positive Solutions

Theorem 3.7. *Let $f(t,u)$ be continuous on $[a,b] \times [0,\infty) \to [0,\infty)$. If there exist constants $0 < a_1 < a_2 \le a_3$ such that the following assumptions*

(i) $f(t,u) < Ma_1$, $(t,u) \in [a,b] \times [0,a_1]$,

(ii) $f(t,u) \le Ma_3$, $(t,u) \in [a,b] \times [0,a_3]$,

(iii) $f(t,u) \ge Na_2$, $(t,u) \in [\tau_1,\tau_2] \times [a_2, a_2/\tau]$, *hold, then BVP* (1.3)-(1.4) *has at least three positive solutions* $u_1,\ u_2$, *and* u_3 *with*

$$\|u_1\| < a_1, \qquad a_2 < \theta(u_2) < \|u_2\| \le a_3,$$
$$a_1 < \|u_3\|, \qquad \theta(u_3) < a_2. \tag{3.39}$$

Proof. We will show that all conditions of Lemma 2.10, are satisfied.

First, if $u \in \overline{P}_{a_3}$, then $\|u\| \le a_3$. So, $0 \le u(t) \le a_3,\ t \in [a,b]$.

By condition (ii), we have

$$\begin{aligned} Tu(t) &= \int_a^b \left[G(t,s) + H(t,s;\eta_1,\dots,\eta_{m-2})\right] f(s,u(s))ds \\ &\le \int_a^b \left[G(s,s) + N_1 q(s)\right] M a_3 ds \\ &= M a_3 \int_a^b \left[G(s,s) + N_1 q(s)\right] ds = a_3, \end{aligned} \tag{3.40}$$

which implies that $\|Tu\| \le a_3,\ u \in \overline{P}_{a_3}$. Hence $T : \overline{P}_{a_3} \to \overline{P}_{a_3}$.

Next, by using the analogous argument, it follows from condition (i) that if $u \in P_{a_1}$, then $\|Tu\| < a_1$.

Choose $u(t) = (a_2 + a_2/\tau)/2,\ t \in [a,b]$, it is easy to see that $u(t) = (a_2 + a_2/\tau)/2 \in P(\theta, a_2, a_3),\ \theta(u) = (a_2 + a_2/\tau)/2 > a_2$.

Therefore, $\{u \in P(\theta, a_2, a_2/\tau) \mid \theta(u) > a_2\} \ne \phi$. On the other hand, if $u \in P(\theta, a_2, a_2/\tau)$, then $a_2 \le u(t) \le a_2/\tau,\ t \in [\tau_1, \tau_2]$. By condition (iii), we have $f(t,u(t)) \ge N a_2$.

Hence,

$$\begin{aligned} \theta(Tu(t)) &= \min_{\tau_1 \le t \le \tau_2} Tu(t) \\ &= \min_{\tau_1 \le t \le \tau_2} \int_a^b \left[G(t,s) + H(t,s;\eta_1,\dots,\eta_{m-2})\right] f(s,u(s))ds \\ &\ge N a_2 \int_a^b \tau \left[G(s,s) + N_2 q(s)\right] ds \\ &> N a_2 \int_{\tau_1}^{\tau_2} \tau \left[G(s,s) + N_2 q(s)\right] ds = a_2, \end{aligned} \tag{3.41}$$

which implies that $\theta(Tu) > a_2$, for $u \in P(\theta, a_2, a_2/\tau)$.

Finally, if $u \in P(\theta, a_2, a_3)$ and $\|Tu\| > a_2/\tau$, then

$$\theta(u) = \min_{\tau_1 \le t \le \tau_2} Tu(t) \ge \tau \|Tu\| > a_2. \tag{3.42}$$

Thus, all the conditions of the Leggett-Williams fixed point theorem are satisfied by taking $d = a_2/\tau$. Hence, the BVPs have at least three solutions in P, that is, three positive solutions u_i $(i = 1, 2, 3)$ such that

$$\|u_1\| < a_1, \qquad a_2 < \theta(u_2) < \|u_2\| \leq a_3,$$
$$a_1 < \|u_3\|, \qquad \theta(u_3) < a_2. \tag{3.43}$$

□

Example 3.8. Consider the problem

$$D_{0^+}^{(4.2)} u(t) + \frac{1}{3}(1 + ue^u) = 0, \quad t \in (0, 1),$$
$$u'(0) = \frac{1}{4}u'\left(\frac{1}{2}\right), \qquad u''(0) = u'''(0) = u''''(0) = 0, \qquad u(1) = \frac{3}{4}u\left(\frac{1}{2}\right), \tag{3.44}$$

where $\alpha = 4.2$, $a = 0$, $b = 1$, $\beta = 0.25$, $\gamma = 0.75$, $\eta = 0.5$, $\tau_1 = 0.25$, $\tau_2 = 0.75$, $N_1 = 2.6$, $N_2 = 0.1758$, $N = 168.9596$, and $M = 1.6968$, $f(t, u) = (1/3)(1 + ue^u)$.

Since $f(t, u) = (1/3)(1 + ue^u)$ is a monotone increasing function on $[0, \infty)$, we take $r_1 = 0.001$, $r_2 = 0.8$. We can get

$$f(t, u) \leq f(0.8) = 0.9268 < Mr_2,$$
$$f(t, u) \geq f(0) = 0.3333 > Nr_1. \tag{3.45}$$

So, conditions (H_1) and (H_2) hold. By Theorem 3.1, the BVP (3.44) has at least one positive solution.

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