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Research Article

Generalized Variational Principles on Oscillation for Nonlinear Nonhomogeneous Differential Equations

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Using the generalized variational principle and the Riccati technique, new oscillation criteria are established for the forced second-order nonlinear differential equation, which improves and generalizes some of the new results in literature.

1. Introduction

In this paper, we consider the oscillatory behavior of the nonlinear nonhomogeneous differential equation of the form

$$\left(p(t)\Psi(y(t))\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + q(t)f(y(t)) = e(t), \quad t \geq t_0, \tag{1.1}$$

where α is a positive constant, $p, q, e \in C([t_0,\infty),\mathbb{R})$ with $p(t) > 0$, $\Psi \in C(\mathbb{R},(0,\infty))$, $f \in C(\mathbb{R},\mathbb{R})$ satisfying $uf(u) > 0$ for $u \neq 0$.

As usual, by a solution of (1.1) we mean a function $y \in C^1[T_y,\infty)$, $T_y \geq t_0$, where $T_y \geq t_0$ depends on the particular solution, which has the property $p(t)\Psi(y(t))|y'(t)|^{\alpha-1}y'(t) \in C^1[T_y,\infty)$ and satisfies (1.1). A nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros; otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Recently, many research works have been done on the oscillatory and asymptotic behavior of solutions of the nonlinear nonhomogeneous differential equation of the form

(1.1) or its special cases (see [1–7] and references cited therein). Using the variational method, oscillation criteria are obtained by Wong [1] for the forced linear differential equation, by Li and Cheng [2] for the forced half-linear differential equation, by Zheng and Meng [3] for the forced quasilinear differential equation, by Çakmak and Tiryaki [4] as well as Zheng and Cheng [5] for the forced nonlinear differential equation (some deficiencies in [2, 4] are pointed out by Zheng and Meng [3]), and Erbe et al. [7], as well as Saker [8, 9] for the dynamic equation on time scales.

Meanwhile, in [6], Komkov gave a generalized Leighton's variational principle, which can also be used to obtain the oscillation criterion.

Theorem 1.1. *Suppose that there exist a C^1 function $u(t)$ defined on $[s_1, t_1]$ and a function $G(u)$ such that $G(u(t))$ is not constant on $[s_1, t_1]$, $G(u(s_1)) = G(u(t_1)) = 0$, $g(u) = G'(u)$ is continuous,*

$$\int_{s_1}^{t_1} \left[q(t)G(u(t)) - p(t)\left(u'(t)\right)^2\right]dt > 0, \tag{1.2}$$

and $(g(u(t)))^2 \le 4G(u(t))$ for $t \in [s_1, t_1]$. Then each solution of the equation

$$\left(p(t)y'(t)\right)' + q(t)y(t) = 0 \tag{1.3}$$

vanishes at least once on $[s_1, t_1]$.

The purpose of our paper is to use the generalized variational principle to study the oscillation for (1.1). These oscillation criteria are closely related to the generalized variational formulae (1.2), which improve the results mentioned above. Examples will also be given to illustrate the effectiveness of our main results.

Before going into the main results, let us state three sets of conditions commonly used in the literature which we rely on:

$$\text{(S1)} \quad 0 < \Psi(u) \le M, \quad \left|f'(u)\right| \ge K\left|f(u)\right|^{(\beta-1)/\beta} > 0, \quad \text{for } u \ne 0, \tag{1.4}$$

$$\text{(S2)} \quad \frac{\left|f'(u)\right|}{\left[\Psi(u)|f(u)|^{\beta-1}\right]^{1/\beta}} \ge \gamma > 0, \quad \text{for } u \ne 0, \tag{1.5}$$

$$\text{(S3)} \quad 0 < \Psi(u) \le M, \quad \frac{\left|f(u)\right|}{|u|^{\beta}} \ge \delta > 0, \quad \text{for } u \ne 0. \tag{1.6}$$

Here, $M, K > 0$, $0 < \alpha \le \beta$, and $\gamma, \delta > 0$ are constants. It is clear that assumption (S1) implies (S2), but not conversely. For example, the function $f(u) = u^3$, $\Psi(u) = u^2$ and $\beta = 1$ do not satisfy (S1), but (S2) holds. In (S1) and (S2), we need f to be differentiable. Clearly, this condition is not required in (S3). These differences force us to study (1.1) under the assumptions (S1), (S2), and (S3) in separate manners.

2. The Case Where $\beta = \alpha$

Firstly, we give an inequality, which is a transformation of Young's inequality.

Lemma 2.1 (see [10]). *Suppose that X and Y are nonnegative. Then*

$$\lambda X Y^{\lambda-1} - X^{\lambda} \leq (\lambda-1)Y^{\lambda}, \quad \lambda > 1, \tag{2.1}$$

where the equality holds if and only if $X = Y$.

Now, we will give our main results.

Theorem 2.2. *Assume (S2) holds. Suppose further that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that*

$$e(t)\begin{cases} \leq 0, & t \in [s_1, t_1], \\ \geq 0, & t \in [s_2, t_2]. \end{cases} \tag{2.2}$$

Let $u \in C^1[s_i, t_i]$, and nonnegative functions G_1, G_2 satisfying $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ are continuous and $(g_i(u(t)))^{\alpha+1} \leq (\alpha+1)^{\alpha+1} G_i^{\alpha}(u(t))$ for $t \in [s_i, t_i]$, $i = 1, 2$. If there exists a positive function $\rho \in C^1([t_0, \infty), \mathbb{R})$ such that

$$Q_i^{\rho}(u) := \int_{s_i}^{t_i} \rho(t)\left[q(t)G_i(u(t)) - \left(\frac{\alpha}{\gamma}\right)^{\alpha} p(t)\left(\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right] dt > 0, \tag{2.3}$$

for $i = 1, 2$. Then (1.1) is oscillatory.

Proof. Suppose to the contrary that there is a nonoscillatory solution $y(t)$ of (1.1). First, we consider the case when $y(t) > 0$ eventually. Assume that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Set

$$w(t) = \frac{\rho(t)p(t)\Psi(y(t))\left|y'(t)\right|^{\alpha-1} y'(t)}{f(y(t))}, \quad t \geq T_0. \tag{2.4}$$

Then differentiating (2.4) and making use of (1.1), it follows that for all $t \geq T_0$,

$$w'(t) = -\rho(t)q(t) + \frac{\rho(t)e(t)}{f(y(t))} + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{|w(t)|^{(\alpha+1)/\alpha} f'(y(t))}{\left[p(t)\rho(t)\Psi(y(t))\left|f(y(t))\right|^{\alpha-1}\right]^{1/\alpha}}. \tag{2.5}$$

By assumptions, we can choose $s_1, t_1 \geq T_0$ with $s_1 < t_1$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$. For $t \in I_1$ and in view of (1.5) and (2.5), $w(t)$ satisfies the inequality

$$\rho(t)q(t) \leq -w'(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \gamma\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)}. \tag{2.6}$$

Multiplying $G_1(u(t))$ through (2.6) and integrating (2.6) from s_1 to t_1, using the fact that $G_1(u(s_1)) = G_1(u(t_1)) = 0$, we obtain

$$\begin{aligned}\int_{s_1}^{t_1} G_1(u(t))\rho(t)q(t)dt &\le \int_{s_1}^{t_1} G_1(u(t))\left[-w'(t)+\frac{\rho'(t)}{\rho(t)}w(t)-\frac{\gamma|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)}\right]dt\\ &= -G_1(u(t))w(t)\big|_{s_1}^{t_1}+\int_{s_1}^{t_1}[G_i(u(t))]'w(t)dt\\ &\quad+\int_{s_1}^{t_1}G_1(u(t))\frac{\rho'(t)}{\rho(t)}w(t)dt-\int_{s_1}^{t_1}G_1(u(t))\frac{\gamma|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)}dt\\ &\le\int_{s_1}^{t_1}\left[\left|g_1(u(t))\right|\left|u'(t)\right|+G_1(u(t))\frac{\left|\rho'(t)\right|}{\rho(t)}\right]|w(t)|dt\\ &\quad-\gamma\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)}dt\\ &\le(\alpha+1)\int_{s_1}^{t_1}\left[\left|G_1^{\alpha/(\alpha+1)}(u(t))\right|\left|u'(t)\right|+G_1(u(t))\frac{\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right]|w(t)|dt\\ &\quad-\gamma\int_{s_1}^{t_1}G_1(u(t))\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)}dt.\end{aligned} \tag{2.7}$$

Let

$$\begin{gathered}X=\frac{\gamma^{\alpha/(\alpha+1)}}{p^{1/(\alpha+1)}(t)\rho^{1/(\alpha+1)}(t)}G_1^{\alpha/(\alpha+1)}(u(t))|w(t)|,\quad \lambda=1+\frac{1}{\alpha},\\ Y=\frac{\alpha^\alpha p^{\alpha/(\alpha+1)}(t)\rho^{\alpha/(\alpha+1)}(t)}{\gamma^{\alpha^2/(\alpha+1)}}\left[\left|u'(t)\right|+\frac{G_1^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right]^\alpha.\end{gathered} \tag{2.8}$$

By Lemma 2.1 and (2.7), we have

$$\int_{s_1}^{t_1} G_1(u(t))\rho(t)q(t)dt \le \int_{s_1}^{t_1}\left(\frac{\alpha}{\gamma}\right)^\alpha p(t)\rho(t)\left(\left|u'(t)\right|+\frac{G_1^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt, \tag{2.9}$$

which contradicts (2.3) with $i = 1$.

When $y(t) < 0$ holds eventually, we assume $y(t) < 0$ for $t \ge T_0 > t_0$. Defining the Riccati transformation as (2.4), we get that (2.5) is true. In this case, we choose $t_2 > s_2 \ge T_0$ so that $e(t) \ge 0$ on the interval $I_2 = [s_2, t_2]$. For a given $t \in I_2$, (1.5) and (2.5) imply (2.6). Multiplying $G_2(u(t))$ through (2.6) and integrating (2.6) from s_2 to t_2, using the fact that $G_2(u(s_2)) = G_2(u(t_2)) = 0$, we obtain a similar contradiction to (2.3) with $i = 2$. This completes the proof. □

Corollary 2.3. *If* $\rho(t) \equiv 1$ *in Theorem 2.2 and* (2.3) *is replaced by*

$$Q_i(u) := \int_{s_i}^{t_i} \left[q(t)G_i(u(t)) - \left(\frac{\alpha}{\gamma}\right)^{\alpha} p(t)\left|u'(t)\right|^{\alpha+1} \right] dt > 0, \tag{2.10}$$

for $i = 1, 2$, *then* (1.1) *is oscillatory.*

Remark 2.4. The left side of inequality (2.10) is closely related to the generalized variational formulae (1.2). Particularly, we obtain $\alpha = \gamma$ for the linear equation; thus, (2.10) reduces to (1.2) for the linear differential equation. So our Theorem 2.2 and Corollary 2.3 are generalizations of the paper by Zheng and Cheng [5] and improvement of papers by Li and Cheng [2] and by Çakmak and Tiryaki [4].

Theorem 2.5. *Assume that (S3) holds. Suppose further that for any* $T \geq t_0$, *there exist* $T \leq s_1 < t_1 \leq s_2 < t_2$ *such that* (2.2) *holds. Let* $u \in C^1[s_i, t_i]$, *and nonnegative functions* G_1, G_2 *satisfying* $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ *are continuous and* $(g_i(u(t)))^{\alpha+1} \leq (\alpha+1)^{\alpha+1} G_i^{\alpha}(u(t))$ *for* $t \in [s_i, t_i]$, $i = 1, 2$. *If there exists a positive function* $\rho \in C^1([t_0, \infty), \mathbb{R})$ *such that*

$$Q_i^{\rho}(u) := \int_{s_i}^{t_i} \rho(t) \left[\delta q(t)G_i(u(t)) - Mp(t)\left(\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} \right] dt > 0, \tag{2.11}$$

for $i = 1, 2$. *Then* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that there is a nonoscillatory solution $y(t)$. Firstly, we assume that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Set

$$w(t) = \frac{\rho(t)p(t)\Psi(y(t))\left|y'(t)\right|^{\alpha-1}y'(t)}{\left|y(t)\right|^{\alpha-1}y(t)}, \quad t \geq T_0. \tag{2.12}$$

Then differentiating (2.12) and making use of (1.1) and (S3), we see that for all $t \geq T_0$, we have

$$\begin{aligned} w'(t) &= \frac{-q(t)\rho(t)f(y(t))}{\left|y(t)\right|^{\alpha-1}y(t)} + \frac{\rho(t)e(t)}{\left|y(t)\right|^{\alpha-1}y(t)} + \frac{\rho'(t)w(t)}{\rho(t)} - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{\left[\rho(t)p(t)\Psi(y(t))\right]^{1/\alpha}} \\ &\leq -\delta q(t)\rho(t) + \frac{\rho'(t)w(t)}{\rho(t)} + \frac{e(t)\rho(t)}{\left|y(t)\right|^{\alpha-1}y(t)} - \frac{\alpha}{M^{1/\alpha}}\frac{|w(t)|^{(\alpha+1)/\alpha}}{\rho^{1/\alpha}(t)p^{1/\alpha}(t)}. \end{aligned} \tag{2.13}$$

By assumptions, we choose $s_1, t_1 \geq T_0$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$. For $t \in I_1$, (2.13) implies that $w(t)$ satisfies the inequality

$$\delta\rho(t)q(t) \leq -w'(t) + \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\alpha}{M^{1/\alpha}} \frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho(t)^{1/\alpha}(t)}. \tag{2.14}$$

A similar method as to that of Theorem 2.2 implies a contradiction to (2.11) with $i = 1$.

When $y(t) < 0$ holds eventually, we assume $y(t) < 0$ for $t \geq T_0 > t_0$. Defining the Riccati transformation as (2.12), we get that (2.13) is true. In this case, we choose $t_2 > s_2 \geq T_0$ so that $e(t) \geq 0$ on the interval $I_2 = [s_2, t_2]$. For a given $t \in I_2$, we get that (2.14) holds. A similar method reaches a similar contradiction to (2.11) with $i = 2$. This completes the proof. □

Now we give two examples to illustrate the efficiency of our results.

Example 2.6. Consider the following forced half-linear differential equation:

$$\left(t^{\lambda}\left|y'(t)\right|^{\alpha-1}y'(t)\right)' + Kt^{\lambda}\left|y(t)\right|^{\alpha-1}y(t) = -\sin t, \tag{2.15}$$

for $t \geq 1$, where $K, \lambda > 0$ are constants and $\alpha = 1$. We may show that (2.15) is oscillatory for $K > 2e(1+\lambda/2)^2$ using Theorem 2.2. Indeed, since the zeros of the forcing term $-\sin t$ are $n\pi$, the constant γ in (1.5) is α, that is, $\gamma = \alpha$. For any $T \geq 1$, we choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. Selecting $u(t) = \sin t \geq 0$, $G_1(u) = u^2\exp(-u)$ (we note that $(G_1'(u))^2 \leq 4G_1(u)$ for $u \geq 0$), $\rho(t) = t^{-\lambda}$, then we have

$$\begin{aligned}
\int_{s_1}^{t_1} \rho(t)q(t)G_1(u(t))dt &= \int_{2k\pi}^{(2k+1)\pi} t^{-\lambda}Kt^{\lambda}(\sin t)^2\exp(-\sin t)dt \\
&= K\int_0^{\pi}(\sin t)^2\exp(-\sin t)dt \\
&\geq \frac{K}{e}\int_0^{\pi}\frac{1-\cos 2t}{2}dt = \frac{K\pi}{2e}, \\
\int_{s_1}^{t_1}\rho(t)\left(\frac{\alpha}{\gamma}\right)^{\alpha}p(t)&\left(\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt \\
&= \int_{2k\pi}^{(2k+1)\pi} t^{-\lambda}\left(\frac{\alpha}{\alpha}\right)^{\alpha}t^{\lambda}\left(|\cos t| + \frac{\lambda\sin t\exp(-\sin t/2)}{t}\right)^2 dt \\
&< \int_0^{\pi}\left(1+\frac{\lambda}{2}\right)^2 dt = \left(1+\frac{\lambda}{2}\right)^2\pi.
\end{aligned} \tag{2.16}$$

So we have $Q_1^{\rho}(u) > 0$ provided $K > 2e(1+\lambda/2)^2$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we select $u(t) = \sin t \leq 0$, $G_2(u) = u^2\exp(u)$, and we note that $(G_2'(u))^2 \leq 4G_2(u)$ for $u \leq 0$; we can show that the integral inequality $Q_2^{\rho}(u) > 0$ for $K > 2e(1+\lambda/2)^2$. So (2.15) is oscillatory for $K > 2e(1+\lambda/2)^2$ by Theorem 2.2.

Example 2.7. Consider the following forced half-linear differential equation:

$$\left[(2+\cos t)t^{-\lambda}\left|y'(t)\right|^{\alpha-1}y'(t)\right]' + Kt^{-\lambda}\exp(\sin t)\left|y(t)\right|^{\alpha-1}y(t) = -\sin t, \tag{2.17}$$

for $t \geq 1$, where $K, \lambda > 0$ are constants and $\alpha = 1$. We may show that (2.17) is oscillatory for $K > 3(1+\lambda)^2$ using Theorem 2.2. Indeed, since the zeros of the forcing term $-\sin t$ are $n\pi$, the constant γ in (1.5) is α, that is, $\gamma = \alpha$. In fact, for any $T \geq 1$, we choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. Selecting $u(t) = \sin t \geq 0$, $G_1(u) = u^2\exp(-u)$ (we note that $(G_1'(u))^2 \leq 4G_1(u)$ for $u \geq 0$), $\rho(t) = t^\lambda$, then we have

$$\begin{aligned}
\int_{s_1}^{t_1}\rho(t)q(t)G_1(u(t))dt &= \int_{2k\pi}^{(2k+1)\pi} t^\lambda K t^{-\lambda}(\sin t)^2\exp(-\sin t)dt \\
&= K\int_0^\pi (\sin t)^2\exp(\sin t - \sin t)dt \\
&\geq K\int_0^\pi \frac{1-\cos 2t}{2}dt = \frac{K\pi}{2}, \\
\int_{s_1}^{t_1}\rho(t)\left(\frac{\alpha}{\gamma}\right)^\alpha p(t)\left(\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}dt \\
&= \int_{2k\pi}^{(2k+1)\pi} t^\lambda \frac{1}{2}\left(\frac{\alpha}{\alpha}\right)^\alpha (2+\cos t)t^{-\lambda}\left(|\cos t| + \frac{\lambda\sin t\exp(-\sin t/2)}{t}\right)^2 dt \\
&< \int_0^\pi \frac{3}{2}(1+\lambda)^2 dt = \frac{3}{2}(1+\lambda)^2\pi.
\end{aligned} \tag{2.18}$$

So, we have $Q_1^\rho(u) > 0$ provided $K > 3(1+\lambda)^2$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we select $u(t) = \sin t \leq 0$, $G_2(u) = u^2\exp(u)$, and we note that $(G_2'(u))^2 \leq 4G_2(u)$ for $u \leq 0$; we can show that the integral inequality $Q_2^\rho(u) > 0$ for $K > 3(1+\lambda)^2$. So, (2.17) is oscillatory for $K > 3(1+\lambda)^2$ by Theorem 2.2.

3. The Case Where $\beta > \alpha$

We now handle the case where $\beta > \alpha$.

Theorem 3.1. *Assume that (S3) holds. Suppose further that for any $T \geq t_0$, there exist $T \leq s_1 < t_1 \leq s_2 < t_2$ such that* (2.2) *holds. Let $u \in C_1[s_i, t_i]$, and the nonnegative functions G_1, G_2 satisfying $G_i(u(s_i)) = G_i(u(t_i)) = 0$, $g_i(u) = G_i'(u)$ are continuous and $(g_i(u(t)))^{\alpha+1} \leq (\alpha+1)^{\alpha+1}G_i^\alpha(u(t))$ for $t \in [s_i, t_i]$, $i = 1, 2$. If there exists a positive function $\rho \in C^1([t_0,\infty),\mathbb{R})$ such that*

$$Q_i^\rho(u) := \int_{s_i}^{t_i}\rho(t)\left[Q_e(t)G_i(u(t)) - Mp(t)\left(\left|u'(t)\right| + \frac{G_i^{1/(\alpha+1)}(u(t))\left|\rho'(t)\right|}{(\alpha+1)\rho(t)}\right)^{\alpha+1}\right]dt > 0, \tag{3.1}$$

for $i = 1, 2$, *then* (1.1) *is oscillatory, where*

$$Q_e(t) = \alpha^{-\alpha/\beta}\beta(\beta-\alpha)^{(\alpha-\beta)/\beta}[\delta q(t)]^{\alpha/\beta}|e(t)|^{(\beta-\alpha)/\beta}. \tag{3.2}$$

Proof. Suppose to the contrary that there is a nontrivial nonoscillatory solution. Firstly, we assume that $y(t) > 0$ on $[T_0, \infty)$ for some $T_0 \geq t_0$. Set

$$w(t) = \frac{\rho(t)p(t)\Psi(y(t))\left|y'(t)\right|^{\alpha-1}y'(t)}{\left|y(t)\right|^{\alpha-1}y(t)}, \quad t \geq T_0. \tag{3.3}$$

Then differentiating (3.3) and making use of (1.1), it follows that for all $t \geq T_0$,

$$\begin{aligned} w'(t) &= \frac{-q(t)\rho(t)f(y(t))}{\left|y(t)\right|^{\alpha-1}y(t)} + \frac{\rho(t)e(t)}{\left|y(t)\right|^{\alpha-1}y(t)} + \frac{\rho'(t)w(t)}{\rho(t)} - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{[\rho(t)p(t)\Psi(y(t))]^{1/\alpha}} \\ &= -\frac{q(t)\rho(t)f(y(t))}{\left|y(t)\right|^{\beta-1}y(t)}\left|y(t)\right|^{\beta-\alpha} + \frac{e(t)\rho(t)}{\left|y(t)\right|^{\alpha-1}y(t)} + \frac{\rho'(t)w(t)}{\rho(t)} - \alpha\frac{|w(t)|^{(\alpha+1)/\alpha}}{[p(t)\rho(t)\Psi(y(t))]^{1/\alpha}} \\ &\leq -\delta q(t)\rho(t)\left|y(t)\right|^{\beta-\alpha} + \frac{e(t)\rho(t)}{\left|y(t)\right|^{\alpha-1}y(t)} + \frac{\rho'(t)w(t)}{\rho(t)} - \frac{\alpha}{M^{1/\alpha}}\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)}. \end{aligned} \tag{3.4}$$

By assumptions, we can choose $t_1 > s_1 \geq T_0$ so that $e(t) \leq 0$ on the interval $I_1 = [s_1, t_1]$. For a given $t \in I_1$, set $F(x) = \delta q(t)x^{\beta-\alpha} - e(t)/x^{\alpha}$, and we have $F'(x^*) = 0$, $F''(x^*) > 0$, where $x^* = [-\alpha e(t)/(\beta-\alpha)\delta q(t)]^{1/\beta}$. So, $F(x)$ attains its minimum at x^* and

$$F(x) \geq F(x^*) = Q_e(t). \tag{3.5}$$

So (3.4) and (3.5) imply that $w(t)$ satisfies

$$\rho(t)Q_e(t) \leq -w'(t) - \frac{\alpha}{M^{1/\alpha}}\frac{|w(t)|^{(\alpha+1)/\alpha}}{p^{1/\alpha}(t)\rho^{1/\alpha}(t)} + \frac{\rho'(t)w(t)}{\rho(t)}. \tag{3.6}$$

The remaining argument is the same as in the proof of Theorem 2.2, so we obtain a desired contradiction to (3.1) with $i = 1$ when $y(t) > 0$ eventually.

On the other hand, if $y(t)$ is a negative solution for $t \geq T_0 \geq t_0$, we define the Riccati transformation (3.3) to yield (3.4). In this case, we choose $t_2 > s_2 \geq T_0$ so that $e(t) \geq 0$ on the interval $I_2 = [s_2, t_2]$. For a given $t \in I_2$, set $F(x) = \delta q(t)x^{\beta-\alpha} - e(t)/x^{\alpha}$, and we have $F(x) \geq F(x^*) = Q_e(t)$. The remaining proof is similar to that of Theorem 2.2; a desired contradiction to (3.1) with $i = 2$ can be obtained. This completes the proof. □

Corollary 3.2. *If* $\rho(t) \equiv 1$ *in Theorem 3.1 and the hypothesis* (3.1) *is replaced by*

$$\widetilde{Q}_i(u) := \int_{s_i}^{t_i} \left[Q_e(t) G_i(u(t)) - M p(t) \left| u'(t) \right|^{\alpha+1} \right] dt > 0, \tag{3.7}$$

for $i = 1, 2$, *then* (1.1) *is oscillatory.*

We remark that Corollary 3.2 is closely related to the generalized variational formulae. Furthermore, in Theorem 3.1, there is no restriction on the positive constant α, so Theorem 2.5 can be treated as its limiting case when $\beta \to \alpha + 0$ with the convention that $0^0 = 1$.

Example 3.3. Consider the following forced quasilinear differential equation:

$$\left(\gamma t^{\lambda/3} y'(t) \right)' + t^{\lambda} \left| y(t) \right|^2 y(t) = -\sin^3 t, \quad t \geq 1, \tag{3.8}$$

where $\gamma, \lambda > 0$ are constants. We see that $\Psi(u) \equiv 1$, which implies that $M = 1$, and $\alpha = 1$, $\beta = 3$ in Theorem 3.1. Since $\alpha < \beta$, Theorem 2.2 cannot be applied. However, we can obtain oscillation for (3.8) using Theorem 3.1. For any $T \geq 1$, choose n sufficiently large so that $n\pi = 2k\pi \geq T$ and $s_1 = 2k\pi$ and $t_1 = (2k+1)\pi$. We choose $u(t) = \sin t \geq 0$, $G_1(u) = u^2 \exp(-u)$ (we note that $(G_1'(u))^2 \leq 4G_1(u)$, for $u \geq 0$), $\rho(t) = t^{\lambda}$. In fact, we can easily verify that $Q_e(t) = (3/2)\sqrt[3]{2} t^{\lambda/3} \sin^2 t$,

$$\begin{aligned}
\int_{s_1}^{t_1} Q_e(t) q(t) G_1(u(t)) dt &= \int_{2k\pi}^{(2k+1)\pi} t^{-\lambda/3} \frac{3}{2} \sqrt[3]{2} \sin t^4 \exp(-\sin t) dt \\
&= \frac{3}{2} \sqrt[3]{2} \int_0^{\pi} (\sin t)^4 \exp(-\sin t) dt \\
&\geq \frac{9\sqrt[3]{2}}{16e} \pi, \\
\int_{s_1}^{t_1} \rho(t) \left(\frac{\alpha}{\gamma} \right)^{\alpha} M p(t) \left(\left| u'(t) \right| + \frac{G_i^{1/(\alpha+1)}(u(t)) \left| \rho'(t) \right|}{(\alpha+1)\rho(t)} \right)^{\alpha+1} dt \\
&= \int_{s_1}^{t_1} t^{-\lambda/3} \gamma t^{\lambda/3} \left(|\cos t| + \frac{(\lambda/3) \sin t \exp(-\sin t/2)}{2t} \right)^2 dt \\
&< \gamma \left(1 + \frac{\lambda}{6} \right)^2 \pi.
\end{aligned} \tag{3.9}$$

So, we have that (3.1) is true for $i = 1$ provided $0 < \gamma < 9\sqrt[3]{2}/16e(1 + \lambda/6)^2$. Similarly, for $s_2 = (2k+1)\pi$ and $t_2 = (2k+2)\pi$, we can show that (3.1) is true for $i = 2$. So (3.8) is oscillatory for $0 < \gamma < 9\sqrt[3]{2}/16e(1 + \lambda/6)^2$ by Theorem 3.1.

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