Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 970659, 12 pages
doi:10.1155/2011/970659

Research Article

Approximation Order for Multivariate Durrmeyer Operators with Jacobi Weights

Jianjun Wang,[1] Chan-Yun Yang,[2] and Shukai Duan[3]

[1] *School of Mathematics and Statistics, Southwest University, Chongqing 400715, China*
[2] *Department of Mechanical Engineering, Technology and Science Institute of Northern Taiwan, No. 2 Xue-Yuan Road, Beitou, Taipei 112, Taiwan*
[3] *School of Electronics and Information Engineering, Southwest University, Chongqing 400715, China*

Correspondence should be addressed to Jianjun Wang, wjj@swu.edu.cn

Received 3 January 2011; Accepted 15 February 2011

Academic Editor: Pavel Drábek

Using the equivalence relation between K-functional and modulus of smoothness, we establish a strong direct theorem and an inverse theorem of weak type for multivariate Bernstein-Durrmeyer operators with Jacobi weights on a simplex in this paper. We also obtain a characterization for multivariate Bernstein-Durrmeyer operators with Jacobi weights on a simplex. The obtained results not only generalize the corresponding ones for Bernstein-Durrmeyer operators, but also give approximation order of Bernstein-Durrmeyer operators.

1. Introduction

Let $S = S_d$ $(d = 1, 2, \ldots)$ be a simplex in R^d defined by

$$S = \left\{ x = (x_1, x_2, \ldots, x_d) : x_i \geq 0,\ i = 1, 2, \ldots, d,\ |x| = \sum_{i=0}^{d} x_i \leq 1 \right\}. \tag{1.1}$$

For $p \geq 1$, we denote by $L^p(S)$ the space of p-order Lebesgue integrable functions on S with

$$\|\omega f\|_p = \begin{cases} \left(\int_S |\omega(x) f(x)|^p dx \right)^{1/p} < \infty & 1 \leq p < +\infty, \\ \max_{x \in S} |\omega(x) f(x)| & p = +\infty, \end{cases} \tag{1.2}$$

where $L^\infty(S) = C(S)$ denote the space of continuous functions on S. For $f \in L(S)$, the multivariate *Bernstein-Durrmeyer* Operators with d variables on S are given by

$$M_{n,d}(f;x) = \sum_{|k|\le n} P_{n\cdot k}(x)\frac{(n+d)!}{n!}\int_S P_{n,k}(u)f(u)du, \tag{1.3}$$

where $P_{n,k}(x) = (n!/(k!(n-|k|)!))x^k(1-|x|)^{n-|k|}$ $(x \in S)$ and $x = (x_1, x_2, \dots, x_d) \in R^d$, $k = (k_1, k_2, \dots, k_d) \in N_0^d$, with the convention

$$|x| = \sum_{i=1}^d x_i, \quad x^k = x_1^{k_1}x_2^{k_2}\cdots x_d^{k_d}, \quad |k| = \sum_{i=1}^d k_i, \quad k! = k_1!k_2!\cdots k_d!. \tag{1.4}$$

For multivariate Jacobi weights $\omega(x) = x^\alpha(1-|x|)^\beta$, $(x \in S, \alpha = (\alpha_1, \dots, \alpha_d) \in R^d, 0 < \alpha_i, \beta < 1, i = 1,2,\dots,d, x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_d^{\alpha_d})$. We give some further notations, for $x \in S$, and we write $\varphi_i(x) = \varphi_{ii}(x) = \sqrt{x_i(1-|x|)}$ $(1 \le i \le d)$, $\varphi_{ij}(x) = \sqrt{x_i x_j}$, $(1 \le i < j \le d)$ and

$$\begin{gathered} D_i = D_{ii} = \frac{\partial}{\partial x_i}, \quad 1 \le i \le d, \qquad D_{ij} = D_i - D_j, \quad 1 \le i < j \le d, \\ D_{ij}^r = D_{ij}\left(D_{ij}^{r-1}\right), \quad 1 \le i \le j \le d,\ r \in N, \qquad D^k = D_1^{k_1}D_2^{k_2}\cdots D_d^{k_d}, \quad k \in N_0^d. \end{gathered} \tag{1.5}$$

For $f \in L^p(S)$, the weighted *Sobolev* space is given by

$$\begin{aligned} W_\phi^{r,p}(S) = \Big\{ & f \in L^p(S) : \omega f \in L^p(S),\ D^k f \in L_{\text{loc}}\left(\overset{0}{S}\right), \\ & \omega\varphi_{ij}^r D_{ij}^r f \in L^p(S),\ |k| \le r,\ 1 \le i \le j \le d,\ r \in N\Big\}, \\ W_\phi^{r,\infty}(S) = \Big\{ & f \in C(S) : \omega f \in C(S),\ f \in C^r\left(\overset{0}{S}\right),\ \omega\varphi_{ij}^r D_{ij}^r f \in C(S),\ 1 \le i \le j \le d,\ r \in N\Big\}, \end{aligned} \tag{1.6}$$

where $\overset{0}{S}$ is the interior of S. To characterize the approximation capability of multivariate Bernstein-Durrmeyer operators, we introduce the weighted K-functional

$$K_\varphi^r(f, t^r)_\omega = \inf_{g \in W_\phi^{r,p}} \left\{ \|\omega(f-g)\|_p + t^r \sum_{1\le i\le j\le d} \left\|\omega\varphi_{ij}^r D_{ij}^r g\right\|_p \right\} \tag{1.7}$$

and a measure of smoothness of f

$$\omega_\varphi^r(f,t)_\omega = \sup_{0<h\le t} \sum_{1\le i\le j\le d} \left\|\omega \Delta_{h\varphi_{ij}e_{ij}}^r f\right\|_p. \tag{1.8}$$

Since 1967, Durrmeyer introduced Bernstein-Durrmeyer operators, and there are many papers which studied theirproperties [1–7]. In 1991, Zhang studied the characterization of convergence for $M_{n,1}(f;x)$ with Jacobi weights. In 1992, Zhou [5] considered multivariate Bernstein-Durrmeyer operators $M_{n,d}(f;x)$ and obtained a characterization of convergence. In 2002, Xuan et al. studied the equivalent characterization of convergence for $M_{n,d}(f;x)$ with Jacobi weights and obtained the following result.

Theorem 1.1. *For $\omega f \in L^p(S)$, $0 < r < 1$, the following results are equivalent:*

(i) $\|\omega(M_{n,d}f - f)\|_p = O(n^{-r})$;

(ii) $K_\varphi^2(f,t)_\omega = O(t^r)$.

In this paper, using the Ditzian-Totik modulus of smoothness, we will give the upper bound and lower bound of approximation function by $M_{n,d}(f;x)$ on simplex. The main results are as follows.

Theorem 1.2. *If $\omega f \in L^p(S)$, then*

$$\|\omega(M_{n,d}f - f)\|_p \le C\left\{\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right)_\omega + \frac{\|\omega f\|_p}{n}\right\}. \tag{1.9}$$

And there exists a positive number δ $(0 < \delta < 1)$ such that the following inequality is satisfied:

$$\omega_\varphi^2\left(f,\frac{1}{\sqrt{n}}\right)_\omega \le \frac{C}{n}\sum_{k=1}^{n}\left(\frac{n}{k}\right)^\delta \|\omega(M_{n,d}f - f)\|_\omega. \tag{1.10}$$

Throughout the paper, the letter C, appearing in various formulas, denotes a positive constant independent of n, x, and f. Its value may be different at different occurrences, even within the same formula.

From Theorem 1.2, we can easily obtain the following corollary.

Corollary 1.3. *If $\omega f \in L^p(S)$, $0 < r < 1$, we has the following equivalent results:*

(i) $\|\omega(M_{n,d}f - f)\|_p = O(n^{-r})$;

(ii) $K_\varphi^2(f,t)_\omega = O(t^r)$;

(iii) $\omega_\varphi^2(f,t)_\omega = O(t^{2r})$.

2. Some Lemmas

To prove Theorem 1.2, we will show some lemmas in this section. For the simplex S, the transformation $T: S \to S^{[10]}$ defined by

$$T(x_1,x_2,\dots,x_d) = (u_1,u_2,\dots,u_d), \quad u_l = \begin{cases} x_j & l = j, \\ 1 - |x| & l \ne j \end{cases} \tag{2.1}$$

satisfies $T^2 = I$, and I is the identity operator. So we have

$$\frac{\partial}{\partial u_l} = \frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_j} \quad (l \neq j), \quad \frac{\partial}{\partial u_j} = -\frac{\partial}{\partial x_j},$$
$$M_{n,d}(f;x) = M_{n,d}(f_T;Tx); \qquad M_{n,d}(f;Tx) = M_{n,d}(f_T;x), \tag{2.2}$$

where $f_T(u) = f(Tx)$.

Lemma 2.1. *If $\omega f \in L^p(S)$, then*

$$\|\omega M_{n,d} f\|_p \leq \|\omega f\|_p,$$
$$\|\omega(M_{n,d} f - f)\|_p \leq \frac{C}{n}\left(\|\omega f\|_p + \sum_{1\leq i\leq j\leq d}\left\|\omega\varphi_{ij}^2 D_{ij}^2 f\right\|_p\right), \quad f \in W_{\phi}^{r,p}(S). \tag{2.3}$$

Proof. Letting $S' = \{\overline{x} : (x_1, \overline{x}) \in S_d\}$, $\overline{x} = (x_2, x_3, \ldots, x_d)$, $\overline{k} = (k_2, k_3 \ldots, k_d)$, $k = (k_1, \overline{k})$, $P_{n,k_1}(x_1) = (n!/k_1!(n-k_1)!)x_1^{k_1}(1-x_1)^{n-k_1}$, then

$$\begin{aligned}
M_{n,d}(f;x) &= \sum_{k_1=0}^{n} P_{n,k_1}(x_1) \sum_{|\overline{k}|\leq n-k_1} P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right)\frac{(n+d)!}{n!} \\
&\quad \times \int_0^1 P_{n,k_1}(u_1)\int_{S'} P_{n-k_1,\overline{k}}\left(\frac{\overline{u}}{1-u_1}\right) f(u)d\overline{u}\, du_1 \\
&= \sum_{k_1=0}^{n} P_{n,k_1}(x_1)\frac{(n+d)!}{n!}\int_0^1 P_{n,k_1}(u_1)(1-u_1)^{d-1}\sum_{|\overline{k}|\leq n-k_1} P_{n-k_1,\overline{k}}\left(\frac{\overline{x}}{1-x_1}\right) \\
&\quad \times \int_{S_{d-1}} P_{n-k_1,\overline{k}}(t) f(u_1,(1-u_1)t)dt\, du_1 \\
&= \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1) M_{n-k_1,d-1}\left(f(u_1,(1-u_1)\cdot);\frac{\overline{x}}{1-x_1}\right)du_1.
\end{aligned} \tag{2.4}$$

Using the transformation T, (2.2), (2.4), the method of [7], we can easily get (2.3). □

Lemma 2.2 (see [8]). *If $f \in L^p(S)$, then*

$$C^{-1}\omega_{\varphi}^r(f,t)_{\omega} \leq K_{\varphi}^r(f,t^r)_{\omega} \leq C\omega_{\varphi}^r(f,t)_{\omega}. \tag{2.5}$$

Proof. Lemma 2.2 is proved when $f \in C(S)$ in [8]. Similarly, we can prove $f \in L^p(S)$. □

Lemma 2.3. *If* $0 < a < 1, b > 0, x \in (0,1)$, $P_{n,k}(x) = C_n^k x^k (1-x)^{n-k}$ *is basis function of the classical Bernstein operators, then*

$$\sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{k}\right)^a \le Cx^{-a},$$
$$\sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{n-k}\right)^b \le C(1-x)^{-b}. \tag{2.6}$$

Proof. The first inequality can be inferred by Hölder inequality. In the following we prove the second inequality.

(i) If $0 < b < 1$, using Hölder inequality, we can easily obtain the result.

(ii) If $b \ge 1$, let $b = m + r$, $m \in N$, $0 \le r < 1$, then

$$\begin{aligned}\sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{n-k}\right)^b &= \sum_{k=1}^{n-1} P_{n,k}(x)\left(\frac{n}{n-k}\right)^m \left(\frac{n}{n-k}\right)^r \\ &\le C(1-x)^{-m}\sum_{k=1}^{n-1} P_{n+m,k}(x)\left(\frac{n+m}{n+m-k}\right)^r \\ &\le C(1-x)^{-m-r} = C(1-x)^{-b}.\end{aligned} \tag{2.7}$$

Lemma 2.3 is completed. □

Lemma 2.4. *If* $f \in L^p(S)$, $1 \le p \le \infty$, *then*

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p \le Cn\|\omega f\|_p \quad 1 \le i \le j \le d. \tag{2.8}$$

Proof. In the following we use the induction on the dimension number d to prove the result. The case $d = 1$ was proved by Lemma 4 of [6]. Next, suppose that Lemma 2.4 is valid for $d = r$ $(r \ge 1)$; we prove it is also true for $d = r + 1$. To observe this, we use a decomposition formula (2.4), and we have

$$\begin{aligned}&\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x) \\ &\quad = x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d} \\ &\qquad \times (1-|z|)^{\beta}\varphi_{11}^2(z)\int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2 M_{n-k_1,d-1}(f(u_1,(1-u_1)\cdot);z)du_1,\end{aligned} \tag{2.9}$$

where $z = (z_1, z_2, \dots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \dots, x_d/(1-x_1))$. Thus we have

$$\begin{aligned}
&\int_S \left|\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x)\right| ds \\
&\quad\le C\int_0^1 x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)(n-k_1) \\
&\qquad\times\int_{z\in S_{d-1}} \left|\omega(z)f(u_1,(1-u_1)z)\right| dz\,dx_1 du_1 \\
&\quad\le C\frac{n+d}{n+1}n\int_0^1\sum_{k_1=0}^{n}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} P_{n+d-1,k_1}(u_1) \\
&\qquad\times\int_{z\in S_{d-1}} \left|\omega(z)f(u_1,(1-u_1)z)\right| dz\,du_1 \\
&\quad\le Cn\int_0^1 u_1^{\alpha_1}(1-u_1)^{|\bar{\alpha}|+\beta}\left(\frac{1}{u_1}\right)^{\alpha_1}(1-u_1)^{-|\bar{\alpha}|-\beta}\int_{z\in S_{d-1}}\left|(\omega f)(u_1,(1-u_1)z)\right| dz\,du_1 \\
&\quad= Cn\|\omega f\|_1.
\end{aligned} \tag{2.10}$$

In the above derivation, we have used the formula [6]

$$\int_0^1 x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}P_{n,k_1}(x_1)dx_1 \le C\frac{1}{n+1}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} \tag{2.11}$$

and the inequality

$$\sum_{k_1=0}^{n}\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar{\alpha}|+\beta} P_{n+d-1,k_1}(u_1) \le Cu_1^{\alpha_1}(1-u_1)^{|\bar{\alpha}|+\beta}. \tag{2.12}$$

When $p = \infty$, we have

$$\begin{aligned}
&\omega(x)\varphi_{22}^2(x)D_{22}^2 M_{n,d}(f;x) \\
&\quad= x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d)z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d} \\
&\qquad\times(1-|z|)^{\beta}\varphi_{11}^2(z)\int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2 M_{n-k_1,d-1}(f(u_1,(1-u_1)\cdot);z)du_1,
\end{aligned} \tag{2.13}$$

where $z = (z_1, z_2, \dots, z_{d-1}) = (x_2/(1-x_1), x_3/(1-x_1), \dots, x_d/(1-x_1))$.

From the Cauchy-Swartz inequality, Hölder inequality, and Lemma 2.3, we have

$$\begin{aligned}
&\left|\omega(x)\varphi_{22}^{2}(x)D_{22}^{2}M_{n,d}(f;x)\right| \\
&\quad\le Cx_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)(n-k_1) \\
&\qquad\times\max_{z\in S_{d-1}}\left|z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d}(1-|z|)^{\beta}f(u_1,(1-u_1)z)\right|du_1 \\
&\quad\le Cn\|\omega f\|_\infty x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)\left(\frac{1}{u_1}\right)^{\alpha_1}(1-u_1)^{-|\bar{\alpha}|-\beta}du_1 \\
&\quad\le Cn\|\omega f\|_\infty x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\left(\int_0^1 P_{n+d-1,k_1}(u_1)u_1^{-2\alpha_1}du_1\right)^{1/2} \\
&\qquad\times\left(\int_0^1 P_{n+d-1,k_1}(u_1)(1-u_1)^{-2|\bar{\alpha}|-2\beta}du_1\right)^{1/2} \\
&\quad\le Cn\|\omega f\|_\infty x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\left(\int_0^1 P_{n+d-1,k_1}(u_1)u_1^{-2}du_1\right)^{\alpha_1/2} \\
&\qquad\times\left(\int_0^1 P_{n+d-1,k_1}(u_1)du_1\right)^{(1-\alpha_1)/2}\left(\int_0^1 P_{n+d-1,k_1}(u_1)(1-u_1)^{-2d}du_1\right)^{(|\bar{\alpha}|+\beta)/2d} \\
&\qquad\times\left(\int_0^1 P_{n+d-1,k_1}(u_1)du_1\right)^{1/2-(|\bar{\alpha}|+\beta)/2d} \\
&\quad\le Cn\|\omega f\|_\infty x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=2}^{n}P_{n,k_1}(x_1)(n+d)\left(\frac{n+d-1}{k_1(k_1-1)}\right)^{\alpha_1/2} \\
&\qquad\times\left(\frac{(n+d-1)!}{(n-d)!}\frac{(n-d-k_1-1)!}{(n+d-k_1-1)!}\right)^{(|\bar{\alpha}|+\beta)/2d}\left((n+d)^{1-(|\bar{\alpha}|+\beta)/2d-\alpha_1/2}\right)^{-1} \\
&\quad\le Cn\|\omega f\|_\infty x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=1}^{n-1}P_{n,k_1}(x_1)\left(\frac{n}{k_1}\right)^{\alpha_1}\left(\frac{n}{n-k_1}\right)^{|\bar{\alpha}|+\beta} \\
&\quad\le Cn\|\omega f\|_\infty.
\end{aligned}\tag{2.14}$$

By *Riesz* interpolation theorem, we get

$$\left\|\omega\varphi_{22}^{2}D_{22}^{2}M_{n,d}f\right\|_p\le Cn\|\omega f\|_p.\tag{2.15}$$

Similarly, the other cases for $i = 1, 3, 4, \ldots, d(=j)$ can be proved. For $i \neq j$, by the transformation T, we have

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p = \left\|\omega_T \varphi_{jj}^2 D_{jj}^2 M_{n,d} f_T\right\|_p \leq Cn\left\|\omega_T f_T\right\|_p = Cn\|\omega f\|_p. \tag{2.16}$$

Lemma 2.4 is completed. □

Lemma 2.5. *If $f \in W_\phi^{r,p}(S) \subset L^p(S)$, $1 \leq p \leq \infty$, then*

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p \leq C\left\|\omega\varphi_{ij}^2 D_{ij}^2 f\right\|_p \quad 1 \leq i \leq j \leq d. \tag{2.17}$$

Proof. We use the induction on the dimension number d to prove Lemma 2.5. The case $d = 1$ was proved by Lemma 3 of [6], that is,

$$\left\|\omega\varphi^2 D^2 M_{n,1} f\right\|_p \leq C\left\|\omega\varphi^2 D^2 f\right\|_p. \tag{2.18}$$

Next, suppose that Lemma 2.5 is valid for $d = r$ $(r \geq 1)$, and we prove it is also true for $d = r + 1$. Noticing formula (2.4), we have

$$\begin{aligned} &\omega(x)\varphi_{22}^2(x) D_{22}^2 M_{n,d}(f;x) \\ &\quad= x_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta} \sum_{k_1=0}^{n} P_{n,k_1}(x_1)(n+d) z_1^{\alpha_2} z_2^{\alpha_3} \cdots z_{d-1}^{\alpha_d} \\ &\qquad\times (1-|z|)^{\beta} \varphi_{11}^2(z) \int_0^1 P_{n+d-1,k_1}(u_1) D_{11}^2 M_{n-k_1,d-1}(f(u_1,(1-u_1)\cdot);z)\,du_1, \end{aligned} \tag{2.19}$$

where $z=(z_1,z_2,\ldots,z_{d-1})=(x_2/(1-x_1),x_3/(1-x_1),\ldots,x_d/(1-x_1))$. When $p=1$, from the inductive assumption of $p=1$, we have

$$\begin{aligned}
&\int_S\left|\omega(x)\varphi_{22}^2(x)D_{22}^2M_{n,d}(f;x)\right|ds\\
&\quad\le C\int_0^1 x_1^{\alpha_1}(1-x_1)^{|\bar\alpha|+\beta}\sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)\\
&\qquad\times\int_{z\in S_{d-1}}\left|\omega(z)\varphi_{11}^2(z)D_{11}^2f(u_1,(1-u_1)z)\right|dz\,dx_1\,du_1\\
&\quad\le C\frac{n+d}{n+1}\int_0^1\sum_{k_1=0}^n\left(\frac{k_1+1}{n+1}\right)^{\alpha_1}\left(\frac{n-k_1+1}{n+1}\right)^{|\bar\alpha|+\beta}P_{n+d-1,k_1}(u_1)\\
&\qquad\times\int_{z\in S_{d-1}}\left|\omega(z)\varphi_{11}^2(z)D_{11}^2f(u_1,(1-u_1)z)\right|dz\,du_1\\
&\quad\le C\int_0^1 u_1^{\alpha_1}(1-u_1)^{|\bar\alpha|+\beta}\left(\frac{1}{u_1}\right)^{\alpha_1}(1-u_1)^{-|\bar\alpha|-\beta}\int_{z\in S_{d-1}}\left|\left(\omega\varphi_{22}^2D_{22}^2f\right)(u_1,(1-u_1)z)\right|dz\,du_1\\
&\quad\le C\left\|\omega\varphi_{22}^2D_{22}^2f\right\|_1.
\end{aligned}\tag{2.20}$$

When $p=\infty$, we have

$$\begin{aligned}
&\omega(x)\varphi_{22}^2(x)D_{22}^2M_{n,d}(f;x)\\
&\quad=x_1^{\alpha_1}(1-x_1)^{|\bar\alpha|+\beta}\sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d)z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d}\\
&\qquad\times(1-|z|)^\beta\varphi_{11}^2(z)\int_0^1 P_{n+d-1,k_1}(u_1)D_{11}^2M_{n-k_1,d-1}(f(u_1,(1-u_1)\cdot);z)du_1,
\end{aligned}\tag{2.21}$$

where $z=(z_1,z_2,\ldots,z_{d-1})=(x_2/(1-x_1),x_3/(1-x_1),\ldots,x_d/(1-x_1))$. From the inductive assumption, the Cauchy-Swartz inequality, Holder inequality, and Lemma 2.4, we get

$$\begin{aligned}
&\left|\omega(x)\varphi_{22}^2(x)D_{22}^2M_{n,d}(f;x)\right|\\
&\quad\le Cx_1^{\alpha_1}(1-x_1)^{|\bar\alpha|+\beta}\sum_{k_1=0}^n P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)\\
&\qquad\times\max_{z\in S_{d-1}}\left|z_1^{\alpha_2}z_2^{\alpha_3}\cdots z_{d-1}^{\alpha_d}(1-|z|)^\beta\varphi_z^2D_z^2f(u_1,(1-u_1)z)\right|du_1
\end{aligned}$$

$$\begin{aligned}
&\le Cx_1^{\alpha_1}(1-x_1)^{|\bar{\alpha}|+\beta}\sum_{k_1=0}^{n}P_{n,k_1}(x_1)(n+d)\int_0^1 P_{n+d-1,k_1}(u_1)x_1^{-\alpha_1}(1-x_1)^{-|\bar{\alpha}|-\beta} \\
&\quad\times\left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_\infty du_1 \\
&\le C\left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_\infty.
\end{aligned} \tag{2.22}$$

By *Riesz* interpolation theorem, we get

$$\left\|\omega\varphi_{22}^2 D_{22}^2 M_{n,d} f\right\|_p \le C\left\|\omega\varphi_{22}^2 D_{22}^2 f\right\|_p. \tag{2.23}$$

Similarly, the other cases for $i = 1, 3, 4, \dots, d(=j)$ can be proved. For $i \ne j$, by the transformation T, we have

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d} f\right\|_p = \left\|\omega_T\varphi_{jj}^2 D_{jj}^2 M_{n,d} f_T\right\|_p \le C\left\|\omega_T\varphi_{jj}^2 D_{jj}^2 f_T\right\|_p \le C\left\|\omega\varphi_{ij}^2 D_{ij}^2 f\right\|_p \tag{2.24}$$

Lemma 2.5 is completed. □

Lemma 2.6 (see [9]). *Let $\{\sigma_n\}$, $\{\phi_n\}$ be nonnegative sequences ($\sigma_1 = 0$, $n \in N$). For $l > 0$, if the sequences $\{\sigma_n\}$, $\{\phi_n\}$ satisfy*

$$\sigma_n \le Q\left(\frac{k}{n}\right)^l \sigma_k + \phi_k \quad (Q \ge 1,\ 1 \le k \le n,\ n \in N), \tag{2.25}$$

one has

$$\sigma_n \le M n^{-s}\sum_{k=1}^{n} k^{s-1}\phi_k. \tag{2.26}$$

If $Q = 1$, then $l = s$. If $Q > 1$, then $0 < s < l$.

3. The Proof of Theorems

Now we prove (1.9) of Theorem 1.2. By using Lemma 2.1, for arbitrary $g \in W_{\phi}^{r,p}(S) \subset L^p(S)$, we have

$$\begin{aligned} \|\omega(M_{n,d}f - f)\|_p &\leq C\left(\|\omega M_{n,d}(f-g)\|_p + \|\omega M_{n,d}g - \omega g\|_p + \|\omega(f-g)\|_p\right) \\ &\leq C\left(\|\omega(f-g)\|_p + \frac{1}{n}\left(\sum_{1\leq i\leq j\leq d}\left\|\omega\varphi_{ij}^2 D_{ij}^2 g\right\|_p + \|\omega g\|_p\right)\right) \\ &\leq C\left(\|\omega(f-g)\|_p + \frac{1}{n}\sum_{1\leq i\leq j\leq d}\left\|\omega\varphi_{ij}^2 D_{ij}^2 g\right\|_p + \frac{1}{n}\|\omega f\|_p\right). \end{aligned} \tag{3.1}$$

Hence, from Lemma 2.2, we obtain

$$\begin{aligned} \|\omega(M_{n,d}f - f)\|_p &\leq C\left(K_{\varphi}^2\left(f, \frac{1}{n}\right)_{\omega} + \frac{1}{n}\|\omega f\|_p\right) \\ &\leq C\left(\omega_{\varphi}^2(f,t)_{\omega} + \frac{1}{n}\|\omega f\|_p\right). \end{aligned} \tag{3.2}$$

Next, we prove (1.10) of Theorem 1.2. Leting $\sigma_n = C(1/n)\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f)\|_p$ $(1 \leq i \leq j \leq d)$, $\phi_n = C\|\omega(M_{n,d}(f) - f)\|_p$, then $\sigma_1 = 0$. By Lemmas 2.4 and 2.5, we have

$$\begin{aligned} \sigma_n &\leq C\frac{1}{n}\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f - M_{k,d}f)\right\|_p + C\frac{1}{n}\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}M_{k,d}f\right\|_p \\ &\leq C\|\omega(f - M_{k,d}f)\|_p + C\frac{1}{n}\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{k,d}f\right\|_p \\ &= C\frac{k}{n}\sigma_k + \phi_k \quad (C > 1). \end{aligned} \tag{3.3}$$

Using Lemma 2.6, we get $\sigma_n \leq C(1/n)\sum_{k=1}^{n}(n/k)^{\delta}\phi_k$ $(0 < \delta < 1)$. That is,

$$\left\|\omega\varphi_{ij}^2 D_{ij}^2 M_{n,d}(f)\right\|_p \leq C\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{\delta}\|\omega(M_{k,d}f - f)\|_p. \tag{3.4}$$

When $n \geq 2$, there exists $(m \in N)$ such that $n/2 \leq m \leq n$ and satisfies the equation

$$\|\omega(M_{m,d}f - f)\|_p = \min_{n/2\leq k\leq n}\|\omega(M_{k,d}f - f)\|_p. \tag{3.5}$$

Thus,

$$\|\omega(M_{m,d}f - f)\|_p \leq \frac{2}{n}\sum_{n/2\leq k\leq n}\|\omega(M_{k,d}f - f)\|_p. \tag{3.6}$$

Using Lemma 2.2, we have

$$\begin{aligned}\omega_{\varphi}^{2}\left(f,\frac{1}{\sqrt{n}}\right)_{\omega} &\leq CK_{\varphi}^{2}\left(f,\frac{1}{n}\right)\\ &\leq C\left(\left\|\omega\left(M_{m,d}f-f\right)\right\|_{p}+\frac{1}{n}\sum_{1\leq i\leq j\leq d}\left\|\omega\varphi_{ij}^{2}D_{ij}^{2}M_{m,d}f\right\|_{p}\right)\\ &\leq C\frac{1}{n}\sum_{k=1}^{n}\left(\frac{n}{k}\right)^{\delta}\left\|\omega\left(M_{k,d}f-f\right)\right\|_{p}.\end{aligned} \tag{3.7}$$

Theorem 1.2 is completed.

Acknowledgments

This paper was supported by the Natural Science Foundation of China (nos. 11001227, 60972155), Natural Science Foundation Project of CQ CSTC (no. CSTC, 2009BB2306, CSTC2009BB2305), and the Fundamental Research Funds for the Central Universities (no. XDJK2010B005, XDJK2010C023).

References

[1] Z. Ditzian and K. Ivanov, "Bernstein-type operators and their derivatives," *Journal of Approximation Theory*, vol. 56, no. 1, pp. 72–90, 1989.
[2] H. Berens and Y. Xu, "K-moduli, moduli of smoothness, and Bernstein polynomials on a simplex," *Indagationes Mathematicae*, vol. 2, no. 4, pp. 411–421, 1991.
[3] M.-M. Derriennic, "On multivariate approximation by Bernstein-type polynomials," *Journal of Approximation Theory*, vol. 45, no. 2, pp. 155–166, 1985.
[4] H. Berens, H. J. Schmid, and Y. Xu, "Bernstein-Durrmeyer polynomials on a simplex," *Journal of Approximation Theory*, vol. 68, no. 3, pp. 247–261, 1992.
[5] D. X. Zhou, "Inverse theorems for multidimensional Bernstein-Durrmeyer operators in L_p," *Journal of Approximation Theory*, vol. 70, no. 1, pp. 68–93, 1992.
[6] Z. Q. Zhang, "On weighted approximation by Bernstein-Durrmeyer operators," *Approximation Theory and its Applications*, vol. 7, no. 2, pp. 51–64, 1991.
[7] P. Xuan, J. Wang, G. You, and R. Song, "The convergence rate of multi-Bernstein-Durrmeyer operators with Jacobi weights," *Approximation Theory and Its Applications*, vol. 18, no. 2, pp. 90–101, 2002.
[8] F. L. Cao and X. D. Zhang, "Degree of convergence with Jacobi weight for d-dimensional Bernstein operators," *Mathematica Numerica Sinica. Jisuan Shuxue*, vol. 23, no. 4, pp. 407–416, 2001.
[9] E. van Wickeren, "Weak-type inequalities for Kantorovitch polynomials and related operators," *Indagationes Mathematicae*, vol. 49, no. 1, pp. 111–120, 1987.