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Research Article

The Critical Strips of the Sums $1 + 2^z + \cdots + n^z$

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We give a partition of the critical strip, associated with each partial sum $1 + 2^z + \cdots + n^z$ of the Riemann zeta function for Re $z < -1$, formed by infinitely many rectangles for which a formula allows us to count the number of its zeros inside each of them with an error, at most, of two zeros. A generalization of this formula is also given to a large class of almost-periodic functions with bounded spectrum.

1. Introduction

Some industrial processes can be modeled [1] by functional equations of the form $f(x) + f(2x) = 0$ or $f(x) + f(2x) + f(3x) = 0$, $x > 0$. The generalization of these functional equations to the complex plane is formally given by

$$f(z) + f(2z) + \cdots + f(nz) = 0, \quad n \geq 2, \tag{1.1}$$

which admits analytic solutions of the form z^α on the open set $\Omega = \mathbb{C} \setminus (-\infty, 0]$ if and only if α is a zero of

$$G_n(z) \equiv 1 + 2^z + \cdots + n^z. \tag{1.2}$$

For each integer $n \geq 2$, each function $G_n(z)$ represents the nth partial sum of the Riemann zeta function $\zeta(z)$ on the half-plane Re $z < -1$, and it belongs to the class of the entire almost-periodic functions of exponential type. In [2], we can see a complete introduction devoted to the study of such class of functions. There, we can also find a theorem of Bohr [2, p. 270] that identifies the functions of the above class having their zeros in a strip (the critical strip) parallel to the real axis with those functions for which the upper and lower bounds of their spectra enter into the spectrum. That is, for instance, the case of the functions defined from (1.2) by means of a rotation of angle $\pi/2$

$$H_n(z) \equiv G_n(iz) = 1 + 2^{iz} + \cdots + n^{iz}, \tag{1.3}$$

whose spectra, for each integer $n \geq 2$, are the finite sets

$$\{i \ln k : k = 1, \ldots, n\}. \tag{1.4}$$

Furthermore, the functions $H_n(z)$ have the property consisting on the existence of some value of $x = \text{Re } z$, say x_0, such that either $\text{Re } H_n(x_0, y) \neq 0$ or $\text{Im } H_n(x_0, y) \neq 0$ for all $y \in \mathbb{R}$. Indeed, $x_0 = 0$ satisfies such property. Therefore, the $H_n(z)$'s belong to a very special class of almost-periodic functions whose study will be our main objective to determine a nonasymptotic formula that allows us to count the amount of zeros that they have inside the rectangles of a certain partition of their critical strips.

One of the most important formulae [2, p. 277] on the number of roots of an almost-periodic function, with closed and bounded spectrum, say $f(z)$, inside a rectangle in the strip where the zeros of $f(z)$ are located, is given by

$$2\pi \lim_{x_2 - x_1 \to \infty} \frac{N(x_1, x_2, y_1, y_2)}{x_2 - x_1} = \varphi'(y_2) - \varphi'(y_1), \tag{1.5}$$

where $N(x_1, x_2, y_1, y_2)$ denotes the number of zeros of $f(z)$ in the rectangle

$$x_1 < \text{Re } z < x_2, \qquad y_1 < \text{Im } z < y_2, \tag{1.6}$$

and $\varphi(y)$ is the mean function associated with $\ln|f(x+iy)|$ defined as

$$\varphi(y) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+\alpha}^{T+\alpha} \ln\left|f(x+iy)\right| dx. \tag{1.7}$$

The formula (1.5) is of asymptotic type, and it is based on the assumption of the existence of derivative of $\varphi(y)$. However, if the spectrum of $f(z)$ is contained in the boundary of a bounded convex polygon of the complex plane and all the vertices of the polygon enter into the spectrum, there exists a formula [2, p. 298] much more explicit than (1.5). For instance, if the polygon is reduced to a segment of the imaginary axis, the formula is, for sufficiently large values of $|y_1|$ and $|y_2|$,

$$N(x_1, x_2, y_1, y_2) = \frac{d}{2\pi}(x_2 - x_1) + O(1), \tag{1.8}$$

where d is the length of the segment.

Formula (1.8) could be used, for instance, to estimate the number of zeros, of our functions $H_n(z)$, inside the rectangle defined by the intersection of its critical strip with the strip $x_1 \le \mathrm{Re}\, z \le x_2$. Indeed, since the spectrum of $H_n(z)$ is contained in the line segment $[0, i \ln n]$ of the imaginary axis, a simple application of (1.8) leads to the formula

$$N(x_1, x_2, y_1, y_2) = \frac{\ln n}{2\pi}(x_2 - x_1) + O(1). \tag{1.9}$$

Nevertheless, it is well known that the term $O(1)$ is an "obscure" function which we only know to represent a bounded quantity. In general, the term $O(1)$ that appears in formula (1.8) depends on the function and the rectangle where we are counting the number of its zeros. Our aim is to give much more precise information about the expression $O(1)$ when the function belongs to that special class of almost-periodic functions which contains, in particular, to our functions $H_n(z)$. In fact, on this subject, we find in [3] the following result.

"There exist infinitely many rectangles $x_1 < \mathrm{Re}\, z < x_2$, $y_1 < \mathrm{Im}\, z < y_2$ in the critical strip of the function $H_n(z)$ for which the number of zeros of $H_n(z)$ is given by the formula

$$N(x_1, x_2, y_1, y_2) = \frac{\ln n}{2\pi}(x_2 - x_1) + \Omega_n, \tag{1.10}$$

where Ω_n is a real number with $|\Omega_n| < 1$."

Now, by following the ideas exhibited in [3], our aim is to demonstrate that for the functions of that special class of almost-periodic functions, there exists a formula similar to that of (1.10) to determine the number of its zeros inside infinitely many rectangles in their critical strips with an error, at most, of two zeros.

In particular, our main result will also prove that the bound $n-1$ which appears in the formula that determines the number of the zeros of an exponential polynomial of degree n inside certain rectangles of its critical strip can be substituted by a universal bound, namely, 2. In fact, to illustrate the scope of our result, we will start recalling an old theorem of Polya [4]:

"if $z = x + iy$

$$\mu_1 < \mu_2 < \cdots < \mu_l, \tag{1.11}$$

and $P_\nu(z)$ is for $\nu = 1, 2, \ldots, l$ a polynomial of degree $\le m_\nu - 1$ with

$$\begin{gathered} m_1 + m_2 + \cdots + m_l = n, \\ P_1(z) P_l(z) \ne 0, \end{gathered} \tag{1.12}$$

then the number $N(g_l, a, a+b)$ of the zeros (according to multiplicity) of the function

$$g_l(z) = \sum_{\nu=1}^{l} P_\nu(z) e^{i\mu_\nu z} \tag{1.13}$$

in the infinite vertical strip

$$a \le x \le a + b \tag{1.14}$$

satisfies the inequality

$$\left|N(g_l, a, a+b) - \frac{b}{2\pi}(\mu_l - \mu_1)\right| \leq n - 1.'' \tag{1.15}$$

Then, under the same hypotheses of the above theorem, our result could be stated as follows.

Under the hypotheses of theorem of Polya with frequencies $\mu_1 < \mu_2 < \cdots < \mu_l$ linearly independent and $m_1 = m_2 = \cdots = m_l = 1$, *there exist infinitely many values for* a, b *such that, independently of* l, *the inequality*

$$\left|N(g_l, a, a+b) - \frac{b}{2\pi}(\mu_l - \mu_1)\right| \leq 2 \tag{1.16}$$

holds.

This result will be an immediate consequence of Lemma 2.6 and Theorem 3.1 of the present paper.

2. Preliminaries

To prove our main theorem, we will use some elementary concepts and results such as the following.

Definition 2.1. A set $\{a_1, a_2, \ldots, a_k\}$ of real numbers is said to be linearly independent if and only if any linear combination

$$\sum_{j=1}^{k} n_j a_j = 0, \tag{2.1}$$

with integers n_j, implies that $n_j = 0$ for all $j = 1, \ldots, k$.

For example, the set

$$\{\ln p_1, \ln p_2, \ldots, \ln p_k\}, \tag{2.2}$$

where $p_1, p_2, \ldots, p_k$ are different prime numbers, is linearly independent. Nevertheless, for a given set of real numbers $\{x_1, x_2, \ldots x_l\}$, we can always suppose the existence of a basis $\{a_1, a_2, \ldots, a_k\}$. That is, on the one hand, $\{a_1, a_2, \ldots, a_k\}$ is linearly independent and, on the other hand, for each $m = 1, \ldots, l$, we can write

$$x_m = \sum_{j=1}^{k} n_{mj} a_j, \tag{2.3}$$

where the n_{mj} are integers.

An important result on linearly independent sets of real numbers is the famous theorem of Kronecker [5, p. 382] which will be used in the following form.

Theorem 2.2 (Kronecker). *Let $\{a_1, a_2, \ldots, a_k\}$ be a linearly independent set of nonnull real numbers. For arbitrary numbers $b_1, b_2, \ldots, b_k$ and T, $\epsilon > 0$, there exists a real number $t > T$ and integers $n_1, n_2, \ldots, n_k$ such that*

$$\left|ta_j - n_j - b_j\right| < \epsilon, \quad \forall j = 1, \ldots, k. \tag{2.4}$$

Given an entire almost-periodic function f with closed and bounded spectrum, a rectangle in its critical strip will be defined as the intersection of the rectangle $x < \operatorname{Re} z < x+T$, $y_1 < \operatorname{Im} z < y_2$, for some $T > 0$ and sufficiently large values of $|y_1|$ and $|y_2|$, with the strip where the zeros of f are situated, of course, by assuming that the critical strip is parallel to the real axis. Then, the number of zeros of $f(z)$ in a rectangle in its critical strip will merely be denoted by $N(f(z); x, x+T)$. Similarly, $N(f(z); y, y+T)$ will denote the number of zeros of $f(z)$ in a rectangle in its critical strip, provided that the critical strip of $f(z)$ to be parallel to the imaginary axis. Nevertheless, noticing the change z by $-iz$ transforms the zeros of a strip parallel to the real axis onto the zeros of a strip parallel to the imaginary axis and conversely, from now on, we will do our study on those functions by assuming that their critical strips are parallel to the imaginary axis.

Because our aim is to study the number of zeros of almost-periodic functions, and noticing these functions, from Bochner's theorem [2, p. 266], are characterized as uniform limits of exponential polynomials, we will start by demonstrating a formula of the type (1.10) assuming that they adopt the normalized form

$$P(z) = 1 + \sum_{j=1}^{n} w_j e^{\mu_j z}, \tag{2.5}$$

where the coefficients w_j are nonnull complex numbers and the frequencies μ_j are positive real numbers so that

$$\mu_1 < \mu_2 < \cdots < \mu_n. \tag{2.6}$$

Then, a normalized exponential polynomial of the form (2.5), not affecting the zeros, will be considered as a prototype of an almost-periodic function whose definition [6, p. 101] we recall.

Definition 2.3. An entire function f is said to be almost periodic if and only if for every $\epsilon > 0$ there exists a length $l = l(\epsilon)$ such that every interval $b < y < b + l$ of length l on the imaginary axis contains at least one translation number τ associated with ϵ satisfying the inequality

$$\left|f(z + i\tau) - f(z)\right| \leq \epsilon, \quad \forall z \in \mathbb{C}. \tag{2.7}$$

From (2.7), we derive the notion of interval of almost periodicity.

Definition 2.4. Let f be an almost-periodic entire function on $\mathbb{C}$ and $\epsilon > 0$. Then, any interval of length l, $l = l(\epsilon, f)$, will be called an ϵ-interval of almost periodicity of f.

In each interval of almost periodicity of an exponential polynomial $P(z)$, the solutions of the equations Re $P(z) = 0$, Im $P(z) = 0$ have a very special form, as we will prove in the following result.

Lemma 2.5. *Let*

$$P(z) = 1 + \sum_{j=1}^{n} w_j e^{\mu_j z}, \quad w_j \in \mathbb{C} \setminus \{0\} \tag{2.8}$$

be an exponential polynomial with increasing positive frequencies $\mu_1 < \cdots < \mu_n$*. Then, there exist two real numbers* x_1*,* x_2 *such that all the zeros of* $P(z)$ *are in the strip*

$$S_{P(z)} = \{z : x_1 < \mathrm{Re}\ z < x_2\}. \tag{2.9}$$

Furthermore, for $n = 1, 2$*, there exists a value for* y*, say* y_0*, such that either*

$$\{z : \mathrm{Re}\ P(z) = 0\} \cap \{z : \mathrm{Im}\ z = y_0\} = \emptyset \tag{2.10}$$

or

$$\{z : \mathrm{Im}\ P(z) = 0\} \cap \{z : \mathrm{Im}\ z = y_0\} = \emptyset. \tag{2.11}$$

Proof. Since

$$\lim_{x \to -\infty} P(x + iy) = 1, \qquad \lim_{x \to +\infty} \frac{P(x + iy)}{w_n e^{\mu_n (x+iy)}} = 1, \tag{2.12}$$

for any value of y, there exist $x_1 < 0 < x_2$ such that

$$|P(z) - 1| < 1, \quad \forall z \text{ with } \mathrm{Re}\ z \le x_1, \qquad \left| \frac{P(z)}{w_n e^{\mu_n z}} - 1 \right| < 1, \quad \forall z \text{ with } \mathrm{Re}\ z \ge x_2. \tag{2.13}$$

Hence, $P(z)$ has no zero neither in the half-plane Re $z \le x_1$ nor in the half-plane Re $z \ge x_2$. Consequently, all the zeros of $P(z)$ are situated in the strip

$$x_1 < \mathrm{Re}\ z < x_2. \tag{2.14}$$

To prove the second part of the lemma, we will only consider the case Im $P(z) = 0$ (the case Re $P(z) = 0$ is completely similar). In this case, for any positive integer n, the equation Im $P(z) = 0$ can be explicitly written as

$$\sum_{j=1}^{n} e^{\mu_j x}\left(\alpha_j \sin(\mu_j y) + \beta_j \cos(\mu_j y)\right) = 0, \tag{2.15}$$

where $\alpha_j = \operatorname{Re} w_j$ and $\beta_j = \operatorname{Im} w_j$. By defining

$$A_j(y) \equiv \alpha_j \sin(\mu_j y) + \beta_j \cos(\mu_j y), \quad \text{for each } j = 1, \ldots, n, \tag{2.16}$$

equation (2.15) becomes

$$\sum_{j=1}^{n} e^{\mu_j x} A_j(y) = 0. \tag{2.17}$$

On the other hand, it is plain that the set of the zeros of each function $A_j(y)$, denoted by B_j, is given by

$$B_j = \left\{ \frac{1}{\mu_j}\left(\pi k_j - \arctan\frac{\beta_j}{\alpha_j}\right) : k_j \in \mathbb{Z} \right\}, \tag{2.18}$$

where $\arctan(\beta_j/\alpha_j)$ is taken as $\pi/2$ when $\alpha_j = 0$. Since $e^{\mu_1 x} > 0$ for all real x, the case $n = 1$ easily follows by taking $y = y_0$, for arbitrary $y_0 \notin B_1$.

Now, assume that $n = 2$. If the sets B_1 and B_2 are distinct, suppose that there exists some $y_0 \in B_1$ such that $y_0 \notin B_2$. Then, the right-line of equation $y = y_0$ does not meet Im $P(z) = 0$. Indeed, if for some real x the point (x, y_0) satisfies the equation Im $P(z) = 0$, then, from (2.17) and taking into account that $y_0 \in B_1$, it necessarily would have $A_2(y_0) = 0$ and, therefore, $y_0 \in B_2$, which is a contradiction. Consequently, the lemma follows for the value y_0. Finally, we analyse the case $B_1 = B_2$. This case means that for each integer k_1 there exists another integer k_2 such that

$$\frac{1}{\mu_1}\left(\pi k_1 - \arctan\frac{\beta_1}{\alpha_1}\right) = \frac{1}{\mu_2}\left(\pi k_2 - \arctan\frac{\beta_2}{\alpha_2}\right), \tag{2.19}$$

and reciprocally. By defining the numbers

$$\begin{gathered} \mu \equiv \frac{\mu_1}{\mu_2}, \\ b \equiv \frac{\arctan(\beta_1/\alpha_1) - \mu\ \arctan(\beta_2/\alpha_2)}{\pi}, \end{gathered} \tag{2.20}$$

equality (2.19) can be written as

$$k_1 - \mu k_2 = b, \tag{2.21}$$

which represents an equation with infinitely many solutions for integers k_1 and k_2. Let k_1, k_2 and k_1', k_2' be integers verifying (2.21). Then, by subtracting in (2.21), one has

$$\left(k_1 - k_1'\right) - \mu\left(k_2 - k_2'\right) = 0, \tag{2.22}$$

which implies that μ must be necessarily a rational number (observe that it means, in particular, that the frequencies μ_1, μ_2 are linearly dependent) and, because of $0 < \mu_1 < \mu_2$, the number μ is a positive rational less than 1. On the other hand, since

$$\left|\arctan\frac{\beta_1}{\alpha_1}\right|, \left|\arctan\frac{\beta_2}{\alpha_2}\right| \leq \frac{\pi}{2}, \tag{2.23}$$

b is a rational number verifying

$$|b| < 1. \tag{2.24}$$

Now, suppose the value $k_2 = 0$ is given. Then, there exists an integer k_1 satisfying (2.19) and, according to (2.21), it follows that $k_1 = b$. Hence, b is an integer and then, noticing (2.24), $b = 0$. Consequently, $k_1 = 0$. Since $B_1 = B_2$, let y_1 be the point of $B_1 = B_2$ corresponding to the values $k_1 = k_2 = 0$. Then, from (2.18), one has

$$y_1 = -\frac{1}{\mu_1}\arctan\frac{\beta_1}{\alpha_1} = -\frac{1}{\mu_2}\arctan\frac{\beta_2}{\alpha_2}. \tag{2.25}$$

Now, assume that, for any real number y, there exists a value of x such that

$$\{z = x + iy : \operatorname{Im} P(x, y) = 0\} \cap \{z : \operatorname{Im} z = y\} \neq \emptyset. \tag{2.26}$$

Thus, in particular, given y_1, there exists a_1 such that $\operatorname{Im} P(a_1, y_1) = 0$. On the other hand, as the set $B_1 = B_2$ is discrete, there exists an open interval (v, y_1) such that one has $y \notin B_1 = B_2$ for any $y \in (v, y_1)$ and, therefore, $A_j(y) \neq 0$ for $j = 1, 2$. Then, by assuming (2.26), if we divide (2.17) by $e^{\mu_1 x} A_1(y)$ one has the following property.

For each $y \in (v, y_1)$, there exists x such that the relation

$$1 + \frac{A_2(y)}{A_1(y)} e^{(\mu_2 - \mu_1)x} = 0, \tag{2.27}$$

holds.

Now, by taking the limit in (2.27), when $y \to y_1$, it follows that the point (a_1, y_1) satisfies

$$1 + \lim_{y \to y_1} \frac{A_2(y)}{A_1(y)} e^{(\mu_2 - \mu_1)a_1} = 0. \tag{2.28}$$

However, since

$$\lim_{y \to y_1} \frac{A_2(y)}{A_1(y)} = \frac{\mu_2 |w_2|}{\mu_1 |w_1|} \tag{2.29}$$

is positive, by substituting in (2.28), we are led to a contradiction. Consequently, the case $B_1 = B_2$ follows, and the proof of the lemma is now completed. $\square$

When the frequencies are linearly independent, the preceding lemma is valid for arbitrary n.

Lemma 2.6. *Let n be an arbitrary positive integer and*

$$P(z) = 1 + \sum_{j=1}^{n} w_j e^{\mu_j z}, \quad w_j \in \mathbb{C} \setminus \{0\} \tag{2.30}$$

an exponential polynomial with increasing positive frequencies $\mu_1 < \cdots < \mu_n$ forming a linearly independent set. Then, there exists a value for y, say y_0, such that either

$$\{z : \operatorname{Re} P(z) = 0\} \cap \{z : \operatorname{Im} z = y_0\} = \emptyset \tag{2.31}$$

or

$$\{z : \operatorname{Im} P(z) = 0\} \cap \{z : \operatorname{Im} z = y_0\} = \emptyset. \tag{2.32}$$

Proof. For the sake of brevity, we will prove the lemma in the case $\operatorname{Im} P(z) = 0$ (the case $\operatorname{Re} P(z) = 0$ is completely similar). Consider the coefficients w_j of the exponential polynomial $P(z)$, since all them are nonnull, the set $J = \{1, 2, \ldots, n\}$ can be partitioned in the following four disjoint sets (some of them could be eventually empty)

$$\begin{aligned}
J_1 &= \{j \in J : \alpha_j \geq 0, \beta_j > 0\}, \\
J_2 &= \{j \in J : \alpha_j > 0, \beta_j \leq 0\}, \\
J_3 &= \{j \in J : \alpha_j \leq 0, \beta_j < 0\}, \\
J_4 &= \{j \in J : \alpha_j < 0, \beta_j \geq 0\}.
\end{aligned} \tag{2.33}$$

Now, define the numbers

$$a_j \equiv \frac{\mu_j}{2\pi}, \quad \forall j \in J,$$
$$b_j \equiv \begin{cases} \dfrac{1}{8}, & \text{if } j \in J_1, \\ \dfrac{3}{8}, & \text{if } j \in J_2, \\ \dfrac{5}{8}, & \text{if } j \in J_3, \\ \dfrac{7}{8}, & \text{if } j \in J_4. \end{cases} \tag{2.34}$$

Let us pick an arbitrary real number T and a positive ϵ such that $\epsilon < 1/4\pi$. Then, by applying Theorem 2.2, there exists $t > T$ and integers n_j such that

$$\left| t a_j - n_j - b_j \right| < \epsilon, \quad \forall j \in J. \tag{2.35}$$

Hence, by substituting the values of a_j and b_j in the preceding inequality and multiplying by 2π, one has

$$t\mu_j = \begin{cases} \dfrac{\pi}{4} + \eta_j + 2\pi n_j, & \text{if } j \in J_1, \\ \dfrac{3\pi}{4} + \eta_j + 2\pi n_j, & \text{if } j \in J_2, \\ \dfrac{5\pi}{4} + \eta_j + 2\pi n_j, & \text{if } j \in J_3, \\ \dfrac{7\pi}{4} + \eta_j + 2\pi n_j, & \text{if } j \in J_4. \end{cases} \tag{2.36}$$

where the η_j's are real numbers such that $|\eta_j| < 1/2$. Then, according to the definition of the J_k's, it is clear that

$$\alpha_j \sin(t\mu_j) + \beta_j \cos(t\mu_j) > 0, \quad \forall j \in J. \tag{2.37}$$

Consequently,

$$\operatorname{Im} P(x,t) = \sum_{j=1}^{n} e^{\mu_j x} \left(\alpha_j \sin(\mu_j t) + \beta_j \cos(\mu_j t) \right) > 0, \quad \forall x \in \mathbb{R}, \tag{2.38}$$

and then the lemma follows by taking $y_0 = t$. □

Corollary 2.7. *Let*

$$P(z) = 1 + \sum_{j=1}^{n} w_j e^{\mu_j z}, \quad w_j \in \mathbb{C} \setminus \{0\} \tag{2.39}$$

be an exponential polynomial with increasing positive frequencies $\mu_1 < \cdots < \mu_n$ *forming a linearly independent set. Then, there exist infinitely many rectangles* $\{R_k\}$ *in the critical strip of* $P(z)$ *such that either* Re $P(z)$ *or* Im $P(z)$ *is always positive at every point of the sides of each* R_k *that are parallel to the real axis.*

Proof. By applying Lemma 2.5, determine the right lines of equations $x = x_1$, $x = x_2$ that define the strip $x_1 < \text{Re } z < x_2$ where all the zeros of $P(z)$ are comprised. Let m be an arbitrary integer, by taking $T_1 = m$ and by applying Theorem 2.2, just as we have done in Lemma 2.6, we determine a value $t_m > T_1$ such that Im $P(x, t_m) > 0$ for all $x \in \mathbb{R}$. Now, again from Theorem 2.2, for $T_2 = t_m$ there exists a value $t_{m+1} > T_2$ such that Im $P(x, t_{m+1}) > 0$. Then, the four right lines of equations $x = x_1$, $y = t_m$, $x = x_2$, $y = t_{m+1}$ define a rectangle, say R_m, such that Im $P(z)$ is positive when z lies on any of the sides of R_m that are parallel to the real axis. By reiterating this process, we will obtain the infinitely many rectangles $\{R_k : k \geq m\}$ desired. A completely analogous result we would have obtained if we had considered Re $P(z)$ instead of Im $P(z)$. Our corollary is then proved. □

3. A Class of Almost-Periodic Functions with Bounded Spectrum Containing the Partial Sums of the Riemann Zeta Function

In this section, we are going to generalize the preceding results to the class of almost-periodic functions $f(z)$ with bounded spectrum having the property of the existence of some value of $y = \text{Im } z$, say y_0, such that either Re $f(x, y_0) \neq 0$ or Im $f(x, y_0) \neq 0$ for all $x \in \mathbb{R}$. By denoting this class by $\mathcal{A}_{\mathcal{S}}$, it follows that $\mathcal{A}_{\mathcal{S}}$ is nonvoid. Indeed, from lemmas 2.5 and 2.6, $\mathcal{A}_{\mathcal{S}}$ contains all exponential polynomials of degree $n = 1, 2$ with increasing positive frequencies and all exponential polynomials of arbitrary degree with linearly independent positive frequencies. Then, in particular, $G_2(z) \equiv 1 + 2^z$ belongs to $\mathcal{A}_{\mathcal{S}}$, and, since the frequencies $\log 2 < \log 3$ are linearly independent, the function $G_3(z) \equiv 1+2^z+3^z$ is also in the class $\mathcal{A}_{\mathcal{S}}$. However, although for any $n \geq 4$, the frequencies $\log 2 < \log 3 < \cdots < \log n$ are always linearly dependent, all the approximants

$$G_n(z) \equiv 1 + 2^z + \cdots + n^z \tag{3.1}$$

of the Riemann zeta function $\zeta(z)$, in the half-plane Re $z < -1$, belong to the class $\mathcal{A}_{\mathcal{S}}$. Likewise, all the derivatives of $G_n(z)$ are in the class $\mathcal{A}_{\mathcal{S}}$. To see that it is enough to check that for the value $y_0 = 0$, Re $G_n^{(k)}(x, 0) > 0$, for all $x \in \mathbb{R}$ and every $k = 0, 1, 2, \ldots$. Then, we are going to obtain our formula to count the zeros of the functions $G_n(z)$ as a consequence of a general result on the functions of the class $\mathcal{A}_{\mathcal{S}}$.

Theorem 3.1. *Let* $f(z)$ *be a function of the class* $\mathcal{A}_{\mathcal{S}}$*. Then, there exist infinitely many rectangles* $\{R_k\}$ *in the critical strip of* $f(z)$ *such that the number of zeros inside each rectangle* R_k, $N(f(z); R_k)$, *satisfies*

$$\left| N(f(z); R_k) - \frac{\mu h_k}{2\pi} \right| < 2, \tag{3.2}$$

where μ *denotes the difference between the upper and the lower bounds of the spectrum of* $f(z)$ *and* h_k *is the height of* R_k.

Proof. As $f(z)$ is an entire function of exponential type with bounded spectrum, from the Bohr theorem [2, p. 270], its zeros are all in a critical strip which we can suppose parallel to the imaginary axis (otherwise, we would consider the function $g(z) \equiv f(iz)$). Hence, without loss of generality, we can assume the existence of two real numbers x_1, x_2 such that all the zeros of $f(z)$ are located in the strip

$$\{z : x_1 < \operatorname{Re} z < x_2\}. \tag{3.3}$$

On the other hand, since $f(z)$ is an almost-periodic function, let $\sum_{j=1}^{\infty} A_j e^{\mu_j z}$ be the Dirichlet series [7, p. 77] associated with $f(z)$, denoted by

$$f(z) \sim \sum_{j=1}^{\infty} A_j e^{\mu_j z}. \tag{3.4}$$

Then, the set of the Fourier exponents of the above series, also called Dirichlet exponents,

$$\{\mu_j : j \in \mathbb{N}\}, \tag{3.5}$$

forms the spectrum of $f(z)$, say $S_{f(z)}$. Now, because the lower and the upper bounds of the spectrum of $f(z)$ enter in the spectrum, let us define

$$\begin{gathered} \mu_1 \equiv \min\{\mu_j : j \in \mathbb{N}\}, \\ \mu_\infty \equiv \max\{\mu_j : j \in \mathbb{N}\}, \\ \mu \equiv \mu_\infty - \mu_1, \end{gathered} \tag{3.6}$$

with $\mu_1, \mu_\infty \in S_f$. Then, by considering the two almost-periodic functions

$$\frac{f(z)}{A_1 e^{\mu_1 z}} - 1, \qquad \frac{f(z)}{A_\infty e^{\mu_\infty z}} - 1, \tag{3.7}$$

from (3.4), it follows that these functions have associated Dirichlet series

$$\sum_{j=1}^{\infty} \frac{A_{j+1}}{A_1} e^{(\mu_{j+1}-\mu_1)z}, \qquad \sum_{j=1}^{\infty} \frac{A_{j+1}}{A_\infty} e^{(\mu_{j+1}-\mu_\infty)z}, \tag{3.8}$$

respectively. Now, as the Dirichlet exponents of the above series, $\mu_{j+1} - \mu_1$ and $\mu_{j+1} - \mu_\infty$ are strictly positive and negative, respectively, because of [7, Theorem 3.21], the functions

$$\frac{f(z)}{A_1 e^{\mu_1 z}} - 1, \qquad \frac{f(z)}{A_\infty e^{\mu_\infty z}} - 1 \tag{3.9}$$

tend to zero as $\mathrm{Re}\, z \to -\infty$ and $\mathrm{Re}\, z \to +\infty$ uniformly with respect to y, respectively. Therefore, there exist two reals a, b with $a \leq x_1 < x_2 \leq b$ such that

$$\left| \frac{f(z)}{A_1 e^{\mu_1 z}} - 1 \right| < 1, \quad \text{for } \mathrm{Re}\, z \leq a,$$
$$\left| \frac{f(z)}{A_\infty e^{\mu_\infty z}} - 1 \right| < 1, \quad \text{for } \mathrm{Re}\, z \geq b. \tag{3.10}$$

On the other hand, as f is of class $\mathcal{A}_S$, let y_0 be a value of $y = \mathrm{Im}\, z$ such that, for instance,

$$\mathrm{Im}\, f(x + iy_0) > 0, \quad \forall x \in \mathbb{R}. \tag{3.11}$$

Then, from continuity, given the interval $K \equiv [a, b]$ there exists $\delta > 0$ such that

$$\mathrm{Im}\, f(x + iy_0) \geq \delta, \quad \forall x \in K. \tag{3.12}$$

Now, from Definition 2.4, by taking $\epsilon_1 = \delta/2$, let $(0, l_1)$ be the ϵ_1-interval of almost periodicity of $f(z)$, with $l_1 > 0$. Then, Definition 2.3 involves the existence of a translation number $\tau_1 \in (0, l_1)$ such that

$$\left| f(z + i\tau_1) - f(z) \right| \leq \epsilon_1, \quad \forall z = x + iy_0, \text{ with } x \in K. \tag{3.13}$$

According to (3.12), inequality (3.13) implies

$$\mathrm{Im}\, f(x + i(y_0 + \tau_1)) \geq \epsilon_1, \quad \forall x \in K. \tag{3.14}$$

Then, the four right lines of equations $x = a$, $y = y_0$, $x = b$, $y = y_0 + \tau_1$ define a rectangle S_1 such that, from (3.12) and (3.14), $\mathrm{Im}\, f(z)$ is positive when z lies on any of the sides of S_1 parallel to the real axis. Furthermore, from (3.10), one has $f(z) \neq 0$ on the sides of S_1 parallel to the imaginary axis.

As above, by now taking $\epsilon_2 = \delta/2^2$, in the ϵ_2-interval of almost periodicity of f, $(0, l_2)$, with $l_2 > 0$, there exists a translation number $\tau_2 \in (0, l_2)$ such that

$$\left| f(z + i\tau_2) - f(z) \right| \leq \epsilon_2, \quad \forall z = x + i(y_0 + \tau_1), \text{ with } x \in K. \tag{3.15}$$

Then, inequalities (3.14) and (3.15) imply

$$\mathrm{Im}\, f(x + i(y_0 + \tau_1 + \tau_2)) \geq \epsilon_2, \quad \forall x \in K. \tag{3.16}$$

Therefore, the four right lines of equations $x = x_1$, $y = y_0 + \tau_1$, $x = x_2$, $y = y_0 + \tau_1 + \tau_2$ define a rectangle S_2 such that $\mathrm{Im}\, f(z)$ is positive on the sides of S_2 that are parallel to the real axis, and $f(z) \neq 0$ on the sides of S_2 that are parallel to the imaginary one. Then, continuing in this way, we obtain a family of rectangles

$$\{S_k : k \in \mathbb{N}\}, \tag{3.17}$$

with the property consisting of $\operatorname{Im} f(z) > 0$ on the sides that are parallel to the real axis and $f(z) \neq 0$ on the sides that are parallel to the imaginary one, for each rectangle S_k. Now, the intersection of each S_k with the critical strip of f, defined by (3.3), is a new rectangle, say R_k. Then, we claim that the set

$$\{R_k : k \in \mathbb{N}\} \tag{3.18}$$

is the desired family of rectangles. Indeed, firstly, we observe that the number of zeros of $f(z)$ inside each rectangle R_k is equal to the number of zeros inside each rectangle S_k. Secondly, R_k and S_k have the same height h_k. Then, the variation of the argument of $f(z)$ on each rectangle S_k is held to the following considerations:

(1) since $\operatorname{Im} f(z) > 0$ on the sides of S_k that are parallel to the real axis, the variation of the argument of $f(z)$ on these sides is less than π;

(2) from (3.10), the variation of the argument of $f(z)$ on each side of S_k defined by the lines $x = a$ and $x = b$ is

$$\begin{aligned} &\mu_1 h_k + \Theta_a, \quad \text{with } |\Theta_a| < \pi, \\ &\mu_\infty h_k + \Theta_b, \quad \text{with } |\Theta_b| < \pi, \end{aligned} \tag{3.19}$$

respectively. Therefore, noticing the previous considerations, the total variation of the argument of $f(z)$ on each rectangle S_k is

$$(\mu_\infty - \mu_1) h_k + \Phi_{f(z),k}, \quad \text{with } \left|\Phi_{f(z),k}\right| < 4\pi. \tag{3.20}$$

Consequently, the number of zeros of $f(z)$ inside each $\mathcal{A}_S$ satisfies the inequality

$$\left| N(f(z); R_k) - \frac{\mu h_k}{2\pi} \right| < 2. \tag{3.21}$$

Therefore, the R_k's are the desired rectangles, as we claimed. Now, the proof of the theorem is completed. □

Corollary 3.2. *The critical strip associated with each partial sum of the Riemann zeta function on the half-plane* $\operatorname{Re} z < -1$, $G_n(z) \equiv 1 + 2^z + \cdots + n^z$, $n \geq 2$, *can be partitioned in infinitely many rectangles* $\{R_k : k \in \mathbb{Z}\}$ *such that the number of its zeros inside each rectangle* R_k, $N(G_n(z); R_k)$, *satisfies*

$$\left| N(G_n(z); R_k) - \frac{h_k \ln n}{2\pi} \right| < 2, \tag{3.22}$$

where h_k *is the height of* R_k.

Proof. Firstly, from Lemma 2.5, for each integer $n \geq 2$, there exist two real numbers a_n, b_n such that the critical strip of the zeros of $G_n(z)$ is defined by

$$S_{G_n(z)} = \{z : a_n < \text{Re } z < b_n\}. \tag{3.23}$$

Now, starting from $y_0 = 0$ and taking into account that $G_n(z) \in \mathcal{A}_S$, determine the family of rectangles $\{R_k : k = 0, 1, 2, \ldots\}$ whose existence is guaranteed from the preceding theorem. It is plain that this family of rectangles forms a partition of the upper critical strip

$$\left\{z \in S_{G_n(z)} : 0 \leq \text{Im } z\right\}. \tag{3.24}$$

Now, defining the rectangle R_{-k} as the conjugate of R_{k-1} for $k = 1, 2, \ldots$, the desired partition of $S_{G_n(z)}$ is formed by the rectangles of the family $\{R_k : k \in \mathbb{Z}\}$. Finally, noticing $G_n(z) = 0$ if and only if $G_n(\overline{z}) = 0$, one has

$$N(G_n(z); R_{-k}) = N(G_n(z); R_{k-1}), \quad k = 1, 2, \ldots, \tag{3.25}$$

and, since the set $\{\ln k : 1 \leq k \leq n\}$ is the spectrum of each $G_n(z)$, inequality (3.22) follows. The corollary is then proved. □

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