Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 874949, 6 pages
doi:10.1155/2011/874949

Research Article

Approximately Multiplicative Functionals on the Spaces of Formal Power Series

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Received 20 February 2011; Accepted 5 May 2011

Academic Editor: John Rassias

We characterize the conditions under which approximately multiplicative functionals are near multiplicative functionals on weighted Hardy spaces.

1. Introduction

Let $\mathcal{A}$ be a commutative Banach algebra and $\widehat{\mathcal{A}}$ the set of all its characters, that is, the nonzero multiplicative linear functionals on $\mathcal{A}$. If φ is a linear functional on $\mathcal{A}$, then define

$$\check{\varphi}(a,b) = \varphi(ab) - \varphi(a)\varphi(b) \tag{1.1}$$

for all $a, b \in \mathcal{A}$. We say that φ is δ-multiplicative if $\|\check{\varphi}\| \leq \delta$.

For each $\varphi \in \mathcal{A}^{\star}$ define

$$d(\varphi) = \inf\left\{ \|\varphi - \psi\| : \psi \in \widehat{\mathcal{A}} \cup \{0\} \right\}. \tag{1.2}$$

We say that $\mathcal{A}$ is an algebra in which approximately multiplicative functionals are near multiplicative functionals or $\mathcal{A}$ is *AMNM* for short if, for each $\varepsilon > 0$, there is $\delta > 0$ such that $d(\varphi) < \varepsilon$ whenever φ is a δ-multiplicative linear functional.

We deal with an algebra in which every approximately multiplicative functional is near a multiplicative functional (*AMNM* algebra). The question whether an almost multiplicative map is close to a multiplicative, constitutes an interesting problem. Johnson has shown that various Banach algebras are *AMNM* and some of them fail to be *AMNM* [1–3]. Also, this property is still unknown for some Banach algebras such as H^{∞}, Douglas algebras, and $R(K)$ where K is a compact subset of $\mathcal{C}$. Here, we want to investigate conditions under which a weighted Hardy space is to be *AMNM*. For some sources on these topics one can refer to [1–8].

Let $\{\beta(n)\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. We consider the space of sequences $f = \{\hat{f}(n)\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^p \beta^p(n) < \infty. \tag{1.3}$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ will be used whether or not the series converges for any value of z. These are called formal power series or weighted Hardy spaces. Let $H^p(\beta)$ denote the space of all such formal power series. These are reflexive Banach spaces with norm $\|\cdot\|_\beta$. Also, the dual of $H^p(\beta)$ is $H^q(\beta^{p/q})$, where $1/p + 1/q = 1$ and $\beta^{p/q} = \{\beta(n)^{p/q}\}_{n=0}^{\infty}$ (see [9]). Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$, and then $\{f_k\}_{k=0}^{\infty}$ is a basis such that $\|f_k\| = \beta(k)$ for all k. For some sources one can see [9–21].

2. Main Results

In this section we investigate the *AMNM* property of the spaces of formal power series. For the proof of our main theorem we need the following lemma.

Lemma 2.1. *Let $1 < p < \infty$ and $1/p + 1/q = 1$. Then, $H^p(\beta)^* = H^q(\beta^{-1})$, where $\beta^{-1} = \{\beta^{-1}(n)\}_{n=0}^{\infty}$.*

Proof. Define $L : H^q(\beta^{p/q}) \to H^q(\beta^{-1})$ by $L(f) = F$, where

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \hat{f}(n) z^n, \\ F(z) &= \sum_{n=0}^{\infty} \hat{f}(n) \beta^p(n) z^n. \end{aligned} \tag{2.1}$$

Then,

$$\begin{aligned} \|F\|_{H^q(\beta^{-1})}^q &= \sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^q \left(\frac{\beta^{pq}(n)}{\beta^q(n)}\right) \\ &= \sum_{n=0}^{\infty} \left|\hat{f}(n)\right|^q \beta^p(n) \\ &= \|f\|_{H^p(\beta^{p/q})}^q. \end{aligned} \tag{2.2}$$

Thus, L is an isometry. It is also surjective because, if

$$F(z) = \sum_{n=0}^{\infty} \hat{F}(n) z^n \in H^q\left(\beta^{-1}\right), \tag{2.3}$$

then $L(f) = F$, where

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{\widehat{F}(n)}{\beta^p(n)} \right) z^n. \tag{2.4}$$

Hence, $H^q(\beta^{p/q})$ and $H^q(\beta^{-1})$ are norm isomorphic. Since $H^p(\beta)^\star = H^q(\beta^{p/q})$, the proof is complete. □

In the proof of the following theorem, our technique is similar to B. E. Johnson's technique in [2].

Theorem 2.2. *Let* $\liminf \beta(n) > 1$ *and* $1 < p < \infty$. *Then,* $H^p(\beta)$ *with multiplication*

$$\left(\sum_{n=0}^{\infty} \widehat{f}(n) z^n \right) \left(\sum_{n=0}^{\infty} \widehat{g}(n) z^n \right) = \sum_{n=0}^{\infty} \widehat{f}(n) \widehat{g}(n) z^n \tag{2.5}$$

is a commutative Banach algebra that is AMNM.

Proof. First note that clearly $H^p(\beta)$ is a commutative Banach algebra. To prove that it is AMNM, let $0 < \varepsilon < 1$ and put $\delta = \varepsilon^2/16$. Suppose that $\varphi \in H^q(\beta^{-1})$ and $\|\breve{\varphi}\| \le \delta$, where $1/p + 1/q = 1$. It is sufficient to show that $d(\varphi) < \varepsilon$. Since $d(\varphi) \le \|\varphi\|$, if $\|\varphi\| < \varepsilon$, then $d(\varphi) \le \|\varphi\|$. So suppose that $\|\phi\| \ge \varepsilon$. For each subset E of $\mathbb{N}_0 (= \mathbb{N} \cup \{0\})$, let

$$n_\varphi(E) = \left(\sum_{j \in E} |\widehat{\varphi}(j)|^q \beta^{-q}(j) \right)^{1/q}, \tag{2.6}$$

where

$$\varphi(z) = \sum_{j=0}^{\infty} \widehat{\varphi}(j) z^j. \tag{2.7}$$

For any subsets E_1 and E_2 of $\mathbb{N}_0$ we have that

$$\begin{aligned} n_\varphi^q(E_1 \cup E_2) &= \sum_{j \in E_1 \cup E_2} |\widehat{\varphi}(j)|^q \beta^{-q}(j) \\ &\le \sum_{j \in E_1} |\widehat{\varphi}(j)|^q \beta^{-q}(j) + \sum_{j \in E_2} |\widehat{\varphi}(j)|^q \beta^{-q}(j) \\ &= n_\varphi^q(E_1) + n_\varphi^q(E_2) \\ &\le \left(n_\varphi(E_1) + n_\varphi(E_2) \right)^q. \end{aligned} \tag{2.8}$$

Hence,

$$n_\varphi(E_1 \cup E_2) \le n_\varphi(E_1) + n_\varphi(E_2) \tag{2.9}$$

for all $E_1, E_2 \subseteq \mathbb{N}_0$. Also if $E_1 \cap E_2 = \emptyset$, then, by considering f, g with support, respectively, in E_1 and E_2, we get that $fg = 0$ and so

$$|\varphi(f)||\varphi(g)| = |\check{\varphi}(f,g)| \leq \delta\|f\|\|g\|. \tag{2.10}$$

By taking supremum over all such f and g with norm one, we see that

$$n_\varphi(E_1) \cdot n_\varphi(E_2) \leq \delta. \tag{2.11}$$

So either $n_\varphi(E_1) \leq \varepsilon/4$ or $n_\varphi(E_2) \leq \varepsilon/4$ whenever $E_1 \cap E_2 = \emptyset$.

For all $E \subseteq \mathbb{N}_0$ we have that

$$\varepsilon \leq \|\varphi\| = n_\varphi(\mathbb{N}_0) \leq n_\varphi(E) + n_\varphi(\mathbb{N}_0 \setminus \mathrm{E}). \tag{2.12}$$

Thus, we get that

$$n_\varphi(\mathbb{N}_0 \setminus \mathrm{E}) \geq \varepsilon - n_\varphi(E). \tag{2.13}$$

Since $(\mathbb{N}_0 \setminus E) \cap E = \emptyset$, as we saw earlier, it should be $n_\varphi(E) \leq \varepsilon/4$ or $n_\varphi(\mathbb{N}_0 \setminus\ E) \leq \varepsilon/4$ and equivalently it should be $n_\varphi(E) \leq \varepsilon/4$ or $n_\varphi(E) \geq 3\varepsilon/4$ for all $E \subseteq \mathbb{N}_0$.

Note that, if $E_1, E_2 \subseteq \mathbb{N}_0$ with $n_\varphi(E_i) \leq \varepsilon/4$ for $i = 1, 2$, then

$$n_\varphi(E_1 \cup E_2) \leq n_\varphi(E_1) + n_\varphi(E_2) \leq \frac{\varepsilon}{2}. \tag{2.14}$$

Thus, the relation $n_\varphi(E_1 \cup E_2) \geq 3\varepsilon/4$ is not true and so it should be

$$n_\varphi(E_1 \cup E_2) \leq \frac{\varepsilon}{4}. \tag{2.15}$$

Since $\|\varphi\| > \varepsilon$, clearly there exists a positive integer n_0 such that $n_\varphi(S_j) > \varepsilon$ for all $j \geq n_0$, where

$$S_j = \{i \in \mathbb{N}_0 : i \leq j\} \tag{2.16}$$

for all $j \in \mathbb{N}_0$. Now, let $n_\varphi(\{i\}) \leq \varepsilon/4$ for $i = 0, 1, 2, \ldots, n_0$. Since $n_\varphi(S_0) \leq \varepsilon/4$ and $n_\varphi(\{1\}) \leq \varepsilon/4$, $n_\varphi(S_1) \leq \varepsilon/4$. By continuing this manner we get that $n_\varphi(S_{n_0}) \leq \varepsilon/4$, which is a contradiction. Hence there exists $m_0 \in S_{n_0}$ such that $n_\varphi(\{m_0\}) \geq 3\varepsilon/4$. On the other hand, since $(\mathbb{N}_0 \setminus \{m_0\}) \cap \{m_0\} = \emptyset$, $n_\varphi(\{m_0\}) \leq \varepsilon/4$ or $n_\varphi(\mathbb{N}_0 \setminus \{m_0\}) \leq \varepsilon/4$. But $n_\varphi(\{m_0\}) \geq 3\varepsilon/4$, and so it should be $n_\varphi(\mathbb{N}_0 \setminus \{m_0\}) \leq \varepsilon/4$.

Remember that $f_j(z) = z^j$ for all $j \in \mathbb{N}_0$. Now we have that

$$\begin{aligned}
|\breve{\varphi}(f_{m_0}, f_{m_0})| &= \left|\varphi\left(f_{m_0}^2\right) - \varphi(f_{m_0})\varphi(f_{m_0})\right| \\
&= \left|\varphi(f_{m_0}) - \varphi^2(f_{m_0})\right| \\
&= |\varphi(f_{m_0})||1 - \varphi(f_{m_0})| \\
&= |\widehat{\varphi}(f_{m_0})||1 - \widehat{\varphi}(f_{m_0})| \\
&\leq \delta\beta^2(m_0).
\end{aligned} \tag{2.17}$$

Therefore,

$$|\widehat{\varphi}(m_0)|\beta^{-1}(m_0)\left(\beta^{-1}(m_0)|1 - \widehat{\varphi}(m_0)|\right) \leq \frac{\varepsilon^2}{16}, \tag{2.18}$$

and so

$$n_\varphi(\{m_0\})\left(\beta^{-1}(m_0)|1 - \widehat{\varphi}(m_0)|\right) \leq \frac{\varepsilon^2}{16}. \tag{2.19}$$

But $n_\varphi(\{m_0\}) \geq 3\varepsilon/4$, and thus

$$\beta^{-1}(m_0)|1 - \widehat{\varphi}(m_0)| \leq \frac{\varepsilon}{12}. \tag{2.20}$$

Define $\psi(z) = z^{m_0}$. Then $\psi \in \widehat{H}^p(\beta)$, and we have that

$$\begin{aligned}
\|\varphi - \psi\| &= \left\|\sum_{n \neq m_0} \widehat{\varphi}(n)z^n + (\widehat{\varphi}(m_0) - 1)z^{m_0}\right\| \\
&= \left(\sum_{n \neq m_0} |\widehat{\varphi}(n)|^q \beta^{-q}(n)\right)^{1/q} + \beta^{-1}(m_0)|1 - \widehat{\varphi}(m_0)| \\
&= n_\varphi(\mathbb{N}_0 \setminus \{m_0\}) + \beta^{-1}(m_0)|1 - \widehat{\varphi}(m_0)| \\
&\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{12} < \varepsilon.
\end{aligned} \tag{2.21}$$

Thus, indeed $d(\varphi) \leq \varepsilon$, and so the proof is complete. □

Disclosure

This is a part of the second author's Doctoral thesis written under the direction of the first author.

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