Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 864296, 16 pages
doi:10.1155/2011/864296

Research Article

Multiple Positive Solutions for Semilinear Elliptic Equations with Sign-Changing Weight Functions in $\mathbb{R}^N$

Tsing-San Hsu

Center for General Education, Chang Gung University, Kwei-Shan, Tao-Yuan 333, Taiwan

Correspondence should be addressed to Tsing-San Hsu, tshsu@mail.cgu.edu.tw

Received 9 March 2011; Accepted 3 May 2011

Academic Editor: Nobuyuki Kenmochi

Existence and multiplicity of positive solutions for the following semilinear elliptic equation: $-\Delta u + u = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u$ in $\mathbb{R}^N$, $u \in H^1(\mathbb{R}^N)$, are established, where $\lambda > 0, 1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$), a, b satisfy suitable conditions, and b maybe changes sign in $\mathbb{R}^N$. The study is based on the extraction of the Palais-Smale sequences in the Nehari manifold.

1. Introduction

In this paper, we deal with the multiplicity of positive solutions for the following semilinear elliptic equation:

$$\begin{gathered} -\Delta u + u = a(x)u^{p-1} + \lambda b(x)u^{q-1} \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{gathered} \tag{$E_{a,\lambda b}$}$$

where $\lambda > 0$, $1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, $2^* = \infty$ if $N = 1, 2$) and $a,\ b$ are measurable functions and satisfy the following conditions:

($a1$) $0 < a \in L^\infty(\mathbb{R}^N)$, where $\lim_{|x|\to\infty} a(x) = 1$, and there exist $C_0 > 0$ and $\delta_0 > 0$ such that

$$a(x) \geq 1 - C_0 e^{-\delta_0 |x|} \quad \forall x \in \mathbb{R}^N. \tag{1.1}$$

$(b1)$ $b \in L^{q^*}(\mathbb{R}^N)$ $(q^* = p/(p-q))$, $b^+ = \max\{b, 0\} \not\equiv 0$, $b^- = \max\{-b, 0\}$ is bounded and b^- has a compact support K in $\mathbb{R}^N$.

$(b2)$ There exist $C_1 > 0$, $0 < \delta_1 < \min\{\delta_0, q\}$ and $R_0 > 0$ such that

$$b^+(x) - b(x) \geq C_1 e^{-\delta_1 |x|} \quad \forall |x| \geq R_0. \tag{1.2}$$

Semilinear elliptic equations with concave-convex nonlinearities in bounded domains are widely studied. For example, Ambrosetti et al. [1] considered the following equation:

$$\begin{gathered} -\Delta u = u^{p-1} + \lambda u^{q-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{E_λ}$$

where $\lambda > 0$, $1 < q < 2 < p < 2^*$. They proved that there exists $\lambda_0 > 0$ such that (E_λ) admits at least two positive solutions for all $\lambda \in (0, \lambda_0)$, has one positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$. Actually, Adimurthi et al. [2], Damascelli et al. [3], Korman [4], Ouyang and Shi [5], and Tang [6] proved that there exists $\lambda_0 > 0$ such that (E_λ) in the unit ball $B^N(0;1)$ has exactly two positive solutions for $\lambda \in (0, \lambda_0)$, has exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. For more general results of (E_λ) (involving sign-changing weights) in bounded domains; see, the work of Ambrosetti et al. in [7], of Garcia Azorero et al. in [8], of Brown and Wu in [9], of Brown and Zhang in [10], of Cao and Zhong in [11], of de Figueiredo et al. in [12], and their references.

However, little has been done for this type of problem in $\mathbb{R}^N$. We are only aware of the works [13–17] which studied the existence of solutions for some related concave-convex elliptic problems (not involving sign-changing weights). Furthermore, we do not know of any results for concave-convex elliptic problems involving sign-changing weight functions except [18, 19]. Wu in [18] have studied the multiplicity of positive solutions for the following equation involving sign-changing weights:

$$\begin{gathered} -\Delta u + u = f_\lambda(x) u^{q-1} + g_\mu(x) u^{p-1} \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{gathered} \tag{E_{f_λ, g_μ}}$$

where $1 < q < 2 < p < 2^*$ the parameters $\lambda, \mu \geq 0$. He also assumed that $f_\lambda(x) = \lambda f_+(x) + f_-(x)$ is sign chaning and $g_\mu(x) = a(x) + \mu b(x)$, where a and b satisfy suitable conditions and proved that (E_{f_λ, g_μ}) has at least four positive solutions.

In a recent work [19], Hsu and Lin have studied $(E_{a,\lambda b})$ in $\mathbb{R}^N$ with a sign-changing weight function. They proved there exists $\lambda_0 > 0$ such that $(E_{a,\lambda b})$ has at least two positive solutions for all $\lambda \in (0, \lambda_0)$ provided that a, b satisfy suitable conditions and b maybe changes sign in $\mathbb{R}^N$.

Continuing our previous work [19], we consider $(E_{a,\lambda b})$ in $\mathbb{R}^N$ involving a sign-changing weight function with suitable assumptions which are different from the assumptions in [19].

In order to describe our main result, we need to define

$$\Lambda_0 = \left(\frac{2-q}{(p-q)\|a\|_{L^\infty}} \right)^{(2-q)/(p-2)} \left(\frac{p-2}{(p-q)\|b^+\|_{L^{q^*}}} \right) S_p^{p(2-q)/2(p-2)+q/2} > 0, \tag{1.3}$$

where $\|a\|_{L^\infty} = \sup_{x\in\mathbb{R}^N} a(x)$, $\|b^+\|_{L^{q^*}} = (\int_{\mathbb{R}^N} |b^+(x)|^{q^*} dx)^{1/q^*}$ and S_p is the best Sobolev constant for the imbedding of $H^1(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$.

Theorem 1.1. *Assume that* $(a1)$, $(b1)$-$(b2)$ *hold. If* $\lambda \in (0, (q/2)\Lambda_0)$, $(E_{a,\lambda b})$ *admits at least two positive solutions in* $H^1(\mathbb{R}^N)$.

This paper is organized as follows. In Section 2, we give some notations and preliminary results. In Section 3, we establish the existence of a local minimum. In Section 4, we prove the existence of a second solution of $(E_{a,\lambda b})$.

At the end of this section, we explain some notations employed. In the following discussions, we will consider $H = H^1(\mathbb{R}^N)$ with the norm $\|u\| = (\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx)^{1/2}$. We denote by S_p the best constant which is given by

$$S_p = \inf_{u\in H\setminus\{0\}} \frac{\|u\|^2}{\left(\int_{\mathbb{R}^N} |u|^p dx\right)^{2/p}}. \tag{1.4}$$

The dual space of H will be denoted by H^*. $\langle\cdot,\cdot\rangle$ denote the dual pair between H^* and H. We denote the norm in $L^s(\mathbb{R}^N)$ by $\|\cdot\|_{L^s}$ for $1 \le s \le \infty$. $B^N(x;r)$ is a ball in $\mathbb{R}^N$ centered at x with radius r. $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$. C, C_i will denote various positive constants, the exact values of which are not important.

2. Preliminary Results

Associated with (1.3), the energy functional $J_\lambda : H \to \mathbb{R}^N$ defined by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} a(x)|u|^p dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} b(x)|u|^q dx, \tag{2.1}$$

for all $u \in H$ is considered. It is well-known that $J_\lambda \in C^1(H,\mathbb{R})$ and the solutions of $(E_{a,\lambda b})$ are the critical points of J_λ.

Since J_λ is not bounded from below on H, we will work on the Nehari manifold. For $\lambda > 0$ we define

$$\mathcal{N}_\lambda = \left\{ u \in H \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0 \right\}. \tag{2.2}$$

Note that $\mathcal{N}_\lambda$ contains all nonzero solutions of $(E_{a,\lambda b})$ and $u \in \mathcal{N}_\lambda$ if and only if

$$\langle J'_\lambda(u), u\rangle = \|u\|^2 - \int_{\mathbb{R}^N} a(x)|u|^p dx - \lambda \int_{\mathbb{R}^N} b(x)|u|^q dx = 0. \tag{2.3}$$

Lemma 2.1. *J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Proof. If $u \in \mathcal{N}_\lambda$, then by $(b1)$, (2.3), and the Hölder and Sobolev inequalities, one has

$$J_\lambda(u) = \frac{p-2}{2p}\|u\|^2 - \lambda\left(\frac{p-q}{pq}\right)\int_{\mathbb{R}^N} b(x)|u|^q dx \tag{2.4}$$

$$\geq \frac{p-2}{2p}\|u\|^2 - \lambda\left(\frac{p-q}{pq}\right)S_p^{-q/2}\|b^+\|_{L^{q^*}}\|u\|^q. \tag{2.5}$$

Since $q < 2 < p$, it follows that J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$. □

The Nehari manifold is closely linked to the behavior of the function of the form $\varphi_u : t \to J_\lambda(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Drábek and Pohozaev in [20] and are also discussed by Brown and Zhang in [10]. If $u \in H$, we have

$$\begin{aligned}
\varphi_u(t) &= \frac{t^2}{2}\|u\|^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} a(x)|u|^p dx - \frac{t^q}{q}\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx,\\
\varphi_u'(t) &= t\|u\|^2 - t^{p-1}\int_{\mathbb{R}^N} a(x)|u|^p dx - t^{q-1}\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx,\\
\varphi_u''(t) &= \|u\|^2 - (p-1)t^{p-2}\int_{\mathbb{R}^N} a(x)|u|^p dx - (q-1)t^{q-2}\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx.
\end{aligned} \tag{2.6}$$

It is easy to see that

$$t\varphi_u'(t) = \|tu\|^2 - \int_{\mathbb{R}^N} a(x)|tu|^p dx - \lambda\int_{\mathbb{R}^N} b(x)|tu|^q dx, \tag{2.7}$$

and so, for $u \in H \setminus \{0\}$ and $t > 0$, $\varphi_u'(t) = 0$ if and only if $tu \in \mathcal{N}_\lambda$ that is, the critical points of φ_u correspond to the points on the Nehari manifold. In particular, $\varphi_u'(1) = 0$ if and only if $u \in \mathcal{N}_\lambda$. Thus, it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima, and points of inflection. Accordingly, we define

$$\begin{aligned}
\mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : \varphi_u''(1) > 0\},\\
\mathcal{N}_\lambda^0 &= \{u \in \mathcal{N}_\lambda : \varphi_u''(1) = 0\},\\
\mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : \varphi_u''(1) < 0\},
\end{aligned} \tag{2.8}$$

and note that if $u \in \mathcal{N}_\lambda$, that is, $\varphi_u'(1) = 0$, then

$$\varphi_u''(1) = (2-q)\|u\|^2 - (p-q)\int_{\mathbb{R}^N} a(x)|u|^p dx, \tag{2.9}$$

$$= (2-p)\|u\|^2 - (q-p)\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx. \tag{2.10}$$

We now derive some basic properties of $\mathcal{N}_\lambda^+$, $\mathcal{N}_\lambda^0$, and $\mathcal{N}_\lambda^-$.

Lemma 2.2. *Suppose that u_0 is a local minimizer for J_λ on $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$, then $J'_\lambda(u_0) = 0$ in H^*.*

Proof. See the work of Brown and Zhang in [10, Theorem 2.3]. □

Lemma 2.3. *If $\lambda \in (0, \Lambda_0)$, then $\mathcal{N}_\lambda^0 = \emptyset$.*

Proof. We argue by contradiction. Suppose that there exists $\lambda \in (0, \Lambda_0)$ such that $\mathcal{N}_\lambda^0 \neq \emptyset$. Then for $u \in \mathcal{N}_\lambda^0$ by (2.9) and the Sobolev inequality, we have

$$\frac{2-q}{p-q}\|u\|^2 = \int_{\mathbb{R}^N} a(x)|u|^p dx \leq \|a\|_{L^\infty} S_p^{-p/2}\|u\|^p, \tag{2.11}$$

and so

$$\|u\| \geq \left(\frac{2-q}{(p-q)\|a\|_{L^\infty}}\right)^{1/(p-2)} S_p^{p/2(p-2)}. \tag{2.12}$$

Similarly, using (2.10), Hölder and Sobolev inequalities, we have

$$\|u\|^2 = \lambda\frac{p-q}{p-2}\int_{\mathbb{R}^N} b(x)|u|^q dx \leq \lambda\frac{p-q}{p-2}\|b^+\|_{L^{q^*}} S_p^{-q/2}\|u\|^q \tag{2.13}$$

which implies

$$\|u\| \leq \left(\lambda\frac{p-q}{p-2}\|b^+\|_{L^{q^*}}\right)^{1/(2-q)} S_p^{-q/2(2-q)}. \tag{2.14}$$

Hence, we must have

$$\lambda \geq \left(\frac{2-q}{(p-q)\|a\|_{L^\infty}}\right)^{(2-q)/(p-2)}\left(\frac{p-2}{(p-q)\|b^+\|_{L^{q^*}}}\right) S_p^{p(2-q)/2(p-2)+q/2} = \Lambda_0 \tag{2.15}$$

which is a contradiction. □

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function $\psi_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\psi_u(t) = t^{2-q}\|u\|^2 - t^{p-q}\int_{\mathbb{R}^N} a(x)|u|^p dx \quad \text{for } t > 0. \tag{2.16}$$

Clearly, $tu \in \mathcal{N}_\lambda$ if and only if $\psi_u(t) = \lambda\int_{\mathbb{R}^N} b(x)|u|^q dx$. Moreover,

$$\psi'_u(t) = (2-q)t^{1-q}\|u\|^2 - (p-q)t^{p-q-1}\int_{\mathbb{R}^N} a(x)|u|^p dx \quad \text{for } t > 0, \tag{2.17}$$

and so it is easy to see that if $tu \in \mathcal{N}_\lambda$, then $t^{q-1}\varphi_u'(t) = \varphi_u''(t)$. Hence, $tu \in \mathcal{N}_\lambda^+$ (or $tu \in \mathcal{N}_\lambda^-$) if and only if $\varphi_u'(t) > 0$ (or $\varphi_u'(t) < 0$).

Let $u \in H \setminus \{0\}$. Then, by (2.17), φ_u has a unique critical point at $t = t_{\max}(u)$, where

$$t_{\max}(u) = \left(\frac{(2-q)\|u\|^2}{(p-q)\int_{\mathbb{R}^N} a(x)|u|^p dx}\right)^{1/(p-2)} > 0, \tag{2.18}$$

and clearly φ_u is strictly increasing on $(0, t_{\max}(u))$ and strictly decreasing on $(t_{\max}(u), \infty)$ with $\lim_{t\to\infty}\varphi_u(t) = -\infty$. Moreover, if $\lambda \in (0, \Lambda_0)$, then

$$\begin{aligned}
\varphi_u(t_{\max}(u)) &= \left[\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} - \left(\frac{2-q}{p-q}\right)^{(p-q)/(p-2)}\right]\frac{\|u\|^{2(p-q)/(p-2)}}{\left(\int_{\mathbb{R}^N} a(x)|u|^p dx\right)^{(2-q)/(p-2)}} \\
&= \|u\|^q\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{2-q/p-2}\left(\frac{\|u\|^p}{\int_{\mathbb{R}^N} a(x)|u|^p dx}\right)^{(2-q)/(p-2)} \\
&\geq \|u\|^q\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} S_p^{p(2-q)/2(p-2)} \\
&> \lambda\|b^+\|_{L^{q^*}} S_p^{-q/2}\|u\|^q \\
&\geq \lambda\int_{\mathbb{R}^N} b^+(x)|u|^q dx \\
&\geq \lambda\int_{\mathbb{R}^N} b(x)|u|^q dx.
\end{aligned} \tag{2.19}$$

Therefore, we have the following lemma.

Lemma 2.4. *Let $\lambda \in (0, \Lambda_0)$ and $u \in H \setminus \{0\}$.*

(i) *If $\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx \leq 0$, then there exists a unique $t^- = t^-(u) > t_{\max}(u)$ such that $t^-u \in \mathcal{N}_\lambda^-$, φ_u is inceasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover,*

$$J_\lambda(t^-u) = \sup_{t\geq 0} J_\lambda(tu). \tag{2.20}$$

(ii) *If $\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx > 0$, then there exist unique $0 < t^+ = t^+(u) < t_{\max}(u) < t^- = t^-(u)$ such that $t^+u \in \mathcal{N}_\lambda^+$, $t^-u \in \mathcal{N}_\lambda^-$, φ_u is decreasing on $(0, t^+)$, inceasing on (t^+, t^-) and decreasing on (t^-, ∞)*

$$J_\lambda(t^+u) = \inf_{0\leq t\leq t_{\max}(u)} J_\lambda(tu), \qquad J_\lambda(t^-u) = \sup_{t\geq t^+} J_\lambda(tu). \tag{2.21}$$

(iii) $\mathcal{N}_\lambda^- = \{u \in H \setminus \{0\} : t^-(u) = (1/\|u\|)t^-(u/\|u\|) = 1\}$.

(iv) *There exists a continuous bijection between $U = \{u \in H \setminus \{0\} : \|u\| = 1\}$ and $\mathcal{N}_\lambda^-$. In particular, t^- is a continuous function for $u \in H \setminus \{0\}$.*

Proof. See the work of Hsu and Lin in [19, Lemma 2.5]. □

We remark that it follows Lemma 2.4, $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \Lambda_0)$. Furthermore, by Lemma 2.4 it follows that $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$ are non-empty and by Lemma 2.1 we may define

$$\alpha_\lambda = \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u), \qquad \alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+} J_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u). \tag{2.22}$$

Theorem 2.5. (i) *If $\lambda \in (0, \Lambda_0)$, then we have $\alpha_\lambda \le \alpha_\lambda^+ < 0$.*

(ii) *If $\lambda \in (0, (q/2)\Lambda_0)$, then $\alpha_\lambda^- > d_0$ for some $d_0 > 0$.*

In particular, for each $\lambda \in (0, (q/2)\Lambda_0)$, we have $\alpha_\lambda^+ = \alpha_\lambda < 0 < \alpha_\lambda^-$.

Proof. See the work of Hsu and Lin in [19, Theorem 3.1]. □

Remark 2.6. (i) If $\lambda \in (0, \Lambda_0)$, then by (2.9), Hölder and Sobolev inequalities, for each $u \in \mathcal{N}_\lambda^+$ we have

$$\begin{aligned} \|u\|^2 &< \frac{p-q}{p-2}\lambda \int_{\mathbb{R}^N} b(x)|u|^q dx \\ &\le \frac{p-q}{p-2}\lambda \|b\|_{L^{q^*}} S_p^{-q/2} \|u\|^q \\ &\le \frac{p-q}{p-2}\Lambda_0 \|b\|_{L^{q^*}} S_p^{-q/2} \|u\|^q, \end{aligned} \tag{2.23}$$

and so

$$\|u\| \le \left(\frac{p-q}{p-2}\Lambda_0 \|b\|_{L^{q^*}} S_p^{-q/2}\right)^{1/(2-q)} \qquad \forall u \in \mathcal{N}_\lambda^+. \tag{2.24}$$

(ii) If $\lambda \in (0, (q/2)\Lambda_0)$, then by Lemma 2.4(i), (ii) and Theorem 2.5(ii), for each $u \in \mathcal{N}_\lambda^-$ we have

$$J_\lambda(u) = \sup_{t \ge 0} J_\lambda(tu) \ge \alpha_\lambda^- > 0. \tag{2.25}$$

3. Existence of a Positive Solution

First, we define the Palais-Smale (simply by (PS)) sequences, (PS)-values, and (PS)-conditions in H for J_λ as follows.

Definition 3.1. (i) For $c \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_c$-sequence in H for J_λ if $J_\lambda(u_n) = c + o_n(1)$ and $J'_\lambda(u_n) = o_n(1)$ strongly in H^* as $n \to \infty$.

(ii) $c \in \mathbb{R}$ is a (PS)-value in H for J_λ if there exists a $(\mathrm{PS})_c$-sequence in H for J_λ.

(iii) J_λ satisfies the $(\mathrm{PS})_c$-condition in H if any $(\mathrm{PS})_c$-sequence $\{u_n\}$ in H for J_λ contains a convergent subsequence.

Now we will ensure that there are $(\mathrm{PS})_{\alpha_\lambda^+}$-sequence and $(\mathrm{PS})_{\alpha_\lambda^-}$-sequencein on $\mathcal{N}_\lambda$ and $\mathcal{N}_\lambda^-$, respectively, for the functional J_λ.

Proposition 3.2. *If $\lambda \in (0,(q/2)\Lambda_0)$, then*

(i) *there exists a $(PS)_{\alpha_\lambda}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda$ in H for J_λ.*

(ii) *there exists a $(PS)_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in H for J_λ.*

Proof. See Wu [21, Proposition 9]. □

Now, we establish the existence of a local minimum for J_λ on $\mathcal{N}_\lambda^+$.

Theorem 3.3. *Assume $(a1)$ and $(b1)$ hold. If $\lambda \in (0,(q/2)\Lambda_0)$, then there exists $u_\lambda \in \mathcal{N}_\lambda^+$ such that*

(i) $J_\lambda(u_\lambda) = \alpha_\lambda = \alpha_\lambda^+ < 0$,

(ii) *u_λ is a positive solution of $(E_{a,\lambda b})$,*

(iii) *$\|u_\lambda\| \to 0$ as $\lambda \to 0^+$.*

Proof. From Proposition 3.2(i) it follows that there exists $\{u_n\} \subset \mathcal{N}_\lambda$ satisfying

$$J_\lambda(u_n) = \alpha_\lambda + o_n(1) = \alpha_\lambda^+ + o_n(1), \quad J_\lambda'(u_n) = o_n(1) \quad \text{in } H^*. \tag{3.1}$$

By Lemma 2.1 we infer that $\{u_n\}$ is bounded on H. Passing to a subsequence (Still denoted by $\{u_n\}$), there exists $u_\lambda \in H$ such that as $n \to \infty$

$$\begin{gathered} u_n \rightharpoonup u_\lambda \quad \text{weakly in } H, \\ u_n \longrightarrow u_\lambda \quad \text{almost everywhere in } \mathbb{R}^N, \\ u_n \longrightarrow u_\lambda \quad \text{strongly in } L^s_{\text{loc}}\left(\mathbb{R}^N\right)\ \forall 1 \le s < 2^*. \end{gathered} \tag{3.2}$$

By $(b1)$, Egorov theorem and Hölder inequality, we have

$$\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx + o_n(1) \quad \text{as } n \longrightarrow \infty. \tag{3.3}$$

By (3.1) and (3.2), it is easy to see that u_λ is a solution of $(E_{a,\lambda b})$. From $u_n \in \mathcal{N}_\lambda$ and (2.4), we deduce that

$$\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \frac{q(p-2)}{2(p-q)}\|u_n\|^2 - \frac{pq}{p-q} J_\lambda(u_n). \tag{3.4}$$

Let $n \to \infty$ in (3.4). By (3.1), (3.3) and $\alpha_\lambda < 0$, we get

$$\lambda \int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx \ge -\frac{pq}{p-q}\alpha_\lambda > 0. \tag{3.5}$$

Thus, $u_\lambda \in \mathcal{N}_\lambda$ is a nonzero solution of $(E_{a,\lambda b})$.

Next, we prove that $u_n \to u_\lambda$ strongly in H and $J_\lambda(u_\lambda) = \alpha_\lambda$. From the fact $u_n, u_\lambda \in \mathcal{N}_\lambda$ and applying Fatou's lemma, we get

$$\begin{aligned}\alpha_\lambda \le J_\lambda(u_\lambda) &= \frac{p-2}{2p}\|u_\lambda\|^2 - \frac{p-q}{pq}\lambda\int_{\mathbb{R}^N} b(x)|u_\lambda|^q dx \\ &\le \liminf_{n\to\infty}\left(\frac{p-2}{2p}\|u_n\|^2 - \frac{p-q}{pq}\lambda\int_{\mathbb{R}^N} b(x)|u_n|^q dx\right) \\ &\le \liminf_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda.\end{aligned} \tag{3.6}$$

This implies that $J_\lambda(u_\lambda) = \alpha_\lambda$ and $\lim_{n\to\infty}\|u_n\|^2 = \|u_\lambda\|^2$. Standard argument shows that $u_n \to u_\lambda$ strongly in H. By Theorem 2.5, for all $\lambda \in (0, (q/2)\Lambda_0)$ we have that $u_\lambda \in \mathcal{N}_\lambda$ and $J_\lambda(u_\lambda) = \alpha_\lambda^+ < \alpha_\lambda^-$ which implies $u_\lambda \in \mathcal{N}_\lambda^+$. Since $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathcal{N}_\lambda^+$, by Lemma 2.2 we may assume that u_λ is a nonzero nonnegative solution of $(E_{a,\lambda b})$. By Harnack inequality [22] we deduce that $u_\lambda > 0$ in $\mathbb{R}^N$. Finally, by (2.10), Hölder and Sobolev inequlities,

$$\|u_\lambda\|^{2-q} < \lambda\frac{p-q}{p-2}\|b^+\|_{L^{q^*}} S_p^{-q/2}, \tag{3.7}$$

and thus we conclude the proof. □

4. Second Positive Solution

In this section, we will establish the existence of the second positive solution of $(E_{a,\lambda b})$ by proving that J_λ satisfies the $(\mathrm{PS})_{\alpha_\lambda^-}$-condition.

Lemma 4.1. *Assume that* $(a1)$ *and* $(b1)$ *hold. If* $\{u_n\} \subset H$ *is a* $(PS)_c$*-sequence for* J_λ*, then* $\{u_n\}$ *is bounded in* H.

Proof. See the work of Hsu and Lin in [19, Lemma 4.1]. □

Let us introduce the problem at infinity associated with $(E_{a,\lambda b})$:

$$-\Delta u + u = u^{p-1} \quad \text{in } \mathbb{R}^N,\ u \in H,\ u > 0 \text{ in } \mathbb{R}^N. \tag{E^∞}$$

We state some known results for problem (E^∞). First of all, we recall that by Lions [23] has studied the following minimization problem closely related to problem (E^∞):

$$S^\infty = \inf\{J^\infty(u) : u \in H,\ u \not\equiv 0,\ (J^\infty)'(u) = 0\} > 0, \tag{4.1}$$

where $J^\infty(u) = (1/2)\|u\|^2 - (1/p)\int_{\mathbb{R}^N}|u|^p dx$. Note that a minimum exists and is attained by a ground state $w_0 > 0$ in $\mathbb{R}^N$ such that

$$S^\infty = J^\infty(w_0) = \sup_{t\ge 0} J^\infty(tw_0) = \left(\frac{1}{2} - \frac{1}{p}\right)S_p^{p/(p-2)}, \tag{4.2}$$

where $S_p = \inf_{u\in H\setminus\{0\}} \|u\|^2/(\int_{\mathbb{R}^N} |u|^p dx)^{2/p}$. Gidas et al. [24] showed that for every $\varepsilon > 0$, there exist positive constants C_ε, C_2 such that for all $x \in \mathbb{R}^N$,

$$\begin{aligned} &C_\varepsilon \exp(-(1+\varepsilon)|x|) \\ &\quad \le w_0(x) \le C_2 \exp(-|x|). \end{aligned} \tag{4.3}$$

We define

$$w_n(x) = w_0(x - ne), \quad \text{where } e = (0,0,\ldots,0,1) \text{ is a unit vector in } \mathbb{R}^N. \tag{4.4}$$

Clearly, $w_n(x) \in H$.

Lemma 4.2. *Let Ω be a domain in $\mathbb{R}^N$. If $f : \Omega \to \mathbb{R}$ satisfies*

$$\int_\Omega \left| f(x) e^{\sigma|x|} \right| dx < \infty \quad \textit{for some } \sigma > 0, \tag{4.5}$$

then

$$\begin{aligned} &\left(\int_\Omega f(x) e^{-\sigma|x-\tilde{x}|} dx \right) e^{\sigma|\tilde{x}|} \\ &\quad = \int_\Omega f(x) e^{\sigma\langle x,\tilde{x}\rangle/|\tilde{x}|} dx + o(1) \quad \textit{as } |\tilde{x}| \longrightarrow \infty. \end{aligned} \tag{4.6}$$

Proof. We know $\sigma|\tilde{x}| \le \sigma|x| + \sigma|x - \tilde{x}|$. Then,

$$\left| f(x) e^{-\sigma|x-\tilde{x}|} e^{\sigma|\tilde{x}|} \right| \le \left| f(x) e^{\sigma|x|} \right|. \tag{4.7}$$

Since $-\sigma|x - \tilde{x}| + \sigma|\tilde{x}| = \sigma\langle x, \tilde{x}\rangle/|\tilde{x}| + o(1)$ as $|\tilde{x}| \to \infty$, then the lemma follows from the Lebesgue dominated convergence theorem. □

Lemma 4.3. *Under the assumptions $(a1)$, $(b1)$-$(b2)$ and $\lambda \in (0, \Lambda_0)$. Then there exists a number $n_0 \in \mathbb{N}$ such that for $n \ge n_0$*

$$\sup_{t\ge 0} J_\lambda(t w_n) < S^\infty. \tag{4.8}$$

In particular, $\alpha_\lambda^- < S^\infty$ for all $\lambda \in (0, \Lambda_0)$.

Proof. (i) First, since $\|w_n\| = \|w_0\|$ for all $n \in \mathbb{N}$ and J_λ is continuous in H and $J_\lambda(0) = 0$, we infer that there exists $t_1 > 0$ such that

$$J_\lambda(t w_n) < S^\infty \quad \forall n \in \mathbb{N},\ t \in [0, t_1]. \tag{4.9}$$

(ii) Since $\lim_{|x|\to\infty} a(x) = 1$, there exists $n_1 \in \mathbb{N}$ such that if $n \geq n_1$, we get $a(x) \geq 1/2$ for $x \in B^N(ne;1)$. Then, for $n \geq n_1$

$$\begin{aligned} J_\lambda(tw_n) &= \frac{t^2}{2}\|w_n\|^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} a(x)|w_n|^p dx - \frac{t^q}{q}\int_{\mathbb{R}^N} \lambda b(x)|w_n|^q dx \\ &\leq \frac{t^2}{2}\|w_0\|^2 - \frac{t^p}{p}\int_{B^N(0;1)} a(x+ne)|w_0|^p dx + \frac{t^q}{q}\lambda\left\|b^-\right\|_{L^\infty}\int_{\mathbb{R}^N} |w_n|^q dx \\ &\leq \frac{t^2}{2}\|w_0\|^2 - \frac{t^p}{2p}\int_{B^N(0;1)} |w_0|^p dx + \frac{t^q}{q}\lambda\left\|b^-\right\|_{L^\infty}\int_{\mathbb{R}^N} |w_0|^q dx \\ &\longrightarrow -\infty \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{4.10}$$

Thus, there exists $t_2 > 0$ such that for any $t > t_2$ and $n > n_1$ we get

$$J_\lambda(tw_n) < 0. \tag{4.11}$$

(iii) By (i) and (ii), we need to show that there exists n_0 such that for $n \geq n_0$

$$\sup_{t_1 \leq t \leq t_2} J_\lambda(tw_n) < S^\infty. \tag{4.12}$$

We know that $\sup_{t\geq 0} J^\infty(tw_0) = S^\infty$. Then, $t_1 \leq t \leq t_2$, we have

$$\begin{aligned} J_\lambda(tw_n) &= \frac{1}{2}\|tw_n\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} a(x)(tw_n)^p dx - \frac{1}{q}\int_{\mathbb{R}^N} \lambda b(x)(tw_n)^q dx \\ &\leq \frac{t^2}{2}\|w_0\|^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} w_0^p dx + \frac{t^p}{p}\int_{\mathbb{R}^N} (1-a(x))w_n^p dx - \frac{t^q}{q}\int_{\mathbb{R}^N} \lambda b(x)w_n^q dx \\ &\leq S^\infty + \frac{t_2^p}{p}\int_{\mathbb{R}^N} (1-a)^+(x)w_n^p dx - \frac{t_1^q}{q}\int_{\mathbb{R}^N} \lambda b^+(x)w_n^q dx + \frac{t_2^q}{q}\int_{\mathbb{R}^N} \lambda b^-(x)w_n^q dx. \end{aligned} \tag{4.13}$$

Suppose a satisfies $(a1)$, we get $(1-a)^+(x) \leq C_0 e^{-\delta_0|x|}$ for all $x \in \mathbb{R}^N$ and some positive constant δ_0. By (4.3) and Lemma 4.3, there exists $n_2 > n_1$ such that for any $n \geq n_2$

$$\int_{\mathbb{R}^N} (1-a)^+(x)w_n^p dx \leq C_3 e^{-\min\{\delta_0,p\}n}. \tag{4.14}$$

By $(b1)$ and (4.3), we get

$$\begin{aligned} \int_{\mathbb{R}^N} \lambda b^-(x)w_n^q dx &\leq \lambda\left\|b^-\right\|_{L^\infty} C_2 \int_K e^{-q|x-ne|} dx \\ &\leq \lambda C_3 e^{-qn}. \end{aligned} \tag{4.15}$$

By $(b2)$, (4.3) and Lemma 4.3, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \lambda b^+(x) w_n^q dx &\geq \lambda C_1 C_\varepsilon \int_{|x|\geq R_0} e^{-\delta_1|x|} e^{-q(1+\varepsilon)|x-ne|} dx \\ &\geq \lambda \overline{C_\varepsilon} e^{-\delta_1 n}.\end{aligned} \tag{4.16}$$

Since $0 < \delta_1 < \min\{\delta_0, q\} \leq \min\{\delta_0, p\}$ and $\lambda \in (0, \Lambda_0)$ and using (4.13)–(4.16), we have there exists $n_0 > n_2$ such that for all $n \geq n_0$, then

$$\sup_{t_1\leq t\leq t_2} J_\lambda(tw_n) < S^\infty, \qquad \lambda \int_{\mathbb{R}^N} b(x)|w_n|^q dx > 0. \tag{4.17}$$

This implies that if $\lambda \in (0, \Lambda_0)$, then for all $n \geq n_0$ we get

$$\sup_{t\geq 0} J_\lambda(tw_n) < S^\infty. \tag{4.18}$$

From $a(x) > 0$ for all $x \in \mathbb{R}^N$ and (4.17), we have

$$\int_{\mathbb{R}^N} a(x)|w_{n_0}|^p dx > 0, \qquad \int_{\mathbb{R}^N} b(x)|w_{n_0}|^q dx > 0. \tag{4.19}$$

Combining this with Lemma 2.4(ii), from the definition of α_λ^- and $\sup_{t\geq 0} J_\lambda(tw_{n_0}) < S^\infty$, for all $\lambda \in (0, \Lambda_0)$, we obtain that there exists $t_0 > 0$ such that $t_0 w_{n_0} \in \mathcal{N}_\lambda^-$ and

$$\alpha_\lambda^- \leq J_\lambda(t_0 w_{n_0}) \leq \sup_{t\geq 0} J_\lambda(tw_{n_0}) < S^\infty. \tag{4.20}$$

□

Lemma 4.4. *Assume that $(a1)$ and $(b1)$ hold. If $\{u_n\} \subset H$ is a $(PS)_c$-sequence for J_λ with $c \in (0, S^\infty)$, then there exists a subsequence of $\{u_n\}$ converging weakly to a nonzero solution of $(E_{a,\lambda b})$ in $\mathbb{R}^N$.*

Proof. Let $\{u_n\} \subset H$ be a $(\mathrm{PS})_c$-sequence for J_λ with $c \in (0, S^\infty)$. We know from Lemma 4.1 that $\{u_n\}$ is bounded in H, and then there exist a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_0 \in H$ such that

$$\begin{gathered} u_n \rightharpoonup u_0 \quad \text{weakly in } H, \\ u_n \longrightarrow u_0 \quad \text{almost everywhere in } \mathbb{R}^N, \\ u_n \longrightarrow u_0 \quad \text{strongly in } L^s_{\text{loc}}\left(\mathbb{R}^N\right)\ \forall 1 \leq s < 2^*. \end{gathered} \tag{4.21}$$

It is easy to see that $J'_\lambda(u_0) = 0$ and by $(b1)$, Egorov theorem and Hölder inequality, we have

$$\lambda \int_{\mathbb{R}^N} b(x)|u_n|^q dx = \lambda \int_{\mathbb{R}^N} b(x)|u_0|^q dx + o_n(1). \tag{4.22}$$

Next we verify that $u_0 \not\equiv 0$. Arguing by contradiction, we assume $u_0 \equiv 0$. By $(a1)$, for any $\varepsilon > 0$, there exists $R_0 > 0$ such that $|a(x) - 1| < \varepsilon$ for all $x \in [B^N(0; R_0)]^C$. Since $u_n \to 0$ strongly in $L^s_{\text{loc}}(\mathbb{R}^N)$ for $1 \le s < 2^*$, $\{u_n\}$ is a bounded sequence in H, therefore $\int_{\mathbb{R}^N}(a(x) - 1)|u_n|^p \le C\int_{B^N(0;R_0)}|u_n|^p + \varepsilon C$. Setting $n \to \infty$, then $\varepsilon \to 0$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)|u_n|^p dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p dx. \tag{4.23}$$

We set

$$\begin{aligned} l &= \lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)|u_n|^p dx \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^p dx. \end{aligned} \tag{4.24}$$

Since $J'_\lambda(u_n) = o_n(1)$ and $\{u_n\}$ is bounded, then by (4.22), we can deduce that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\langle J'_\lambda(u_n), u_n\rangle \\ &= \lim_{n\to\infty}\left(\|u_n\|^2 - \int_{\mathbb{R}^N} a(x)|u_n|^p dx\right) \\ &= \lim_{n\to\infty}\|u_n\|^2 - l, \end{aligned} \tag{4.25}$$

that is,

$$\lim_{n\to\infty}\|u_n\|^2 = l. \tag{4.26}$$

If $l = 0$, then we get $c = \lim_{n\to\infty} J_\lambda(u_n) = 0$, which contradicts to $c > 0$. Thus we conclude that $l > 0$. Furthermore, by the definition of S_p we obtain

$$\|u_n\|^2 \ge S_p\left(\int_{\mathbb{R}^N}|u_n|^p dx\right)^{2/p}. \tag{4.27}$$

Then, as $n \to \infty$, we have

$$l = \lim_{n\to\infty}\|u_n\|^2 \ge S_p l^{2/p}, \tag{4.28}$$

which implies that

$$l \ge S_p^{p/(p-2)}. \tag{4.29}$$

Hence, from (4.2) and (4.22)–(4.29), we get

$$\begin{aligned} c &= \lim_{n\to\infty} J_\lambda(u_n) \\ &= \frac{1}{2}\lim_{n\to\infty}\|u_n\|^2 - \frac{1}{p}\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)|u_n|^p dx - \frac{\lambda}{q}\lim_{n\to\infty}\int_{\mathbb{R}^N} b(x)|u_n|^q dx \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)l \\ &\geq \frac{p-2}{2p}S_p^{p/(p-2)} = S^\infty. \end{aligned} \tag{4.30}$$

This is a contradiction to $c < S^\infty$. Therefore, u_0 is a nonzero solution of $(E_{a,\lambda b})$. □

Now, we establish the existence of a local minimum of J_λ on $\mathcal{N}_\lambda^-$.

Theorem 4.5. *Assume that* $(a1)$ *and* $(b1)$*-*$(b2)$ *hold. If* $\lambda \in (0,(q/2)\Lambda_0)$*, then there exists* $U_\lambda \in \mathcal{N}_\lambda^-$ *such that*

(i) $J_\lambda(U_\lambda) = \alpha_\lambda^-$,

(ii) U_λ *is a positive solution of* $(E_{a,\lambda b})$.

Proof. If $\lambda \in (0,(q/2)\Lambda_0)$, then by Theorem 2.5(ii), Proposition 3.2(ii) and Lemma 4.3(ii), there exists a $(\mathrm{PS})_{\alpha_\lambda^-}$-sequence $\{u_n\} \subset \mathcal{N}_\lambda^-$ in H for J_λ with $\alpha_\lambda^- \in (0,S^\infty)$. From Lemma 4.4, there exist a subsequence still denoted by $\{u_n\}$ and a nonzero solution $U_\lambda \in H$ of $(E_{a,\lambda b})$ such that $u_n \rightharpoonup U_\lambda$ weakly in H.

First, we prove that $U_\lambda \in \mathcal{N}_\lambda^-$. On the contrary, if $U_\lambda \in \mathcal{N}_\lambda^+$, then by $\mathcal{N}_\lambda^-$ is closed in H, we have $\|U_\lambda\|^2 < \liminf_{n\to\infty}\|u_n\|^2$. From (2.9) and $a(x) > 0$ for all $x \in \mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} b(x)|U_\lambda|^q dx > 0, \qquad \int_{\mathbb{R}^N} a(x)|U_\lambda|^p dx > 0. \tag{4.31}$$

By Lemma 2.4(ii), there exists a unique t_λ^- such that $t_\lambda^- U_\lambda \in \mathcal{N}_\lambda^-$. If $u \in \mathcal{N}_\lambda$, then it is easy to see that

$$J_\lambda(u) = \frac{p-2}{2p}\|u\|^2 - \frac{p-q}{pq}\lambda\int_{\mathbb{R}^N} b(x)|u|^q dx. \tag{4.32}$$

From (3.1), $u_n \in \mathcal{N}_\lambda^-$ and (4.32), we can deduce that

$$\alpha_\lambda^- \leq J_\lambda(t_\lambda^- U_\lambda) < \lim_{n\to\infty} J_\lambda(t_\lambda^- u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \alpha_\lambda^- \tag{4.33}$$

which is a contradiction. Thus, $U_\lambda \in \mathcal{N}_\lambda^-$.

Next, by the same argument as that in Theorem 3.3, we get that $u_n \to U_\lambda$ strongly in H and $J_\lambda(U_\lambda) = \alpha_\lambda^- > 0$ for all $\lambda \in (0,(q/2)\Lambda_0)$. Since $J_\lambda(U_\lambda) = J_\lambda(|U_\lambda|)$ and $|U_\lambda| \in \mathcal{N}_\lambda^-$, by Lemma 2.2 we may assume that U_λ is a nonzero nonnegative solution of $(E_{a,\lambda b})$. Finally, by the Harnack inequality [22] we deduce that $U_\lambda > 0$ in $\mathbb{R}^N$. □

Now, we complete the proof of Theorem 1.1. By Theorems 3.3, 4.5, we obtain $(E_{a,\lambda b})$ has two positive solutions u_λ and U_λ such that $u_\lambda \in \mathcal{N}_\lambda^+$, $U_\lambda \in \mathcal{N}_\lambda^-$. Since $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$, this implies that u_λ and U_λ are distinct. It completes the proof of Theorem 1.1.

References

[1] A. Ambrosetti, H. Brézis, and G. Cerami, "Combined effects of concave and convex nonlinearities in some elliptic problems," *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.

[2] Adimurthi, F. Pacella, and S. L. Yadava, "On the number of positive solutions of some semilinear Dirichlet problems in a ball," *Differential and Integral Equations*, vol. 10, no. 6, pp. 1157–1170, 1997.

[3] L. Damascelli, M. Grossi, and F. Pacella, "Qualitative properties of positive solutions of semilinear elliptic equations in symmetric domains via the maximum principle," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 16, no. 5, pp. 631–652, 1999.

[4] P. Korman, "On uniqueness of positive solutions for a class of semilinear equations," *Discrete and Continuous Dynamical Systems*, vol. 8, no. 4, pp. 865–871, 2002.

[5] T. Ouyang and J. Shi, "Exact multiplicity of positive solutions for a class of semilinear problem. II," *Journal of Differential Equations*, vol. 158, no. 1, pp. 94–151, 1999.

[6] M. Tang, "Exact multiplicity for semilinear elliptic Dirichlet problems involving concave and convex nonlinearities," *Proceedings of the Royal Society of Edinburgh*, vol. 133, no. 3, pp. 705–717, 2003.

[7] A. Ambrosetti, J. Garcia Azorero, and I. Peral, "Multiplicity results for some nonlinear elliptic equations," *Journal of Functional Analysis*, vol. 137, no. 1, pp. 219–242, 1996.

[8] J. Garcia Azorero, J. J. Manfredi, and I. Peral Alonso, "Sobolev versus Hölder local minimizers and global multiplicity for some quasilinear elliptic equations," *Communications in Contemporary Mathematics*, vol. 2, no. 3, pp. 385–404, 2000.

[9] K. J. Brown and T. F. Wu, "A fibering map approach to a semilinear elliptic boundary value problem," *Electronic Journal of Differential Equations*, vol. 2007, no. 69, pp. 1–9, 2007.

[10] K. J. Brown and Y. Zhang, "The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function," *Journal of Differential Equations*, vol. 193, no. 2, pp. 481–499, 2003.

[11] D. M. Cao and X. Zhong, "Multiplicity of positive solutions for semilinear elliptic equations involving critical Sobolev exponents," *Nonlinear Analysis*, vol. 29, no. 4, pp. 461–483, 1997.

[12] D. G. de Figueiredo, J. P. Gossez, and P. Ubilla, "Multiplicity results for a family of semilinear elliptic problems under local superlinearity and sublinearity," *Journal of the European Mathematical Society*, vol. 8, no. 2, pp. 269–286, 2006.

[13] J. Chabrowski and J. M. Bezzera do Ó, "On semilinear elliptic equations involving concave and convex nonlinearities," *Mathematische Nachrichten*, vol. 233-234, pp. 55–76, 2002.

[14] K. J. Chen, "Combined effects of concave and convex nonlinearities in elliptic equation on $\mathbb{R}^N$," *Journal of Mathematical Analysis and Applications*, vol. 355, no. 2, pp. 767–777, 2009.

[15] J. V. Goncalves and O. H. Miyagaki, "Multiple positive solutions for semilinear elliptic equations in $\mathbb{R}^N$ involving subcritical exponents," *Nonlinear Analysis*, vol. 32, no. 1, pp. 41–51, 1998.

[16] Z. Liu and Z. Q. Wang, "Schrödinger equations with concave and convex nonlinearities," *Zeitschrift fur Angewandte Mathematik und Physik*, vol. 56, no. 4, pp. 609–629, 2005.

[17] T. F. Wu, "Multiplicity of positive solutions for semilinear elliptic equations in $\mathbb{R}^N$," *Proceedings of the Royal Society of Edinburgh*, vol. 138, no. 3, pp. 647–670, 2008.

[18] T. F. Wu, "Multiple positive solutions for a class of concave-convex elliptic problems in $\mathbb{R}^N$ involving sign-changing weight," *Journal of Functional Analysis*, vol. 258, no. 1, pp. 99–131, 2010.

[19] T. S. Hsu and H. L. Lin, "Multiple positive solutions for semilinear elliptic equations in $\mathbb{R}^N$ involving concave-convex nonlinearities and sign-changing weight functions," *Abstract and Applied Analysis*, vol. 2010, Article ID 658397, 21 pages, 2010.

[20] P. Drábek and S. I. Pohozaev, "Positive solutions for the p-Laplacian: application of the fibering method," *Proceedings of the Royal Society of Edinburgh*, vol. 127, no. 4, pp. 703–726, 1997.

[21] T. F. Wu, "On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 253–270, 2006.

[22] N. S. Trudinger, "On Harnack type inequalities and their application to quasilinear elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 721–747, 1967.

[23] P. L. Lions, "On positive solutions of semilinear elliptic equations in unbounded domains," in *Nonlinear Diffusion Equations and Their Equilibrium States II*, vol. 13 of *Mathematical Sciences Research Institute*, pp. 85–122, Springer, New York, NY, USA, 1988.

[24] B. Gidas, W. M. Ni, and L. Nirenberg, "Symmetry and related properties via the maximum principle," *Communications in Mathematical Physics*, vol. 68, no. 3, pp. 209–243, 1979.