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Research Article

Oscillation of Second-Order Nonlinear Delay Dynamic Equations on Time Scales

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In this work, we use the generalized Riccati transformation and the inequality technique to establish some new oscillation criteria for the second-order nonlinear delay dynamic equation $(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + q(t)f(x(\tau(t))) = 0$, on a time scale $\mathbb{T}$, where γ is the quotient of odd positive integers and *p(t)* and *q(t)* are positive right-dense continuous (rd-continuous) functions on $\mathbb{T}$. Our results improve and extend some results established by Sun et al. 2009. Also our results unify the oscillation of the second-order nonlinear delay differential equation and the second-order nonlinear delay difference equation. Finally, we give some examples to illustrate our main results.

1. Introduction

The theory of time scales was introduced by Hilger [1] in order to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. Many authors have expounded on various aspects of this new theory, see [2–4]. A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers, If the time scale equals the real numbers or integer numbers, it represents the classical theories of the differential and difference equations. Many other interesting time scales exist and give rise to many applications. The new theory of the so-called "dynamic equation" not only unify the theories of differential equations and difference equations, but also extends these classical cases to the so-called q-difference equations (when $\mathbb{T} = q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0$ for $q > 1\}$ or $\mathbb{T} = q^{\overline{\mathbb{Z}}} = q^{\mathbb{Z}} \cup \{0\})$ which have important applications in quantum theory (see [5]). Also it can be applied on different types of time scales like $\mathbb{T} = h\mathbb{Z}, \mathbb{T} = \mathbb{N}_0^2$, and the space of the harmonic numbers $\mathbb{T} = \mathbb{T}_n$. In the last two decades, there has been increasing interest in obtaining sufficient conditions for oscillation (nonoscillation) of the solutions of different classes of dynamic equations on time scales, see [6–9]. In this paper, we deal with the oscillation behavior of all solutions of the second-order nonlinear delay dynamic equation

$$\left(p(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)f(x(\tau(t)))=0,\quad t\in\mathbb{T},\ t\geq t_0, \tag{1.1}$$

subject to the hypotheses

(H$_1$) $\mathbb{T}$ is a time scale which is unbounded above, and $t_0\in\mathbb{T}$ with $t_0>0$. We define the time scale interval $[t_0,\infty)_{\mathbb{T}}$ by $[t_0,\infty)_{\mathbb{T}}=[t_0,\infty)\bigcap\mathbb{T}$.

(H$_2$) γ is the quotient of odd positive integers.

(H$_3$) p and q are positive rd-continuous functions on an arbitrary time scale $\mathbb{T}$, and

$$\int_{t_0}^{\infty}\frac{\Delta(t)}{p^{1/\gamma}(t)}=\infty \tag{1.2}$$

(H$_4$) $\tau:\mathbb{T}\to\mathbb{T}$ is a strictly increasing and differentiable function such that $\tau(t)\leq t$, $\lim_{t\to\infty}\tau(t)=\infty$.

(H$_5$) $f\in C(\mathbb{R},\mathbb{R})$ is a continuous function such that for some positive constant L, it satisfies $f(x)/x^{\gamma}\geq L$ for all $x\neq 0$.

By a solution of (1.1), we mean that a nontrivial real valued function x satisfies (1.1) for $t\in\mathbb{T}$. A solution x of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. (1.1) is said to be oscillatory if all of its solutions are oscillatory. We concentrate our study to those solutions of (1.1) which are not identically vanishing eventually.

It is easy to see that (1.1) can be transformed into a half linear dynamic equation

$$\left(p(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+q(t)x^{\gamma}(t)=0,\quad t\in\mathbb{T},\ t\geq t_0, \tag{1.3}$$

where $f(x)=x^{\gamma}$, $\tau(t)=t$. If $\gamma=1$, then (1.1) is transformed into the equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta}+q(t)f(x(\tau(t)))=0,\quad t\in\mathbb{T},\ t\geq t_0. \tag{1.4}$$

If $p(t)=1$, then (1.4) has the form

$$x^{\Delta\Delta}(t)+q(t)f(x(\tau(t)))=0,\quad t\in\mathbb{T},\ t\geq t_0. \tag{1.5}$$

If $f(x)=x$, then (1.5) becomes

$$x^{\Delta\Delta}(t)+q(t)x(\tau(t))=0,\quad t\in\mathbb{T},\ t\geq t_0. \tag{1.6}$$

Recently, Zhang et al. [10] have considered the nonlinear delay (1.1) and established some sufficient conditions for oscillation of (1.1) when $\gamma\geq 1$. Also Grace et al. [11] introduced some new sufficient conditions for oscillation of the half linear dynamic equation (1.3). In 2009, Sun et al. [12] extended and improved the results of [6, 13, 14] to (1.1) when $\gamma \geq 1$, but their results can not be applied for $0 < \gamma < 1$. In 2008, Hassan [15] considered the half linear dynamic equation (1.3) and established some sufficient conditions for oscillation of (1.3). In 2007, Erbe et al. [13] considered the nonlinear delay dynamic equation (1.4) and obtained some new oscillation criteria which improve the results of Şahiner [14]. In 2005, Agarwal et al. [6] studied the linear delay dynamic equation (1.6), also Şahiner [14] considered the nonlinear delay dynamic equation (1.5) and gave some sufficient conditions for oscillation of (1.6) and (1.5). In this work, we give some new oscillation criteria of (1.1) by using the generalized Riccati transformation and the inequality technique. Our results are general cases for some results of [12, 15].

This paper is organized as follows. In Section 2, we present some preliminaries on time scales. In Section 3, we give several lemmas. In Section 4, we establish some new sufficient conditions for oscillation of (1.1). Finally, in Section 5, we present some examples to illustrate our results.

2. Some Preliminaries on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T}, s < t\}. \tag{2.1}$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$ is said to be left dense if $\rho(t) = t$, right dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left scattered if $\rho(t) < t$, and right scattered if $\sigma(t) > t$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided that it is continuous at right-dense points of $\mathbb{T}$, and its left-sided limits exist (finite) at left-dense points of $\mathbb{T}$. The set of rd-continuous functions is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. By $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, we mean the set of functions whose delta derivative belongs to $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced with any Banach space), the delta derivative f^{Δ} is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{2.2}$$

provided that f is continuous at t, and t is right scattered. If t is not right scattered, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^{+}} \frac{f(\sigma(t)) - f(t)}{t - s} = \lim_{s \to t^{+}} \frac{f(t) - f(s)}{t - s}, \tag{2.3}$$

provided that this limit exists.

A function $f : [a,b] \to \mathbb{R}$ is said to be differentiable if its derivative exists. The derivative f^{Δ} and the shift f^{σ} of a function f are related by the equation

$$f^{\sigma} = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{2.4}$$

The derivative rules of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g are given by

$$\begin{gathered} (f \cdot g)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t), \\ \left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)}. \end{gathered} \tag{2.5}$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t = \left[f(t)g(t)\right]_a^b - \int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t \tag{2.6}$$

or

$$\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t = \left[f(t)g(t)\right]_a^b - \int_a^b f^{\Delta}(t)g(t)\Delta t \tag{2.7}$$

and the infinite integral is defined by

$$\int_b^{\infty} f(s)\Delta s = \lim_{t \to \infty} \int_b^t f(s)\Delta s. \tag{2.8}$$

Note that in case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \qquad \mu(t) = 0, \qquad f^{\Delta}(t) = f'(t), \qquad \int_a^b f(t)\Delta t = \int_a^b f(t)dt, \tag{2.9}$$

and in case $\mathbb{T} = \mathbb{Z}$, we have

$$\begin{gathered} \sigma(t) = t+1, \qquad \rho(t) = t-1, \qquad \mu(t) = 1, \qquad f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t) \\ \text{if } a < b, \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t). \end{gathered} \tag{2.10}$$

Throughout this paper, we use

$$d_+(t) := \max\{0, d(t)\}, \qquad d_-(t) := \max\{0, -d(t)\},$$
$$\beta(t) := \begin{cases} \alpha(t) & 0 < \gamma \le 1 \\ \alpha^{\gamma}(t) & \gamma > 1, \end{cases} \tag{2.11}$$

where

$$\alpha(t) := \frac{R(t)}{R(t) + \mu(t)}, \quad R(t) := p^{1/\gamma}(t) \int_{t_0}^{t} \frac{\Delta s}{p^{1/\gamma}(s)}, \quad \text{for } t \ge t_0. \tag{2.12}$$

3. Several Lemmas

In this section, we present some lemmas that we need in the proofs of our results in Section 4.

Lemma 1 (Bohner and Peterson [3, Theorem 1.90]). *If x(t) is delta differentiable and eventually positive or negative, then*

$$\left((x(t))^{\gamma}\right)^{\Delta} = \gamma \int_0^1 [hx(\sigma(t)) + (1-h)x(t)]^{\gamma-1} x^{\Delta}(t) dh. \tag{3.1}$$

Lemma 2 (Hardy et al. [16, Theorem 41]). *If A and B are nonnegative real numbers, then*

$$\lambda A B^{\lambda-1} - A^{\lambda} \le (\lambda - 1)\, B^{\lambda}, \quad \lambda > 1, \tag{3.2}$$

where the equality holds if and only if $A = B$.

Lemma 3. *If (H_1)–(H_3) and* (1.2) *hold and* (1.1) *has a positive solution x on $[t_0, \infty)_{\mathbb{T}}$, then*

$$\left(p(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} < 0, \quad x^{\Delta}(t) > 0, \quad \frac{x(t)}{x^{\sigma}(t)} > \alpha(t), \quad \textit{for } t \in [t_0, \infty). \tag{3.3}$$

Proof. The proof is similar to the proof of Lemma 2.1 in [15] and, hence, is omitted. □

4. Main Results

Theorem 1. *Assume that (H_1)–(H_5),* (1.2), *Lemma 3 hold and $\tau \in C^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$, $\tau([t_0, \infty)_{\mathbb{T}}) = [t_0, \infty)_{\mathbb{T}}$. Furthermore, assume that there exists a positive Δ-differentiable function $\delta(t)$ such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[L\alpha^{\gamma}(\tau(s)) q(s) \delta^{\sigma}(s) - \frac{p(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}} \right] \Delta s = \infty. \tag{4.1}$$

Then every solution of (1.1) *is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.*

Proof. Assume that (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$, and there is $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of Lemma 3 on $[T,\infty)_{\mathbb{T}}$. Consider the generalized Riccati substitution

$$w(t) = \delta(t)p(t)\left(\frac{x^{\Delta}(t)}{x(\tau(t))}\right)^{\gamma}. \tag{4.2}$$

Using the delta derivative rules of the product and quotient of two functions, we have

$$\begin{aligned} w^{\Delta}(t) &= \delta^{\Delta}(t)\frac{p(t)(x^{\Delta}(t))^{\gamma}}{(x(\tau(t)))^{\gamma}} + \delta^{\sigma}(t)\left(\frac{p(t)(x^{\Delta}(t))^{\gamma}}{(x(\tau(t)))^{\gamma}}\right)^{\Delta} \\ &= \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) + \frac{\delta^{\sigma}(t)\left(p(t)(x^{\Delta}(t))^{\gamma}\right)^{\Delta}}{(x(\tau(\sigma(t))))^{\gamma}} - \frac{\delta^{\sigma}(t)p(t)(x^{\Delta}(t))^{\gamma}((x(\tau(t)))^{\gamma})^{\Delta}}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}}, \end{aligned} \tag{4.3}$$

using the fact $f(x)/x^{\gamma} \ge L$ and $x(t)/x^{\sigma}(t) > \alpha(t)$, we have

$$w^{\Delta}(t) \le \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\delta^{\sigma}(t)p(t)(x^{\Delta}(t))^{\gamma}((x(\tau(t)))^{\gamma})^{\Delta}}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}}. \tag{4.4}$$

If $0 < \gamma \le 1$, then using the chain rule and the fact that $x(t)$ is strictly increasing on $[T,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} \left((x(\tau(t)))^{\gamma}\right)^{\Delta} &= \gamma\int_0^1 \left[x(\tau(t)) + h\mu(\tau(t))(x(\tau(t)))^{\Delta}\right]^{\gamma-1} dh(x(\tau(t)))^{\Delta} \\ &= \gamma\int_0^1 [(1-h)x(\tau(t)) + hx^{\sigma}(\tau(t))]^{\gamma-1} dh(x(\tau(t)))^{\Delta} \\ &\ge \gamma(x^{\sigma}(\tau(t)))^{\gamma-1}(x(\tau(t)))^{\Delta}\tau^{\Delta}(t) \end{aligned} \tag{4.5}$$

which implies

$$\begin{aligned} w^{\Delta}(t) &\le \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\gamma\delta^{\sigma}(t)p(t)(x^{\Delta}(t))^{\gamma}(x^{\sigma}(\tau(t)))^{\gamma-1}(x(\tau(t)))^{\Delta}\tau^{\Delta}(t)}{(x(\tau(t)))^{\gamma}(x(\tau(\sigma(t))))^{\gamma}} \\ &\le \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\gamma\delta^{\sigma}(t)(x(\tau(t)))^{\Delta}\alpha(\tau(t))\tau^{\Delta}(t)}{\delta(t)x(\tau(t))}w(t), \end{aligned} \tag{4.6}$$

since $(p(t)(x^{\Delta}(t))^{\gamma})^{\Delta} < 0$, then by integrating from t to $\tau(t)$, we get

$$(x(\tau(t)))^{\Delta} > \frac{(p(t))^{1/\gamma}}{((p\tau(t)))^{1/\gamma}} x^{\Delta}(t) \tag{4.7}$$

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\gamma\delta^{\sigma}(t)(p(t))^{1/\gamma}x^{\Delta}(t)\alpha(\tau(t))\tau^{\Delta}(t)}{\delta(t)x(\tau(t))(p(\tau(t)))^{1/\gamma}} w(t), \tag{4.8}$$

that is,

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\gamma\delta^{\sigma}(t)\alpha(\tau(t))\tau^{\Delta}(t)}{\delta^{(\gamma+1)/\gamma}(t)(p(\tau(t)))^{1/\gamma}} w^{(\gamma+1)/\gamma}(t)\ . \tag{4.9}$$

If $\gamma > 1$, then using the chain rule and the fact that $x(t)$ is strictly increasing on $[T,\infty)_{\mathbb{T}}$, we obtain

$$\left((x(\tau(t)))^{\gamma}\right)^{\Delta} \geq (x(\tau(t)))^{\gamma-1}(x(\tau(t)))^{\Delta}\tau^{\Delta}(t). \tag{4.10}$$

From (4.4), (4.7), and (4.10), we have

$$w^{\Delta}(t) \leq \frac{\delta^{\Delta}(t)}{\delta(t)} w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\gamma\delta^{\sigma}(t)\alpha^{\gamma}(\tau(t))\tau^{\Delta}(t)}{\delta^{(\gamma+1)/\gamma}(t)(p(\tau(t)))^{1/\gamma}} w^{(\gamma+1)/\gamma}(t). \tag{4.11}$$

By (4.9), (4.11), and the definition of $\beta(t)$, we have for $\gamma > 0$

$$w^{\Delta}(t) \leq \frac{(\delta^{\Delta}(t))_{+}}{\delta(t)} w(t) - L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)}{\delta^{\lambda}(t)p^{\lambda-1}(\tau(t))} w^{\lambda}(t), \tag{4.12}$$

where $\lambda = (\gamma+1)/\gamma$. Defining $A \geq 0$ and $B \geq 0$ by

$$A^{\lambda} = \frac{\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)}{\delta^{\lambda}(t)p^{\lambda-1}(\tau(t))} w^{\lambda}(t), \qquad B^{\lambda-1} = \frac{p^{(\lambda-1)/\lambda}(\tau(t))(\delta^{\Delta}(t))_{+}}{\lambda(\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t))^{1/\lambda}}, \tag{4.13}$$

then using Lemma 2, we get

$$\frac{(\delta^{\Delta}(t))_{+}}{\delta(t)} w(t) - \frac{\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)}{\delta^{\lambda}(t)p^{\lambda-1}(\tau(t))} w^{\lambda}(t) \leq \frac{p(\tau(t))((\delta^{\Delta}(t))_{+})^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}(\beta(\tau(t))\delta^{\sigma}(t)\tau^{\Delta}(t))^{\gamma}}. \tag{4.14}$$

From this last inequality and (4.12), we get

$$w^{\Delta}(t) \leq -L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) + \frac{p(\tau(t))\left(\left(\delta^{\Delta}(t)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(t))\delta^{\sigma}(t)\tau^{\Delta}(t)\right)^{\gamma}}. \tag{4.15}$$

Integrating both sides from T to t, we get

$$\int_{T}^{t}\left[L\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)q(s) - \frac{p(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}}\right]\Delta s \leq w(T) - w(t) \leq w(T), \tag{4.16}$$

which contradicts the assumption (4.1). This contradiction completes the proof. □

Theorem 2. *Assume that (H_1)–(H_5), (1.2), Lemma 3 hold and $\tau \in C^{1}_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$, $\tau([t_0,\infty)_{\mathbb{T}}) = [t_0,\infty)_{\mathbb{T}}$. Furthermore, assume that there exist functions $H, h \in C_{rd}(\mathbb{D}, \mathbb{R})$ (where $\mathbb{D} \equiv \{(t,s) : t \geq s \geq t_0\}$) such that*

$$H(t,t) = 0, \quad t \geq t_0, \qquad H(t,s) > 0, \quad t > s \geq t_0, \tag{4.17}$$

and H has a nonpositive continuous Δ-partial derivative with respect to the second variable $H^{\Delta_s}(t,s)$ which satisfies

$$H^{\Delta_s}(\sigma(t),s) + H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(t)}{\delta(t)} = -\frac{h(t,s)}{\delta(t)}\left(H(\sigma(t),\sigma(s))\right)^{\gamma/(\gamma+1)}, \tag{4.18}$$

$$\limsup_{t\to\infty}\frac{1}{H(\sigma(t),t_0)}\int_{t_0}^{\sigma(t)} K(t,s)\Delta s = \infty, \tag{4.19}$$

where $\delta(t)$ is positive Δ-differentiable function and

$$K(t,s) = H(\sigma(t),\sigma(s))L\alpha^{\gamma}(\tau(s))q(s)\delta^{\sigma}(s) - \frac{p(\tau(s))\left(h_{-}(t,s)\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}}. \tag{4.20}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$, and there is $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of Lemma 3 on $[T,\infty)_{\mathbb{T}}$. Define $w(t)$ as in the proof of Theorem 1. Replacing $(\delta^{\Delta}(t))_{+}$ with $\delta^{\Delta}(t)$ in (4.12), we have

$$L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) \leq -w^{\Delta}(t) + \frac{\delta^{\Delta}(t)}{\delta(t)}w(t) - \frac{\gamma\delta^{\sigma}(t)\beta(\tau(t))\tau^{\Delta}(t)}{\delta^{\lambda}(t)p^{\lambda-1}(\tau(t))}w^{\lambda}(t). \tag{4.21}$$

Multiplying (4.21) by $H(\sigma(t),\sigma(s))$, and integrating with respect to s from T to $\sigma(t)$, we get

$$\begin{aligned}\int_T^{\sigma(t)} & H(\sigma(t),\sigma(s))L\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)q(s)\Delta s\\ &\le -\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))w^{\Delta}(s)\Delta s\\ &\quad+\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\delta^{\Delta}(s)}{\delta(s)}w(s)\Delta s\\ &\quad-\int_T^{\sigma(t)} H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)}{\delta^{\lambda}(s)p^{\lambda-1}(\tau(s))}w^{\lambda}(s)\Delta s.\end{aligned} \tag{4.22}$$

Integrating by parts and using (4.17) and (4.18), we obtain

$$\begin{aligned}\int_T^{\sigma(t)} & H(\sigma(t),\sigma(s))L\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)q(s)\Delta s\le H(\sigma(t),T)w(T)\\ &+\int_T^{\sigma(t)}\left[\frac{h_-(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{1/\lambda}w(s)-H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)}{\delta^{\lambda}(s)p^{\lambda-1}(\tau(s))}w^{\lambda}(s)\right]\Delta s.\end{aligned} \tag{4.23}$$

Defining $A\ge 0$ and $B\ge 0$ by

$$A^{\lambda}=H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)}{\delta^{\lambda}(s)p^{\lambda-1}(\tau(s))}w^{\lambda}(s),\qquad B^{\lambda-1}=\frac{p^{(\lambda-1)/\lambda}(\tau(s))h_-(t,s)}{\lambda\left(\gamma\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)\right)^{1/\lambda}}, \tag{4.24}$$

then using Lemma 2, we get

$$\begin{aligned}&\frac{h_-(t,s)}{\delta(s)}(H(\sigma(t),\sigma(s)))^{1/\lambda}w(s)-H(\sigma(t),\sigma(s))\frac{\gamma\delta^{\sigma}(s)\beta(\tau(s))\tau^{\Delta}(s)}{\delta^{\lambda}(s)p^{\lambda-1}(\tau(s))}w^{\lambda}(s)\\ &\quad\le\frac{p(\tau(s))(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}},\end{aligned} \tag{4.25}$$

therefore,

$$\begin{aligned}\int_T^{\sigma(t)} & H(\sigma(t),\sigma(s))L\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)q(s)\Delta s\le H(\sigma(t),T)w(T)\\ &+\int_T^{\sigma(t)}\frac{p(\tau(s))(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}}\Delta s.\end{aligned} \tag{4.26}$$

By the definition of $K(t,s)$, we get

$$\int_T^{\sigma(t)} K(t,s)\Delta s \le H(\sigma(t),T)w(T), \tag{4.27}$$

and this implies that

$$\frac{1}{H(\sigma(t),T)}\int_T^{\sigma(t)} K(t,s)\Delta s \le w(T), \tag{4.28}$$

which contradicts the assumption (4.19). This contradiction completes the proof. □

Theorem 3. *Assume that (H_1)–(H_5),* (1.2)*, Lemma 3 hold and $\tau \in C_{rd}^1([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau([t_0,\infty)_{\mathbb{T}}) = [t_0,\infty)_{\mathbb{T}}$. Furthermore, assume that there exists a positive Δ-differentiable function $\delta(t)$ such that for $m \ge 1$*

$$\limsup_{t\to\infty}\frac{1}{t^m}\int_{t_0}^{t}(t-s)^m\left[L\alpha^{\gamma}(\tau(s))q(s)\delta^{\sigma}(s) - \frac{p(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\delta+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}}\right]\Delta s = \infty. \tag{4.29}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$, and there is $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of Lemma 3 on $[T,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 1, we get (4.15) from which we have

$$L\alpha^{\gamma}(\tau(t))\delta^{\sigma}(t)q(t) - \frac{p(\tau(t))\left(\left(\delta^{\Delta}(t)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(t))\delta^{\sigma}(t)\tau^{\Delta}(t)\right)^{\gamma}} \le -w^{\Delta}(t), \tag{4.30}$$

therefore,

$$\begin{aligned}&\int_{t_1}^{t}(t-s)^m\left(L\alpha^{\gamma}(\tau(s))\delta^{\sigma}(s)q(s) - \frac{p(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}}\right)\Delta s.\\ &\quad\le -\int_{t_1}^{t}(t-s)^m w^{\Delta}(t)\Delta s.\end{aligned} \tag{4.31}$$

The right hand side of the above inequality gives

$$\int_{t_1}^{t}(t-s)^m w^{\Delta}(s)\Delta s = (t-s)^m w(s)_{t_1}^{t} - \int_{t_1}^{t}\left((t-s)^m\right)^{\Delta_s} w(\sigma(s))\Delta s. \tag{4.32}$$

Since $((t-s)^m)^{\Delta_s} \leq -m(t-\sigma(s))^{m-1} \leq 0$ for $t \geq \sigma(s)$, $m \geq 1$, then we have

$$\int_{t_1}^{t} (t-s)^m \left[L\alpha^\gamma(\tau(s))q(s)\delta^\sigma(s) - \frac{p(\tau(s))\left(\left(\delta^\Delta(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^\sigma(s)\tau^\Delta(s)\right)^\gamma} \right] \Delta s \leq (t-t_1)^m w(t_1), \tag{4.33}$$

then,

$$\frac{1}{t^m}\int_{t_1}^{t} (t-s)^m \left[L\alpha^\gamma(\tau(s))q(s)\delta^\sigma(s) - \frac{p(\tau(s))\left(\left(\delta^\Delta(s)\right)_+\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^\sigma(s)\tau^\Delta(s)\right)^\gamma} \right] \Delta s \leq \left(\frac{t-t_1}{t}\right)^m w(t_1) \tag{4.34}$$

which contradicts (4.29). This contradiction completes the proof. □

Theorem 4. *Assume that* $\int_{t_0}^{\infty} \Delta t / p^{1/\gamma}(t) = \infty$ *and*

$$\limsup_{t\to\infty} \frac{LR^\gamma(\tau(t))}{p(\tau(t))} \int_t^\infty q(s)\Delta s > 1, \ \textit{hold}. \tag{4.35}$$

Then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Assume that (1.1) has a nonoscillatory solution on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, we assume that $x(t) > 0$, $x(\tau(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, $t_1 \in [t_0,\infty)_{\mathbb{T}}$, and there is $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t)$ satisfies the conclusion of Lemma 3 on $[T,\infty)_{\mathbb{T}}$. From (1.1), we have

$$\left(p(t)\left(x^\Delta(t)\right)^\gamma\right)^\Delta = -q(t)f(x(\tau(t))) \leq -Lq(t)x^\gamma(\tau(t)). \tag{4.36}$$

Integrating last equation from $\tau(t)$ to ∞, we obtain

$$\int_{\tau(t)}^{\infty} Lq(s)x^\gamma(\tau(s))\Delta s < p(\tau(t))\left(x^\Delta(\tau(t))\right)^\gamma - \lim_{s\to\infty} p(s)\left(x^\Delta(s)\right)^\gamma. \tag{4.37}$$

Since $p(s)(x^\Delta(s))^\gamma$ decreasing and $p(s)(x^\Delta(s))^\gamma > 0$, then we have

$$\frac{1}{p(\tau(t))}\int_{\tau(t)}^{\infty} Lq(s)x^\gamma(\tau(s))\Delta s < \left(x^\Delta(\tau(t))\right)^\gamma. \tag{4.38}$$

Since $x(t) > R(t)x^{\Delta}(t)$, then $x(\tau(t)) > R(\tau(t))x^{\Delta}(\tau(t))$, and consequently

$$\frac{L}{p(\tau(t))}\int_{\tau(t)}^{\infty} q(s)x^{\gamma}(\tau(s))\Delta s < \left(\frac{x(\tau(t))}{R(\tau(t))}\right)^{\gamma},$$
$$\frac{LR^{\gamma}(\tau(t))}{p(\tau(t))}\int_{\tau(t)}^{\infty} q(s)x^{\gamma}(\tau(s))\Delta s < x^{\gamma}(\tau(t)), \tag{4.39}$$

but

$$\frac{LR^{\gamma}(\tau(t))}{p(\tau(t))}\int_{t}^{\infty} q(s)x^{\gamma}(\tau(s))\Delta s < \frac{LR^{\gamma}(\tau(t))}{p(\tau(t))}\int_{\tau(t)}^{\infty} q(s)x^{\gamma}(\tau(s))\Delta s < x^{\gamma}(\tau(t)). \tag{4.40}$$

Since $x(t)$ and $\tau(t)$ are strictly increasing, then we get that

$$\frac{LR^{\gamma}(\tau(t))}{p(\tau(t))}\int_{t}^{\infty} q(s)\Delta s < 1, \tag{4.41}$$

therefore,

$$\frac{LR^{\gamma}(\tau(t))}{p(\tau(t))}\int_{t}^{\infty} q(s)\Delta s \leq 1. \tag{4.42}$$

This contradiction completes the proof. □

5. Examples

In this section, we give some examples to illustrate our main results.

Example 1. Consider the second-order nonlinear delay dynamic equation

$$\left(t^{\gamma}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{\lambda}{t\alpha^{\gamma}(\tau(t))}x^{\gamma}(\tau(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}},\ t_0 \geq 0, \tag{5.1}$$

where λ is a positive constant,l and γ is the quotient of odd positive integers.
Here,

$$p(t) = t^{\gamma}, \qquad q(t) = \frac{\lambda}{t\alpha^{\gamma}(\tau(t))}, \qquad f(x) = x^{\gamma}, \qquad L = 1. \tag{5.2}$$

If $\delta(t) = 1$, then

$$\int_{t_0}^{\infty} \frac{\Delta t}{(p(t))^{1/\gamma}} = \int_{t_0}^{\infty} \frac{\Delta t}{t} = \infty,$$

$$\begin{aligned}
\limsup_{t\to\infty} \int_{t_0}^{t} \left[L\alpha^{\gamma}(\tau(s))q(s)\delta^{\sigma}(s) - \frac{p(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}} \right] \Delta s \\
= \limsup_{t\to\infty} \int_{t_0}^{t} \frac{\lambda}{s} \Delta s = \infty.
\end{aligned} \tag{5.3}$$

Therefore, by Theorem 1, every solution of (5.1) is oscillatory.

Example 2. Consider the second-order nonlinear delay dynamic equation

$$\left(\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + \frac{\lambda\sigma^{\gamma-1}(t)}{t^{\gamma}\tau^{\gamma}(t)} x^{\gamma}(\tau(t))\left[x^{2\gamma}(\tau(t)) + 1\right] = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},\ t_0 \geq 0, \tag{5.4}$$

where λ is a positive constant, and $0 < \gamma \leq 1$ is the quotient of odd positive integers, that is, $\alpha(t) = \beta(t)$.
Here,

$$p(t) = 1, \qquad q(t) = \frac{\lambda\sigma^{\gamma-1}(t)}{t^{\gamma}\tau^{\gamma}(t)}, \qquad f(x) = x^{\gamma}\left(x^{2\gamma} + 1\right), \qquad L = 1, \qquad \tau(t) = \frac{t}{2}. \tag{5.5}$$

It is clear that (1.2) holds.

Since $R(\tau(t)) = p^{1/\gamma}(\tau(t)) \int_{t_0}^{\tau(t)} \Delta s / p^{1/\gamma}(s) = \tau(t) - t_0$, then we can find $0 < b < 1$ such that

$$\begin{aligned}
\alpha(\tau(t)) &= \frac{R(\tau(t))}{R(\tau(t)) + \mu(\tau(t))} = \frac{\tau(t) - t_0}{\tau(t) - t_0 + \sigma(\tau(t)) - \tau(t)} \\
&= \frac{\tau(t) - t_0}{\sigma(\tau(t)) - t_0} > \frac{b\tau(t)}{\sigma(\tau(t))}, \qquad \text{for } t \geq t_b > t_0.
\end{aligned} \tag{5.6}$$

If $\delta(t) = t$, then

$$\begin{aligned}
&\limsup_{t\to\infty} \int_{t_0}^{t} \left[L\alpha^{\gamma}(\tau(s))q(s)\delta^{\sigma}(s) - \frac{p(\tau(s))\left(\left(\delta^{\Delta}(s)\right)_{+}\right)^{\gamma+1}}{(\gamma+1)^{(\gamma+1)}\left(\beta(\tau(s))\delta^{\sigma}(s)\tau^{\Delta}(s)\right)^{\gamma}} \right] \Delta s \\
&\quad > \limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{b^{\gamma}\tau^{\gamma}(s)\lambda\sigma^{\gamma-1}(s)\sigma(s)}{\sigma^{\gamma}(\tau(s))\tau^{\gamma}(s)s^{\gamma}} - \frac{2^{2\gamma}\sigma^{\gamma}(\tau(s))}{(\gamma+1)^{(\gamma+1)}b^{\gamma}s^{\gamma}\sigma^{\gamma}(s)} \right] \Delta s
\end{aligned}$$

$$
\begin{aligned}
&> \limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{b^{\gamma}\lambda\sigma^{\gamma}(\tau(s))}{s^{\gamma}\sigma^{\gamma}(\tau(s))}-\frac{2^{2\gamma}\sigma^{\gamma}(s)}{(\gamma+1)^{(\gamma+1)}b^{\gamma}s^{\gamma}\sigma^{\gamma}(s)}\right]\Delta s \\
&= \left(b^{\gamma}\lambda-\frac{2^{2\gamma}}{(\gamma+1)^{(\gamma+1)}b^{\gamma}}\right)\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{s^{\gamma}}\Delta s=\infty,
\end{aligned}
\tag{5.7}
$$

if $\lambda > 2^{2\gamma}/(b^{2\gamma}(\gamma+1)^{(\gamma+1)})$. Then by Theorem 1, every solution of (5.4) is oscillatory if $\lambda > 2^{2\gamma}/(b^{2\gamma}(\gamma+1)^{(\gamma+1)})$.

Example 3. Consider the second-order nonlinear delay dynamic equation

$$
\left(t^{\gamma-1}\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta}+\frac{\lambda}{t\sigma(t)}x^{\gamma}(\tau(t))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},\ t_0\geq 0, \tag{5.8}
$$

where λ is a positive constant and $\gamma \geq 1$ is the quotient of odd positive integers.
Here,

$$
p(t)=t^{\gamma-1},\qquad q(t)=\frac{\lambda}{t\sigma(t)},\qquad f(x)=x^{\gamma},\qquad L=1. \tag{5.9}
$$

It is clear that $\int_{t_0}^{\infty}\Delta t/p^{1/\gamma}(t)=\int_{t_0}^{\infty}\Delta t/t^{(\gamma-1)/\gamma}=\infty$, for $\gamma\geq 1$, (i.e., (1.2) holds) and $R(\tau(t))\geq \tau(t)-t_0\geq k\tau(t)$ for $0<k<1$, and $t\geq t_0\geq 1$.
Then,

$$
\begin{aligned}
\limsup_{t\to\infty}\frac{LR^{\gamma}(\tau(t))}{p(\tau(t))}\int_{t}^{\infty}q(s)\Delta s &\geq \limsup_{t\to\infty}\frac{k^{\gamma}\tau^{\gamma}(t)}{\tau^{\gamma-1}(t)}\int_{t}^{\infty}\frac{\lambda}{s\sigma(s)}\Delta s \\
&= \lambda\limsup_{t\to\infty}k^{\gamma}\tau(t)\int_{t}^{\infty}\left(\frac{-1}{s}\right)^{\Delta}\Delta s=\frac{\lambda k^{\gamma}\tau(t)}{t}>1,
\end{aligned}
\tag{5.10}
$$

if $\lambda > t/k^{\gamma}\tau(t)$. Then by Theorem 4, every solution of (5.8) is oscillatory if $\lambda > t/k^{\gamma}\tau(t)$.

Remarks 1. (1) The recent results due to Hassan [15], Grace et al. [11] and Agarwal et al. [7] cannot be applied to (5.1), (5.4), and (5.8) as they deal with ordinary equations without delay.

(2) If $0<\gamma\leq 1$, the results of Sun et al. [12] cannot be applied to (5.1) and (5.4).

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