Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 846736, 13 pages
doi:10.1155/2011/846736

Research Article

On the Composition and Neutrix Composition of the Delta Function with the Hyperbolic Tangent and Its Inverse Functions

Brian Fisher[1] and Adem Kılıçman[2]

[1] *Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK*
[2] *Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, Serdang, 43400 Selangor, Malaysia*

Correspondence should be addressed to Adem Kılıçman, akilicman@putra.upm.edu.my

Received 20 April 2011; Accepted 13 June 2011

Academic Editor: Andrew Pickering

Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. The composition $F(f(x))$ of F and f is said to exist and be equal to the distribution $h(x)$ if the limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. It is proved that the neutrix composition $\delta^{(rs-1)}((\tanh x_+)^{1/r})$ exists and $\delta^{(rs-1)}((\tanh x_+)^{1/r}) = \sum_{k=0}^{s-1}\sum_{i=0}^{K_k}((-1)^k c_{s-2i-1,k}(rs)!/2sk!)\delta^{(k)}(x)$ for $r, s = 1, 2, \ldots$, where K_k is the integer part of $(s-k-1)/2$ and the constants $c_{j,k}$ are defined by the expansion $(\tanh^{-1}x)^k = \{\sum_{i=0}^{\infty}(x^{2i+1}/(2i+1))\}^k = \sum_{j=k}^{\infty} c_{j,k}x^j$, for $k = 0, 1, 2, \ldots$. Further results are also proved.

1. Introduction

In the following, we let $\mathfrak{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathfrak{D}[a,b]$ be the space of infinitely differentiable functions with support contained in the interval $[a,b]$. A *distribution* is a continuous linear functional defined on $\mathfrak{D}$. The set of all distributions defined on $\mathfrak{D}$ is denoted by $\mathfrak{D}'$ and the set of all distributions defined on $\mathfrak{D}[a,b]$ is denoted by $\mathfrak{D}'[a,b]$.

Now let $\rho(x)$ be a function in $\mathfrak{D}[-1,1]$ having the following properties:

(i) $\rho(x) \geq 0$,

(ii) $\rho(x) = \rho(-x)$,

(iii) $\int_{-1}^{1}\rho(x)dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is an arbitrary distribution in $\mathfrak{D}'$ and $F_n(x) = F(x) * \delta_n(x) = \langle F(x-t), \varphi(t)\rangle$, then $\{F_n(x)\}$ is a regular sequence converging to $F(x)$.

Since the theory of distributions is a linear theory, we can extend some operations which are valid for ordinary functions to space of distributions; such operations may be called regular operations, and among them are addition and multiplication by scalars, see [1]. Other operations can be defined only for particular class of distributions; these may be called irregular, and among them are multiplication of distributions, see [2], and composition [3, 4], convolution products, see [5–7], further in [8], where some singular integrals were defined as distributions. Note that it is a difficult task to give a meaning to the expression $F(f(x))$, if F and f are singular distributions.

Thus there have been several attempts recently to define distributions of the form $F(f(x))$ in $\mathfrak{D}'$, where F and f are distributions in $\mathfrak{D}'$, see for example [9–12]. In the following, we are going to consider an alternative approach. As a starting point, we look at the following definition which is a generalization of Gel'fand and Shilov's definition of the composition involving the delta function [13], and was given in [10].

Definition 1.1. Let F be a distribution in $\mathfrak{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b), with $-\infty < a < b < \infty$, if

$$N - \lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle, \tag{1.1}$$

for all φ in $\mathfrak{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [14], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda}\ln^{r-1} n, \quad \ln^{r}\mathrm{n}\colon\ \lambda > 0,\ r = 1, 2, \ldots \tag{1.2}$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle, \tag{1.3}$$

for all φ in $\mathfrak{D}[a, b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$. If f, g are two distributions then in the ordinary sense the composition $f(g)$ does not necessarily exist. Thus the definition of the neutrix composition of distributions was originally given in [10] but was then simply called the composition of distributions.

We also note that, Ng and van Dam applied the neutrix calculus, in conjunction with the Hadamard integral, developed by van der Corput, to the quantum field theories,

in particular, to obtain finite results for the coefficients in the perturbation series. They also applied neutrix calculus to quantum field theory, and obtained finite renormalization in the loop calculations, see [15, 16].

The following two theorems involving derivatives of the Dirac-delta function were proved in [17] and [12], respectively.

Theorem 1.2. *The neutrix composition* $\delta^{(s)}(\operatorname{sgn} x|x|^{\lambda})$ *exists and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = 0, \tag{1.4}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 3, \ldots$ *and*

$$\delta^{(s)}\left(\operatorname{sgn} x|x|^{\lambda}\right) = \frac{(-1)^{(s+1)(\lambda+1)}s!}{\lambda[(s+1)\lambda-1]!}\delta^{((s+1)\lambda-1)}(x), \tag{1.5}$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 2, 4, \ldots$.

Theorem 1.3. *The compositions* $\delta^{(2s-1)}(\operatorname{sgn} x|x|^{1/s})$ *and* $\delta^{(s-1)}(|x|^{1/s})$ *exist and*

$$\begin{aligned}\delta^{(2s-1)}\left(\operatorname{sgn} x|x|^{1/s}\right) &= \frac{1}{2}(2s)!\delta'(x),\\ \delta^{(s-1)}\left(|x|^{1/s}\right) &= (-1)^{s-1}\delta(x),\end{aligned} \tag{1.6}$$

for $s = 1, 2, \ldots$.

The following two theorems were also proved in [18].

Theorem 1.4. *The neutrix composition* $\delta^{(s)}(\ln^{r}(1+|x|))$ *exists and*

$$\delta^{(s)}(\ln^{r}(1+|x|)) = \sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^{s-i}\left[1+(-1)^{k}\right]s!(i+1)^{rs+r-1}}{2r(rs+r-1)!k!}\delta^{(k)}(x), \tag{1.7}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 1, 2, \ldots$.

In particular, the composition $\delta(\ln(1+|x|))$ *exists and*

$$\delta(\ln[1+|x|]) = \delta(x). \tag{1.8}$$

Theorem 1.5. *The neutrix composition* $\delta^{(s)}(\ln(1+|x|^{1/r}))$ *exists and*

$$\delta^{(s)}\left(\ln\left(1+|x|^{1/r}\right)\right) = \sum_{k=0}^{m-1}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{r+s+i-1}\left[1+(-1)^{k}\right]r(i+1)^{s}}{2k!}\delta^{(k)}(x), \tag{1.9}$$

for $s = 0, 1, 2, \ldots$ *and* $r = 2, 3, \ldots$, *where* m *is the smallest nonnegative integer greater than* $(s-r+1)r^{-1}$.

In particular, the composition $\delta^{(s)}(\ln(1+|x|^{1/r}))$ *exists and*

$$\delta^{(s)}\left(\ln\left(1+|x|^{1/r}\right)\right)=0, \tag{1.10}$$

for $s=0,1,2,\ldots,r-2$ *and* $r=2,3,\ldots$ *and*

$$\delta^{(r-1)}\left(\ln\left(1+|x|^{1/r}\right)\right)=(-1)^{r-1}r!\delta(x), \tag{1.11}$$

for $r=2,3,\ldots$.

The following two theorems were proved in [4].

Theorem 1.6. *The neutrix composition* $\delta^{(s)}((\sinh^{-1}x_+)^r)$ *exists and*

$$\delta^{(s)}\left(\left(\sinh^{-1}x_+\right)^r\right)=\sum_{k=0}^{sr+r-1}\sum_{i=0}^{k}\binom{k}{i}\frac{(-1)^k r c_{s,k,i}}{2^{k+1}k!}\delta^{(k)}(x), \tag{1.12}$$

for $s=0,1,2,\ldots$ *and* $r=1,2,\ldots$, *where*

$$c_{r,s,k,i}=\frac{(-1)^s s!\left[(k-2i+1)^{rs+r-1}+(k-2i-1)^{rs+r-1}\right]}{2(rs+r-1)!}. \tag{1.13}$$

In particular, the neutrix composition $\delta(\sinh^{-1}x_+)$ *exists and*

$$\delta\left(\sinh^{-1}x_+\right)=\frac{1}{2}\delta(x). \tag{1.14}$$

Theorem 1.7. *The neutrix composition* $\delta^{(2s-1)}(\sinh^{-1}(\operatorname{sgn}x\cdot x^2))$ *exists and*

$$\delta^{(2s-1)}\left(\sinh^{-1}\left(\operatorname{sgn}x\cdot x^2\right)\right)=\sum_{k=0}^{2s-1}\sum_{i=0}^{i+k+1}\binom{k}{i}\frac{(-1)^k b_{s,k,i}}{2^{k+1}(2k+1)!}\delta^{(k)}(x), \tag{1.15}$$

for $s=1,2,\ldots$, *where*

$$b_{s,k,i}=(k-2i+1)^{2s-1}+(k-2i-1)^{2s-1}. \tag{1.16}$$

In particular

$$\delta'\left(\sinh^{-1}\left(\operatorname{sgn}x\cdot x^2\right)\right)=\frac{\delta'(x)}{4.3!}-2\delta(x). \tag{1.17}$$

2. Main Result

In the next theorem, the constants $c_{j,k}$ are defined by the expansion

$$\left(\tanh^{-1}x\right)^k = \left\{\sum_{i=0}^{\infty}\frac{x^{2i+1}}{2i+1}\right\}^k = \sum_{j=k}^{\infty}c_{j,k}x^j, \tag{2.1}$$

for $k = 0, 1, 2, \ldots$.

Theorem 2.1. *The neutrix composition $\delta^{(rs-1)}((\tanh x_+)^{1/r})$ exists and*

$$\delta^{(rs-1)}\left((\tanh x_+)^{1/r}\right) = \sum_{k=0}^{s-1}\sum_{i=0}^{K_k}\frac{(-1)^k c_{s-2i-1,k}(rs)!}{2sk!}\delta^{(k)}(x), \tag{2.2}$$

for $r, s = 1, 2, \ldots$, where K_k is the integer part of $(s-k-1)/2$.

In particular, the neutrix compositions $\delta^{(r-1)}((\tanh x_+)^{1/r})$ and $\delta^{(2r-1)}((\tanh x_+)^{1/r})$ exist and

$$\delta^{(r-1)}\left((\tanh x_+)^{1/r}\right) = \frac{(-1)^{r-1}r!}{2}\delta(x), \tag{2.3}$$

$$\delta^{(2r-1)}\left((\tanh x_+)^{1/r}\right) = \frac{(2r)!}{4}\left[\delta(x) - \delta'(x)\right], \tag{2.4}$$

for $r = 1, 2, \ldots$.

Proof. To prove (2.2), first of all we evaluate

$$\int_{-1}^{1}\delta_n^{(rs-1)}\left[(\tanh x_+)^{1/r}\right]x^k dx. \tag{2.5}$$

We have

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(rs-1)}\left[(\tanh x_+)^{1/r}\right]x^k dx &= n^{rs}\int_{-1}^{1}\rho^{(rs-1)}\left[n(\tanh x_+)^{1/r}\right]x^k dx\\ &= n^{rs}\int_{0}^{1}\rho^{(rs-1)}\left[n(\tanh x)^{1/r}\right]x^k dx\\ &\quad + n^{rs}\int_{-1}^{0}\rho^{(rs-1)}(0)x^k dx\\ &= I_{1,k} + I_{2,k}.\end{aligned} \tag{2.6}$$

It is obvious that

$$N-\lim_{n\to\infty} I_{2,k} = N-\lim_{n\to\infty}\int_{-1}^{0}\delta_n^{(rs-1)}\left[(\tanh x_+)^{1/r}\right]x^k dx = 0, \tag{2.7}$$

for $k = 0, 1, 2, \ldots$.

Making the substitution $t = n(\tanh x)^{1/r}$, we have for large enough n

$$\begin{aligned} I_{1,k} &= rn^{rs-r}\int_0^1 t^{r-1}\left[\tanh^{-1}\left(\frac{t}{n}\right)^r\right]^k\left[1-\left(\frac{t}{n}\right)^{2r}\right]^{-1}\rho^{(rs-1)}(t)dt \\ &= r\sum_{i=0}^{\infty}\sum_{j=k}^{\infty}\int_0^1 \frac{c_{j,k}t^{rj+2ri+r-1}}{n^{rj+2ri+r-rs}}\rho^{(rs-1)}(t)dt. \end{aligned} \tag{2.8}$$

It follows that

$$\begin{aligned} N-\lim_{n\to\infty} I_{1,k} &= N-\lim_{n\to\infty} n^{rs}\int_0^1 \rho^{(rs-1)}\left[n(\tanh x)^{1/r}\right]x^k dx \\ &= r\sum_{i=0}^{k_0}\int_0^1 c_{s-2i-1,k}t^{rs-1}\rho^{(rs-1)}(t)dt \end{aligned} \tag{2.9}$$

$$= \sum_{i=0}^{k_0}\frac{(-1)^{rs-1}rc_{s-2i-1,k}(rs-1)!}{2}, \tag{2.10}$$

for $k = 0, 1, 2, \ldots, s-1$, where k_0 denotes the integer part of $(s-k-1)/2$ for $k = 0, 1, 2, \ldots$.

In particular, when $s = 1$, we have $k = 0 = k_0$. It follows from (2.9) that

$$I_{1,0} = r\int_0^1 c_{0,0}t^{2ri+r-1}\rho^{(r-1)}(t)dt, \tag{2.11}$$

and so ordinary limit exists

$$\lim_{n\to\infty} I_{1,0} = r\int_0^1 t^{r-1}\rho^{(r-1)}(t)dt = \frac{(-1)^{r-1}r!}{2}, \tag{2.12}$$

for $r = 1, 2, \ldots$, since $c_{0,0} = 1$.

Further, when $s = 2$, we have $k = 0, 1$ and $k_0 = 0$. It follows from (2.10) that

$$\begin{aligned} N-\lim_{n\to\infty} I_{1,0} &= \frac{rc_{1,0}(2r-1)!}{2} = \frac{(2r)!}{4}, \\ N-\lim_{n\to\infty} I_{1,1} &= \frac{rc_{1,1}(2r-1)!}{2} = \frac{(2r)!}{4}, \end{aligned} \tag{2.13}$$

for $r = 1, 2, \ldots$, since $c_{1,0} = 0$, and $c_{1,1} = 1$.

When $k = s$, we have

$$\begin{aligned}
|I_{1,s}| &= rn^{rs-r}\int_0^1 \left| t^{r-1}\left[\tanh^{-1}\left(\frac{t}{n}\right)^r\right]^s \left[1-\left(\frac{t}{n}\right)^{2r}\right]^{-1} \rho^{(rs-1)}(t)\right| dt \\
&\le rn^{rs-r}\int_0^1 \left| \left[\tanh^{-1}\left(\frac{1}{n}\right)^r\right]^s \left[1-\left(\frac{t}{n}\right)^{2r}\right]^{-1} \rho^{(rs-1)}(t)\right| dt \\
&= rn^{rs-r}\int_0^1 \left| \left[n^{-rs}+O\left(n^{-rs-2r}\right)\right]\rho^{(rs-1)}(t)\right| dt \\
&= O\left(n^{-r}\right).
\end{aligned} \tag{2.14}$$

Thus, if ψ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right] x^s\psi(x)dx = 0. \tag{2.15}$$

We also have

$$\int_{-1}^0 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\psi(x)dx = n^{rs}\int_{-1}^0 \rho^{(rs-1)}(0)\psi(x)dx, \tag{2.16}$$

and it follows that

$$N-\lim_{n\to\infty}\int_{-1}^0 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\psi(x)dx = 0. \tag{2.17}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^s}{s!}\varphi^{(s)}(\xi x), \tag{2.18}$$

where $0 < \xi < 1$, and so

$$\begin{aligned}
&N-\lim_{n\to\infty}\left\langle \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right], \varphi(x)\right\rangle \\
&\quad= N-\lim_{n\to\infty}\sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^k dx \\
&\qquad+ N-\lim_{n\to\infty}\sum_{k=0}^{s-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^0 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^k dx \\
&\qquad+ \lim_{n\to\infty}\frac{1}{s!}\int_0^1 \delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^s\varphi^{(s)}(\xi x)dx
\end{aligned}$$

$$
\begin{aligned}
&+ N-\lim_{n\to\infty}\frac{1}{s!}\int_{-1}^{0}\delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^s\varphi^{(s)}(\xi x)dx\\
&=\sum_{k=0}^{s-1}\sum_{i=0}^{k_0}\frac{rc_{s-2i-1,k}(rs-1)!}{2}\frac{\varphi^{(k)}(0)}{k!}+0\\
&=\sum_{k=0}^{s-1}\sum_{i=0}^{k_0}\frac{c_{s-2i-1,k}(rs)!}{2s}\frac{(-1)^k}{k!}\left\langle\delta^{(k)}(x),\varphi(x)\right\rangle,
\end{aligned}
\tag{2.19}
$$

on using (2.6), (2.7), (2.10), (2.15), and (2.17). This proves (2.2) on the interval $(-1,1)$.

It is also clear that $\delta^{(rs-1)}((\tanh^{-1}x_+)^{1/r})=0$ for $x>0$ and so (2.2) holds for $x>-1$.

Now suppose that φ is an arbitrary function in $\mathfrak{D}[a,b]$, where $a<b<0$. Then

$$
\int_a^b\delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\varphi(x)dx=n^{rs}\int_a^b\rho^{(rs-1)}(0)\varphi(x)dx,
\tag{2.20}
$$

and so

$$
N-\lim_{n\to\infty}\int_a^b\delta_n^{(rs-1)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]\varphi(x)dx=0.
\tag{2.21}
$$

It follows that $\delta^{(rs-1)}((\tanh^{-1}x_+)^{1/r})=0$ on the interval (a,b). Since a and b are arbitrary, we see that (2.2) holds on the real line.

Equations (2.3) and (2.4) are just particular cases of (2.2). Equation (2.3) follows on using (2.12) and (2.4) follows on using (2.13). This completes the proof of the theorem. □

Corollary 2.2. *The neutrix composition* $\delta^{(rs-1)}((\tanh|x|)^{1/r})$ *exists and*

$$
\delta^{(rs-1)}\left((\tanh|x|)^{1/r}\right)=\sum_{k=0}^{s-1}\sum_{i=0}^{K_k}\frac{\left[1+(-1)^k\right]rc_{s-2i-1,k}(rs-1)!}{2k!}\delta^{(k)}(x),
\tag{2.22}
$$

for $r,s=1,2,\ldots$, *where* K_k *is the integer part of* $(s-k-1)/2$.

In particular, the composition $\delta^{(r-1)}((\tanh|x|)^{1/r})$ *exists and the neutrix composition* $\delta^{(2r-1)}((\tanh|x|)^{1/r})$ *exists and*

$$
\begin{aligned}
\delta^{(r-1)}\left((\tanh|x|)^{1/r}\right)&=(-1)^{r-1}r!\delta(x),\\
\delta^{(2r-1)}\left((\tanh|x|)^{1/r}\right)&=\frac{(2r)!}{2}\delta(x),
\end{aligned}
\tag{2.23}
$$

for $r=1,2,\ldots$.

Proof. To prove (2.22), we note that

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(rs-1)}\left[\left(\tanh^{-1}|x|\right)^{1/r}\right]x^k dx &= n^{rs}\int_{-1}^{1}\rho^{(rs-1)}\left[\left(n\tanh^{-1}|x|\right)^{1/r}\right]x^k dx\\ &= n^{rs}\left[1+(-1)^k\right]\int_0^1\rho^{(s)}\left[n\left(\tanh^{-1}x\right)^{1/r}\right]x^k dx,\end{aligned} \tag{2.24}$$

and (2.22) now follow as above. Further, (2.23) are particular cases of (2.22) and so follows immediately. Note that in the particular case $s = 1$, the ordinary limit exists in (2.12) and so the composition $\delta^{(r-1)}((\tanh|x|)^{1/r})$ exists in this case. This completes the proof of the corollary. □

In the next theorem, the constants $b_{j,k}$ are defined by the following expansion

$$\tanh^k x = \sum_{j=k}^{\infty} b_{j,k}x^j, \tag{2.25}$$

for $k = 0, 1, 2, \ldots$.

Theorem 2.3. *The neutrix composition $\delta^{(s)}(\tanh^{-1}x_+^{1/r})$ exists and*

$$\delta^{(s)}\left(\tanh^{-1}x_+^{1/r}\right) = \sum_{k=0}^{K-1}\frac{(-1)^{s+k}b_{s+1,kr+r}(s+1)!}{2k!}\delta^{(k)}(x), \tag{2.26}$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where K is the smallest integer for which $s < Kr + r - 1$.

Proof. To prove (2.26), first of all we evaluate

$$\int_{-1}^{1}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx. \tag{2.27}$$

We have

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx &= n^{s+1}\int_{-1}^{1}\rho^{(s)}\left(n\tanh^{-1}x_+^{1/r}\right)x^k dx\\ &= n^{s+1}\int_0^1\rho^{(s)}\left(n\tanh^{-1}x^{1/r}\right)x^k dx + n^{s+1}\int_{-1}^{0}\rho^{(s)}(0)x^k dx\\ &= J_{1,k} + J_{2,k}.\end{aligned} \tag{2.28}$$

It is obvious that

$$N-\lim_{n\to\infty}J_{2,k} = N-\lim_{n\to\infty}\int_{-1}^{0}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx = 0. \tag{2.29}$$

Making the substitution $t = n(\tanh^{-1}x^{1/r})$, we have for large enough n

$$\begin{aligned} J_{1,k} &= rn^s \int_0^1 \tanh^{kr+r-1}\left(\frac{t}{n}\right)\operatorname{sech}^2\left(\frac{t}{n}\right)\rho^{(s)}(t)dt \\ &= \frac{n^{s+1}}{k+1}\int_0^1 \rho^{(s)}(t)d\tanh^{kr+r}\left(\frac{t}{n}\right) \\ &= -\frac{n^{s+1}}{k+1}\int_0^1 \tanh^{kr+r}\left(\frac{t}{n}\right)\rho^{(s+1)}(t)dt \\ &= -\frac{1}{k+1}\sum_{j=kr+r}^{\infty} n^{s-j+1}\int_0^1 b_{j,kr+r}t^j\rho^{(s+1)}(t)dt, \end{aligned} \tag{2.30}$$

and it follows that

$$\begin{aligned} N-\lim_{n\to\infty} J_{1,k} &= N-\lim_{n\to\infty} rn^s \int_0^1 \tanh^{kr+r-1}\left(\frac{t}{n}\right)\operatorname{sech}^2\left(\frac{t}{n}\right)\rho^{(s)}(t)dt \\ &= -\frac{b_{s+1,kr+r}}{k+1}\int_0^1 t^{s+1}\rho^{(s+1)}(t)dt \\ &= \frac{(-1)^s b_{s+1,kr+r}(s+1)!}{2(k+1)}. \end{aligned} \tag{2.31}$$

In particular, when $s = 0$ and $r = 1$, we have $K = 1$ and then

$$\lim_{n\to\infty} J_{1,0} = \frac{b_{1,1}}{2} = \frac{1}{2}. \tag{2.32}$$

When $k = K$, we have

$$\begin{aligned} |J_{1,k}| &\le \frac{1}{K+1}\sum_{j=Kr+r}^{\infty} n^{s-j+1}\int_0^1 \left|b_{j,Kr+r}t^j\rho^{(s+1)}(t)dt\right| \\ &= O\left(n^{s-Kr-r+1}\right). \end{aligned} \tag{2.33}$$

Thus, if ψ is an arbitrary continuous function, then

$$\int_0^1 \delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^K\psi(x)dx = O\left(n^{s-Kr-r+1}\right), \tag{2.34}$$

and so

$$\lim_{n\to\infty}\int_0^1 \delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)\, x^K\psi(x)dx = \lim_{n\to\infty} rn^s\int_0^1 \tanh^{Kr+r-1}\left(\frac{t}{n}\right)\operatorname{sech}^2\left(\frac{t}{n}\right)\rho^{(s)}(t)\psi(t)dt = 0, \tag{2.35}$$

since $s - Kr - r + 1 < 0$.

We also have

$$\int_{-1}^{0} \delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)\psi(x)dx = n^{s+1}\int_{-1}^{0}\rho^{(s)}(0)\psi(x)dx, \tag{2.36}$$

and it follows that

$$N-\lim_{n\to\infty}\int_{-1}^{0}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)\psi(x)dx = 0. \tag{2.37}$$

If now φ is an arbitrary function in $\mathfrak{D}[-1,1]$, then by Taylor's Theorem, we have

$$\varphi(x) = \sum_{k=0}^{K-1}\frac{\varphi^{(k)}(0)}{k!}x^k + \frac{x^K}{K!}\varphi^{(K)}(\xi x), \tag{2.38}$$

where $0<\xi<1$, and so

$$\begin{aligned}
&N-\lim_{n\to\infty}\left\langle \delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right),\varphi(x)\right\rangle \\
&\quad= N-\lim_{n\to\infty}\sum_{k=0}^{K-1}\frac{\varphi^{(k)}(0)}{k!}\int_0^1\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^k dx \\
&\qquad+ N-\lim_{n\to\infty}\sum_{k=0}^{K-1}\frac{\varphi^{(k)}(0)}{k!}\int_{-1}^{0}\delta_n^{(s)}\left[\left(\tanh^{-1}x_+\right)^{1/r}\right]x^k dx \\
&\qquad+ \lim_{n\to\infty}\frac{1}{K!}\int_0^1\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^K\varphi^{(K)}(\xi x)dx \\
&\qquad+ N-\lim_{n\to\infty}\frac{1}{K!}\int_{-1}^{0}\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)x^K\varphi^{(K)}(\xi x)dx \\
&\quad= \sum_{k=0}^{K-1}\frac{(-1)^s b_{s+1,kr+r}(s+1)!\varphi^{(k)}(0)}{2k!}+0 \\
&\quad= \sum_{k=0}^{K-1}\frac{(-1)^{s+k}b_{s+1,kr+r}(s+1)!}{2k!}\left\langle\delta^{(k)}(x),\varphi(x)\right\rangle,
\end{aligned} \tag{2.39}$$

on using (2.9) and (2.12). This proves (2.26) on the interval $(-1,1)$. It is clear that $\delta^{(s)}(\tanh^{-1}x_+^{1/r})=0$ for $x>0$ and so (2.2) holds for $x>-1$.

Now suppose that φ is an arbitrary function in $[a,b]$, where $a<b<0$. Then

$$\int_a^b\delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)\varphi(x)dx = n^{s+1}\int_a^b\rho^{(s)}(0)\varphi(x)dx, \tag{2.40}$$

and so

$$N-\lim_{n\to\infty}\int_a^b \delta_n^{(s)}\left(\tanh^{-1}x_+^{1/r}\right)\varphi(x)dx=0. \tag{2.41}$$

It follows that $\delta^{(s)}(\tanh^{-1}x_+^{1/r})=0$ on the interval (a,b). Since a and b are arbitrary, we see that (2.26) holds on the real line. □

Corollary 2.4. *The neutrix composition $\delta^{(s)}(\tanh^{-1}|x|^{1/r})$ exists and*

$$\delta^{(s)}\left(\tanh^{-1}|x|^{1/r}\right)=\sum_{k=0}^{K-1}\frac{\left[(-1)^{s+k}+(-1)^s\right]b_{s+1,kr+r}(s+1)!}{2k!}\delta^{(k)}(x), \tag{2.42}$$

for $s=0,1,2,\ldots$ and $r=1,2,\ldots$, where K is the smallest integer for which $s<Kr+r-1$. In particular, the composition $\delta(\tanh^{-1}|x|)$ exists and

$$\delta\left(\tanh^{-1}|x|\right)=\delta(x). \tag{2.43}$$

Proof. To prove (2.42), we note that

$$\begin{aligned}\int_{-1}^{1}\delta_n^{(s)}\left[\left(\tanh^{-1}|x|\right)^{1/r}\right]x^k dx&=n^{s+1}\int_{-1}^{1}\rho^{(s)}\left[\left(n\tanh^{-1}|x|\right)^{1/r}\right]x^k dx\\&=n^{s+1}\left[(-1)^{s+k}+(-1)^s\right]\int_0^1\rho^{(s)}\left[n\left(\tanh^{-1}x\right)^{1/r}\right]x^k dx,\end{aligned} \tag{2.44}$$

and (2.42) now follows as above. Equation (2.43) is a particular case of (2.42) and so follows immediately. Note that in the particular case $s=0$ and $r=1$, the ordinary limit exists in (2.12), and so the composition $\delta(\tanh|x|)$ exists in this case. For some related results on the neutrix composition of distributions, see [19–22]. □

Acknowledgments

The paper was prepared when the first author visited University Putra Malaysia, therefore the authors gratefully acknowledge that this research was partially supported by the University Putra Malaysia under the Research University Grant Scheme 05-01-09-0720RU. The authors would also like to thank the referees for valuable remarks and suggestions on the previous version of the paper.

References

[1] H. Eltayeb, A. Kılıçman, and B. Fisher, "A new integral transform and associated distributions," *Integral Transforms and Special Functions*, vol. 21, no. 5-6, pp. 367–379, 2010.

[2] B. Fisher and A. Kılıçman, "A commutative neutrix product of ultradistributions," *Integral Transforms and Special Functions*, vol. 4, no. 1-2, pp. 77–82, 1996.

[3] B. Fisher and A. Kılıçman, "On the composition and neutrix composition of the delta function and powers of the inverse hyperbolic sine function," *Integral Transforms and Special Functions*, vol. 21, no. 12, pp. 935–944, 2010.

[4] B. Fisher and A. Kılıçman, "On the neutrix composition of the delta and inverse hyperbolic sine functions," *Journal of Applied Mathematics*, vol. 2011, Article ID 612353, 12 pages, 2011.

[5] A. Kılıçman, "On the commutative neutrix product of distributions," *The Indian Journal of Pure and Applied Mathematics*, vol. 30, no. 8, pp. 753–762, 1999.

[6] A. Kılıçman, "A comparison on the commutative neutrix convolution of distributions and the exchange formula," *Czechoslovak Mathematical Journal*, vol. 51, no. 3, pp. 463–471, 2001.

[7] A. Kılıçman, B. Fisher, and S. Pehlivan, "The neutrix convolution product of $x_+^\lambda \ln^r x_+$ and $x_-^\mu \ln^s x_-$," *Integral Transforms and Special Functions*, vol. 7, no. 3-4, pp. 237–246, 1998.

[8] A. Kılıçman and H. Eltayeb, "A note on defining singular integral as distribution and partial differential equations with convolution term," *Mathematical and Computer Modelling*, vol. 49, no. 1-2, pp. 327–336, 2009.

[9] P. Antosik, "Composition of distributions," Tech. Rep. 9, University of Wisconsin, 1989.

[10] B. Fisher, "On defining the change of variable in distributions," *Rostocker Mathematisches Kolloquium*, no. 28, pp. 33–40, 1985.

[11] B. Fisher, "On defining the distribution $(x_+^r)_-^{-s}$," *Univerzitet u Novom Sadu. Zbornik Radova Prirodno-Matematičkog Fakulteta. Serija za Matemati*, vol. 15, no. 1, pp. 119–129, 1985.

[12] B. Fisher, "The composition and neutrix composition of distributions," in *Mathematical Methods in Engineering*, pp. 59–69, Springer, New York, NY, USA, 2007.

[13] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 1, Academic Press, New York, NY, USA, 1st edition, 1964.

[14] J. G. van der Corput, "Introduction to the neutrix calculus," *Journal d'Analyse Mathématique*, vol. 7, pp. 281–398, 1959.

[15] Y. J. Ng and H. van Dam, "Neutrix calculus and finite quantum field theory," *Journal of Physics A*, vol. 38, no. 18, pp. L317–L323, 2005.

[16] Y. J. Ng and H. van Dam, "An application of neutrix calculus to quantum field theory," *International Journal of Modern Physics A*, vol. 21, no. 2, pp. 297–312, 2006.

[17] B. Fisher, "The delta function and the composition of distributions," *Demonstratio Mathematica*, vol. 35, no. 1, pp. 117–123, 2002.

[18] B. Fisher, T. Kraiweeradechachai, and E. Özçağ, "Results on the neutrix composition of the delta function," *Hacettepe Journal of Mathematics and Statistics*, vol. 36, no. 2, pp. 147–156, 2007.

[19] B. Fisher, B. Jolevska-Tuneska, and E. Özçağ, "Further results on the compositions of distributions," *Integral Transforms and Special Functions*, vol. 13, no. 2, pp. 109–116, 2002.

[20] B. Fisher, A. Kananthai, G. Sritanatana, and K. Nonlaopon, "The composition of the distributions $x_-^{ms} \ln x_-$ and $x_+^{r-p/m}$," *Integral Transforms and Special Functions*, vol. 16, no. 1, pp. 13–20, 2005.

[21] B. Fisher and K. Taş, "On the composition of the distributions x_+^{-r} and x_+^μ," *The Indian Journal of Pure and Applied Mathematics*, vol. 36, no. 1, pp. 11–22, 2005.

[22] B. Fisher and K. Taş, "On the composition of the distributions $x^{-1}\ln|x|$ and x_+^r," *Integral Transforms and Special Functions*, vol. 16, no. 7, pp. 533–543, 2005.