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Research Article

3-Adic System and Chaos

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Let $(Z(3), \tau)$ be a 3-adic system. we prove in $(Z(3), \tau)$ the existence of uncountable distributional chaotic set of $A(\tau)$, which is an almost periodic points set, and further come to a conclusion that τ is chaotic in the sense of Devaney and Wiggins.

1. Introduction

In 1975, Li and Yorke introduced in [1] a new definition of chaos for interval maps. The central point in their definition is the existence of a scrambled set. Later, it was observed that positive topological entropy of interval map implies the existence of a scrambled set [2]. Many sharpened results come into being in succession (see [3–11]). One can find in [3, 4, 12] equivalent conditions for f to be chaotic and in [13] or [14] a chaotic map with topological entropy zero, which showed that positive topological entropy and Li-Yorke chaos are not equivalent.

By the result, it became clear that the positive topological entropy is a much stronger notion than the definition of chaos in the sense of Li and Yorke. To remove this disadvantage, Zhou [15] introduced the notion of measure center and showed importantly dynamical properties of system on its measure center. To decide the concept of measure center, he defined weakly almost periodic point, too, showing that the closure of a set of weakly almost periodic points equals to its measure center and the set of weakly almost periodic points is a set of absolutely ergodic measure 1. These show that it is more significant to discuss problems on a set of weakly almost periodic points. On the other hand, one important extensions of Li-Yorke definition were developed by Schweizer and Smítal in [16]; this paper introduced the definition of distributional chaos and prove that this notion is equivalent to positive topological entropy for interval maps. And many scholars (such as Liao, Du, and Zhou, Wang) proved that the positive topological entropy of interval map is equivalent to the uncountable Li-Yorke chaotic set and the uncountable distributional chaotic set for $A(f)$, $W(f)$, and $R(f)$. Meanwhile Liao showed that the equivalent characterization is no longer valid when f acts on more general compact metric spaces.

In this paper, we discuss the existence of uncountable distributional chaotic set of $A(f)$ in 3-adic system.

The main results are stated as follows.

Main Theorem. Let $(Z(3), \tau)$ be a 3-adic system. Then

(1) $A(\tau)$ contains an uncountable distributional chaotic set of τ;

(2) τ is chaotic in the sense of Devaney;

(3) τ is chaotic in the sense of Wiggins.

2. Basic Definitions and Preparations

Throughout this paper, X will denote a compact metric space with metric d, I is the closed interval [0, 1].

For a continuous map $f\colon X \to X$, we denote the set of almost periodic points of f by $A(f)$ and denote the topological entropy of f by $\mathrm{ent}(f)$, whose definitions are as usual; f^n will denote the n-fold iteration of f.

For x, y in X, any real number t and positive integer n, let

$$\xi_n(f, x, y, t) = \#\left\{i \mid d\left(f^i(x), f^i(y)\right) < t, 1 \leq i \leq n\right\}, \tag{2.1}$$

where we use $\#(\cdot)$ to denote the cardinality of a set. Let

$$F(f, x, y, t) = \liminf_{n\to\infty} \frac{1}{n} \xi_n(f, x, y, t) \qquad F^*(f, x, y, t) = \limsup_{n\to\infty} \frac{1}{n} \xi_n(f, x, y, t). \tag{2.2}$$

Definition 2.1. Call $x, y \in X$ a pair of points displaying distributional chaos, if

(1) $F(f, x, y, t) = 0$ for some $t > 0$;

(2) $F(f, x, y, t) = 1$ for any $t > 0$.

Definition 2.2. f is said to display distributional chaos, if there exists an uncountable set $D \subset X$ such that any two different points in D display distributional chaos.

Definition 2.3. Let X be a metric space and $f : X \to X$ be a continuous map. The dynamical system (X, f) is called chaotic in the sense of Devaney, if

(1) (X, f) is transitive;

(2) the periodic points are dense in X;

(3) (X, f) is sensitive to initial conditions.

Definition 2.4. Let X be a metric space and $f : X \to X$ be a continuous map. The dynamical system (X, f) is called chaotic in the sense of Wiggins, if there exists a compact invariant subset $Y \subset X$ such that

(1) $f|_Y$ is sensitive to initial conditions;

(2) $f|_Y$ is transitive.

Definition 2.5. Let (X, f) and (Y, g) be dynamical systems; if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, then f and g are said to be topologically conjugate.

The notion of adic system is defined as follows.

Definition 2.6. Put

$$Z(3) = \left\{ \sum_{i=1}^{\infty} a_i 3^{i-1} \mid a_i = 0, 1, 2 \right\}. \tag{2.3}$$

We use the sequence $a = a_1 a_2 \cdots$ to denote simply the member $\sum_{i=1}^{\infty} a_i 3^{i-1}$ in $Z(3)$. Define $\rho : Z(3) \times Z(3) \to R$ as follows: for any $a, b \in Z(3)$, if $a = a_1 a_2 \cdots$, $b = b_1 b_2 \cdots$, then

$$\rho(a, b) = \begin{cases} 0, & \text{if } a = b, \\ \dfrac{1}{3^k}, & \text{if } a \neq b,\ k = \min\{m \geq 1 \mid a_m \neq b_m\}. \end{cases} \tag{2.4}$$

It is not difficult to check that ρ is a metric on $Z(3)$ and $(Z(3), \rho)$ is a compact abelian group. Define $\tau : Z(3) \to Z(3)$ by $\tau(a) = a + 1$ for $a = a_1 a_2 \cdots \in Z(3)$; τ or $(Z(3), \tau)$ is called the 3-adic system. (see [17])

Call an invariant closed set $A \subset I$ 3-adic, if the restriction $f|_A$ is topologically conjugate to the 3-adic system.

Consider the following functional equation:

$$\begin{aligned} &f^3(\lambda x) = \lambda f(x), \\ &f(0) = 1, \quad 0 \leq f(x) \leq 1, \end{aligned} \tag{2.5}$$

where $\lambda \in$(0,1) is to be determined, $x \in [0, 1]$ and f^3 is the 3-fold iteration of f.

By $\mathcal{F}$ we denote the set of continuous solutions of (2.5) such that any $f \in \mathcal{F}$ satisfies: (p_1) there exists $\alpha \in (\lambda, 1)$ such that $f(\alpha) = 0$; the restrictions $f|_{[\lambda,\alpha]}$ and $f|_{[\alpha,1]}$ are both once continuously differentiable, and $f'(x) \geq 1$ on $[\alpha, 1]$, $f'(x) < -2$ on $[\lambda, \alpha]$; $(p_2) f(\lambda\alpha) < f(\lambda)$.

The following Lemma can be concluded by in [18, Theorem 2.1].

Lemma 2.7. *Let $0 < \lambda < 1, \alpha \in (\lambda, 1)$. Let $f_0 : [\lambda, 1] \to [0, 1]$ be C^1 on each of the interval $[\lambda, \alpha]$ and $[\alpha, 1]$, and satisfy*

(1) *$f_0(\alpha) = 0$;*

(2) *$f_0'(x) < -2$ on $[\lambda, \alpha]$ and $f_0'(x) \geq 1$ on $[\alpha, 1]$;*

(3) *there exists $\alpha_0 \in (\alpha, 1)$ such that $f_0(\alpha_0) = \alpha$ and $\alpha < f_0(1) < \alpha_0 < f_0(\lambda) < 1$;*

(4) *$f_0^2(1) = \lambda$, $f_0^3(\lambda) = \lambda f_0(1)$.*

Then there exists a unique $f \in \mathcal{F}$ with $f|_{[\lambda,1]} = f_0$. Conversely, if f_0 is the restriction on $[\lambda, 1]$ of some $f \in \mathcal{F}$, then it must satisfy (1)–(4).

Proposition 2.8. $\mathcal{F} \neq \emptyset$.

Proof. Let $\lambda = 2/9$, $\alpha = 1/2$. Define $f_0 : [\lambda, 1] \to [0,1]$ by

$$f_0(x) = \begin{cases} \dfrac{27}{16} - \dfrac{27x}{8}, & \dfrac{2}{9} \leq x \leq \dfrac{1}{2}, \\ \dfrac{4x}{3} - \dfrac{2}{3}, & \dfrac{1}{2} \leq x \leq 1. \end{cases} \tag{2.6}$$

It is not difficult to check that f_0 satisfies the condition (1)–(4) in Lemma 2.7. So $\mathcal{F} \neq \emptyset$.

We will be concerned in the notions of Hausdorff metric and Hausdorff dimension, whose definitions can be found in [19]. □

Lemma 2.9 (see [19, Theorem 8.3]). *Let $\phi_1, \phi_2, \ldots, \phi_m$ be contractions on R^n. Then there exists a unique nonempty compact set E such that*

$$E = \phi(E) = \bigcup_{i=1}^{m} \phi_i(E), \tag{2.7}$$

where

$$\phi = \bigcup_{i=1}^{m} \phi_i \tag{2.8}$$

is a transformation of subsets of R^n. Furthermore, for any nonempty compact subset F of R^n, the iterates $\phi^k(F)$ converge to E in the Hausdorff metric as $k \to \infty$.

Lemma 2.10 (see [19, Theorem 8.8]). *Let $\{\phi_i\}_1^m$ be contractions on R for which the open set condition holds; that is, there is an open interval V such that*

(1) $\phi(V) = \bigcup_{i=1}^{m} \phi_i(V) \subset V$,

(2) $\phi_1(V), \phi_2(V), \ldots, \phi_m(V)$ *are pairwise disjoint.*

Moreover, suppose that for each i, there exists r_i, such that $|\phi_i(x) - \phi_i(y)| \leq r_i|x - y|$ for all $x, y \in \overline{V}$. Then $\dim E \leq t$, where $\dim(\cdot)$ denotes the Hausdorff dimension and t is defined by

$$\sum_{i=1}^{m} r_i^t = 1. \tag{2.9}$$

Lemma 2.11 (see [20, Theorem 3.2], [21]). *Let $f : I \to I$ be continuous. Then the followings are equivalent:*

(1) $\mathrm{ent}(f) > 0$;

(2) *$A(f)$ contains an uncountable distributional chaotic set of f.*

Lemma 2.12 (see [21]). *Let $f : X \to X$, $g : Y \to Y$ be continuous, where X, Y are compact metric spaces. If there exists a continuous surjection $h : X \to Y$ such that $g \circ h = h \circ f$, then $h(A(f)) = A(g)$.*

Lemma 2.13 (see [22]). *Let $\mathrm{ent}(f) = 0$ and $x \in I$ be recurrent but not periodic such that $f(x) > x$. Then the inequality $f^m(x) < f^n(x)$ holds for all even m and all odd n.*

Lemma 2.14 (see [23, Theorem 6.1.4]). *Let $f : I \to I$ be an interval map. Then $\mathrm{ent}(f) > 0$ if and only if there exists a closed invariant subset $\wedge \subset I$ such that $f|_\wedge$ is chaotic in the sense of Devaney.*

Lemma 2.15 (see [23, Theorem 6.2.4]). *Let $f : I \to I$ be an interval map. If $\mathrm{ent}(f) > 0$, then f is chaotic in the sense of Wiggins.*

3. Proof of Main Theorem

In the sequel, we always suppose that $f \in \mathcal{F}$ and f take the minimum at $\alpha \in (\lambda, 1)$.
Let $I = [0,1]$, $f_+ = f|_{[\alpha,1]}$. For $i = 0,1,2$, define $\phi_i : I \to I$ by $\phi_2(x) = \lambda x$, $\phi_1(x) = f_+^{-1}(\phi_2(x))$, $\phi_0(x) = f_+^{-1}(\phi_1(x))$. Then ϕ_i is a contraction for every $i = 0,1,2$. Let $\phi(x) = \bigcup_{i=0}^{2} \phi_i(x)$. By Lemma 2.9, there exists a unique nonempty compact set E with

$$\phi(E) = E. \tag{3.1}$$

For simplicity, we write $\phi_{i_1 \cdots i_k}$ for $\phi_{i_1} \circ \phi_{i_2} \circ \cdots \circ \phi_{i_k}$.

Step 1. Prove that for any $x \in I$, $f \circ \phi_0(x) = \phi_1(x)$, $f \circ \phi_1(x) = \phi_2(x)$, $f \circ \phi_2(x) = \phi_0 \circ f(x)$.

Proof. Letting f act on both sides of the equality $\phi_0(x) = f_+^{-1}(\phi_1(x))$, we get immediately the first equality. A similar argument yields the second equality. To show the third equality, we write (2.5) as $f(f(f(\phi_2(x)))) = \lambda f(x)$. Since $\phi_2(x) \in [0,\lambda]$, it follows from Lemma 2.7 that $f \circ \phi_2(x) \in [\alpha, 1]$ and $f^2 \circ \phi_2(x) \in [\alpha, 1]$. By this and definitions of ϕ_0 and ϕ_1, we get

$$f \circ \phi_2(x) = f_+^{-1}\left(f_+^{-1}(\lambda f(x))\right) = f_+^{-1}(\phi_1(f(x))) = \phi_0 \circ f(x). \tag{3.2}$$

□

Step 2. Prove that for any subsets $\phi_{i_1 \cdots i_k}(I)$ and $\phi_{j_1 \cdots j_k}(I)$, there is an $n > 0$ such that $f^n \circ \phi_{i_1 \cdots i_k}(I) = \phi_{j_1 \cdots j_k}(I)$.

Proof. If $x \in I$, $i = 0,1,2$, then $f^3 \circ \phi_i(x) = \phi_i \circ f(x)$ by Step 1. Using this repeatedly, we get for any $k > 0$

$$f^{3^k} \circ \phi_i(x) = \phi_i \circ f^{3^{k-1}}(x). \tag{3.3}$$

If for each $r = 1, 2, \ldots, k$, we all have $i_r = j_r$, then from (3.3),

$$f^{3^k} \circ \phi_{i_1 \cdots i_k}(I) = \phi_{i_1} \circ f^{3^{k-1}} \circ \phi_{i_2 \cdots i_k}(I) = \cdots = \phi_{i_1 \cdots i_k} \circ f(I) = \phi_{j_1 \cdots j_k}(I) \tag{3.4}$$

(nothing that $f(I) = I$). Thus the lemma holds for this special case. Assume that there exists some r, $1 \leq r \leq k$, such that $i_q = j_q$ for $q < r$, but $i_r \leq j_r$. Then by using (3.3) repeatedly, we know that $f^{3^{r-1}} \circ \phi_{i_1\cdots i_k}(I)$ or $f^{2\cdot 3^{r-1}} \circ \phi_{i_1\cdots i_k}(I)$ has the form $\phi_{l_1\cdots l_r l_{r+1}\cdots l_k}(I)$, where $l_q = j_q$ for $q = 1, \ldots, r$. Continuing this procedure, we must get some n, such that $f^n \circ \phi_{i_1\cdots i_k}(I) = \phi_{j_1\cdots j_k}(I)$.

In, Steps 3, 5, and 6, we always suppose that the notation E is as in (3.1). □

Step 3. Prove that

$$E = \bigcap_{k=0}^{\infty} \phi^k(I). \tag{3.5}$$

Proof. Since $\phi(I) \subset I$, we have $\phi^{k+1}(I) = \phi^k \circ \phi(I) \subset \phi^k(I)$ for any $k > 0$. So from Lemma 2.9 we get

$$\bigcap_{k=0}^{\infty} \phi^k(I) = \lim_{k \to \infty} \phi^k(I) = E. \tag{3.6}$$

□

Step 4. Prove that for any $k > 0$, $\phi^k(I) = \bigcup_{i_1\cdots i_k=0}^{2}(I)$ is an invariant set of f, that is, $f(\phi^k(I)) \subset \phi^k(I)$.

Proof. Note that each $\phi_{i_1\cdots i_k}$ has the form $\phi_{22\cdots 2}$ or $\phi_{22\cdots 20 i_r\cdots i_k}$ or $\phi_{22\cdots 21 i_r\cdots i_k}$. Then, by using Step 1 repeatedly, we have

$$f \circ \phi_{22\cdots 2} = \phi_{00\cdots 0} \circ f, \qquad f \circ \phi_{22\cdots 20 i_r\cdots i_k} = \phi_{00\cdots 01 i_r\cdots i_k}, \qquad f \circ \phi_{22\cdots 21 i_r\cdots i_k} = \phi_{00\cdots 2 i_r\cdots i_k}. \tag{3.7}$$

Thus by $f(I) \subset I$, we have $f \circ \phi_{i_1\cdots i_k}(I) \subset \phi^k(I)$. Moreover,

$$f\left(\phi^k(I)\right) \subset \bigcup_{i_1\cdots i_k=0}^{2} f \circ \phi_{i_1\cdots i_k}(I) \subset \phi^k(I). \tag{3.8}$$

□

Step 5. Prove that the restriction $f|_E$ is topologically conjugate to τ, where τ is the 3-adic system as defined in Section 1.

Proof. By the definition of ϕ, we have $\phi(I) = \bigcup_{i=0}^{2} \phi_i(I)$ with this union disjoint. Then transforming by $\phi_{i_1\cdots i_k}$,

$$\bigcup_{i=0}^{2} \phi_{i_1\cdots i_k i}(I) \subset \phi_{i_1\cdots i_k}(I) \tag{3.9}$$

again with a disjoint union. Thus the sets $\{\phi_{i_1\cdots i_k}(I)\}$ (with k arbitrary) form a net in the sense that any pair of sets from the collection are either disjoint or such that one is included in the other. It follows from Step 3 that for any $a = a_1 a_2 \cdots \in Z(3)$, if let

$$\phi_a(I) = \bigcap_{k=1}^{\infty} \phi_{a_1\cdots a_k}(I), \tag{3.10}$$

then $\phi_a(I) \subset E$ is nonempty, and if $x \in E$, then there exists a unique $a \in Z(3)$ with $x \in \phi_a(I)$.

We now define a map H of E onto $Z(3)$ by setting $H(x) = a$ if $x \in \phi_a(I)$. Then H is well defined. It is easy to see that for each $i = 0, 1, 2$, the contraction ratio of $\phi_i \leq \lambda$, so the contraction ratio of $\phi_{i_1 \cdots i_k} \leq \lambda^k$. It follows that diam $\phi_{i_1 \cdots i_k}(I)$ converges to zero uniformly for $i_r \in \{0, 1, 2\}$ as $k \to \infty$ (where diam denotes diameter). Thus $\phi_a(I)$ is a single point for each $a \in Z(3)$. And so H is injective. Moreover the map H is continuous. Let $\delta_k > 0$ be the least distance between any two of the 3^k interval $\phi_{a_1 \cdots a_k}(I)$. If $x \in \phi_\alpha(I)$, $y \in \phi_\beta(I)$, and $|x - y| < \delta$, then $\rho(\alpha, \beta) < 3^{-k}$. Finally, since $f(\phi_a(I)) = \phi_{\tau(a)}(I)$ by (3.7), we have $H \circ f(x) = \tau \circ H(x)$ for each $x \in E$. □

Step 6. Prove that if f has an n-adic set and the n is not a power of 2, then $\mathrm{ent}(f) > 0$.

Proof. Write $n = k \cdot 2^m$, where $k \geq 3$ is odd and $m \geq 0$ is an integer. Let A be the n-adic set of f and $p = \min A$. There exists a homeomorphism $H : A \to Z(n)$ such that for $x \in A$, $\tau \circ H(x) = H \circ f(x)$. We may assume without loss of generality that $H(p) = a = 0a_2a_3 \cdots$. Put

$$V = \{z \in Z(n) \mid z_1 = 0\}. \tag{3.11}$$

Then $V \subset Z(n)$ is an open neighborhood of the sequence a. There exists an $\varepsilon > 0$, such that for any $q \in A$, if $q - p < \varepsilon$, there $H(q) \in V$. Note that for $l \to \infty, \tau^{n^l}(a) \to a$ and furthermore $f^{n^l}(p) \to p$, we have that there exists an $l \geq 0$ such that

$$f^{n^l}(p) - p < \varepsilon. \tag{3.12}$$

Let $g = f^{2^{lm}}$. Since we easily see that $H(f^s(p)) = \tau^s(H(p)) \in V$ if and only if n divides s, it follows that $H(f^{2^{lm}}(p)) \notin V$, since n can not divide 2^{lm}. And so $g(p) = f^{2^{lm}}(p) \geq p + \varepsilon$. In particular, $g(p) > p$. By the same argument, we also have $g^2(p) = f^{2^{lm+1}}(p) \geq p+\varepsilon$. In particular, $g^2(p) > p$. Since $n^l = (k \cdot 2^m)^l = k^l \cdot 2^{lm}$, from (3.12), $g^{k^l}(p) - p = f^{n^l}(p) - p < \varepsilon$, that is, $g^{k^l}(p) < p + \varepsilon$. Thus we have for the odd k^l,

$$g^{k^l}(p) < g^2(p). \tag{3.13}$$

Note that a is current and nonperiodic for $\tau^{2^{lm}}$, and so is p for g. By Lemma 2.13 we get $\mathrm{ent}(g) > 0$. Moreover $\mathrm{ent}(f) > 0$.

Finally, we prove that $A(\tau)$ contains an uncountable distributional chaotic set of τ. By Step 5, the restriction $f|_E$ is topologically conjugate to τ. Thus there is a homeomorphism $h : Z(3) \to E$ such that for any $x \in Z(3)$,

$$f \circ h(x) = h \circ \tau(x). \tag{3.14}$$

According to Lemma 2.11, there is an uncountable set $\wedge \subset A(f)$, which is distributional chaotic. By Lemma 2.12 for any $y \in \wedge$, there exists $x \in A(\tau)$ such that $h(x) = y$. Let

$$D = \{x \mid x \in A(\tau), h(x) = y,\ y \in \wedge\}. \tag{3.15}$$

Then D is an uncountable set.

To complete the proof, it suffices to show that D is a distributional chaotic set for τ.

First of all, we prove that for any $x_1, x_2 \in D$, if $F(f, h(x_1), h(x_2), t) = 0$ for some $t > 0$, then $F(\tau, x_1, x_2, s) = 0$ for some $s > 0$.

For given $t > 0$, by uniform continuity of h, there exists $s > 0$, such that for any $p, q \in D$, $|h(p) - h(q)| < t$, provided $\rho(p, q) < s$. Since we easily see that $h \circ \tau^i = f^i \circ h$, it follows that if $\rho(\tau^i(x_1), \tau^i(x_2)) < s$, then

$$\left| f^i \circ h(x_1) - f^i \circ h(x_2) \right| < t. \tag{3.16}$$

This implies

$$\xi_n(\tau, x_1, x_2, s) \leq \xi_n(f, h(x_1), h(x_2), t) \tag{3.17}$$

for any $n \geq 0$. Thus by the definition of F, we immediately have the following result:

$$F(\tau, x_1, x_2, s) = 0. \tag{3.18}$$

Secondly, we prove that if $F^*(f, h(x_1), h(x_2), s) = 1$ for all $s > 0$, then $F^*(\tau, x_1, x_2, t) = 1$ for all $t > 0$. Since h is homeomorphism, $h^{-1} : E \to Z(n)$ is a surjective continuous map. By the first proof, we have

$$\xi_n(f, h(x_1), h(x_2), s) \leq \xi_n(\tau, x_1, x_2, t), \tag{3.19}$$

which gives

$$F^*(\tau, x_1, x_2, t) = 1. \tag{3.20}$$

By (3.18), (3.20), and the arbitrariness of x_1 and x_2, we conclude that D is an uncountable distributional chaotic set of τ.

The proofs of (2) and (3) of the Main Theorem are obvious. □

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