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Research Article

A Generalization of Suzuki's Lemma

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Let $\{z_n\}$, $\{w_n\}$, and $\{v_n\}$ be bounded sequences in a metric space of hyperbolic type (X, d), and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$. If $z_{n+1} = \alpha_n w_n \oplus (1 - \alpha_n) v_n$ for all $n \in \mathbb{N}$, $\lim_n d(z_n, v_n) = 0$, and $\limsup_n (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \leq 0$, then $\lim_n d(w_n, z_n) = 0$. This is a generalization of Lemma 2.2 in (T. Suzuki, 2005). As a consequence, we obtain strong convergence theorems for the modified Halpern iterations of nonexpansive mappings in CAT(0) spaces.

1. Introduction

Suppose that (X, d) is a metric space which contains a family $\mathcal{L}$ of metric segments (isometric images of real line segments) such that distinct points $x, y \in X$ lie on exactly one member $S[x, y]$ of $\mathcal{L}$. Let $\alpha \in [0, 1]$, we use the notation $\alpha x \oplus (1 - \alpha) y$ to denote the point of the segment $S[x, y]$ with distance $\alpha d(x, y)$ from y, that is,

$$d(\alpha x \oplus (1 - \alpha) y, y) = \alpha d(x, y). \tag{1.1}$$

We will say that $(X, d, \mathcal{L})$ is of *hyperbolic type* if for each $p, x, y \in X$ and $\alpha \in [0, 1]$,

$$d(\alpha p \oplus (1 - \alpha) x, \alpha p \oplus (1 - \alpha) y) \leq (1 - \alpha) d(x, y). \tag{1.2}$$

It is proved in [1] that (1.2) implies

$$d(p, (1 - \alpha) x \oplus \alpha y) \leq (1 - \alpha) d(p, x) + \alpha d(p, y). \tag{1.3}$$

It is well-known that Banach spaces are of hyperbolic type. Notice also that CAT(0) spaces and hyperconvex metric spaces are of hyperbolic type (see [2, 3]).

In 1983, Goebel and Kirk [4] proved that if $\{z_n\}$ and $\{w_n\}$ are sequences in a metric space of hyperbolic type (X, d) and $\{\alpha_n\} \subset [0,1]$ which satisfy for all $i, n \in \mathbb{N}$, (i) $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n)z_n$, (ii) $d(w_{n+1}, w_n) \le d(z_{n+1}, z_n)$, (iii) $d(w_{i+n}, x_i) \le a < \infty$, (iv) $\alpha_n \le b < 1$, and (v) $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\lim_n d(w_n, z_n) = 0$. It was proved by Suzuki [5] that one obtains the same conclusion if the conditions (i)–(v) are replaced by the conditions (S1)–(S4) as follows:

(S1) $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n)z_n$,

(S2) $\limsup_n (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \le 0$,

(S3) $\{z_n\}$ and $\{w_n\}$ are bounded sequences,

(S4) $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$.

Both Goebel-Kirk's and Suzuki's results have been used to prove weak and strong convergence theorems for approximating fixed points of various types of mappings. The purpose of this paper is to generalize Suzuki's result by relaxing the condition (S1), namely, we can define z_{n+1} in terms of w_n and v_n such that $\lim_n d(z_n, v_n) = 0$. Precisely, we are going to prove the following lemma.

Lemma 1.1. *Let $\{z_n\}$, $\{w_n\}$, and $\{v_n\}$ be bounded sequences in a metric space of hyperbolic type (X, d), and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$. Suppose that $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$, $\lim_n d(z_n, v_n) = 0$, and $\limsup_n (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \le 0$, then $\lim_n d(w_n, z_n) = 0$.*

2. Proof of Lemma 1.1

We begin by proving a crucial lemma.

Lemma 2.1. *Let $\{z_n\}$, $\{w_n\}$, and $\{v_n\}$ be sequences in a metric space of hyperbolic type (X, d), and let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\limsup_n \alpha_n < 1$. Put*

$$r = \limsup_{n\to\infty} d(w_n, z_n) \quad \text{or} \quad r = \liminf_{n\to\infty} d(w_n, z_n). \tag{2.1}$$

Suppose that $r < \infty$, $z_{n+1} = \alpha_n w_n \oplus (1-\alpha_n)v_n$ for all $n \in \mathbb{N}$, $\lim_n d(z_n, v_n) = 0$, and

$$\limsup_{n\to\infty} (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)) \le 0, \tag{2.2}$$

then

$$\liminf_{n\to\infty} |d(w_{n+k}, z_n) - (1 + \alpha_n + \alpha_{n+1} + \cdots + \alpha_{n+k-1})r| = 0 \tag{2.3}$$

holds for all $k \in \mathbb{N}$.

Proof. (This proof is patterned after the proof of [5, Lemma 1.1]). For each $n \in \mathbb{N}$, let $u_n = \alpha_n w_n \oplus (1-\alpha_n) z_n$, then by (1.2), we have

$$d(u_n, z_{n+1}) \leq (1-\alpha_n) d(z_n, v_n) \leq d(z_n, v_n). \tag{2.4}$$

This implies

$$\begin{aligned} d(w_{n+1}, z_{n+1}) &\leq d(w_{n+1}, w_n) + d(w_n, u_n) + d(u_n, z_{n+1}) \\ &\leq d(w_{n+1}, w_n) + d(w_n, u_n) + d(z_n, v_n). \end{aligned} \tag{2.5}$$

Since $d(w_n, u_n) + d(u_n, z_n) = d(w_n, z_n)$, then

$$\begin{aligned} d(w_{n+1}, z_{n+1}) - d(w_n, z_n) &\leq d(w_{n+1}, w_n) + d(w_n, u_n) + d(z_n, v_n) - d(w_n, u_n) - d(u_n, z_n) \\ &= d(w_{n+1}, w_n) + d(z_n, v_n) - d(u_n, z_n). \end{aligned} \tag{2.6}$$

This fact and (2.4) yield

$$\begin{aligned} d(w_{n+1}, z_{n+1}) - d(w_n, z_n) - d(z_n, v_n) &\leq d(w_{n+1}, z_{n+1}) - d(w_n, z_n) - d(u_n, z_{n+1}) \\ &\leq d(w_{n+1}, w_n) + d(z_n, v_n) - d(u_n, z_n) - d(u_n, z_{n+1}) \\ &\leq d(w_{n+1}, w_n) + d(z_n, v_n) - d(z_{n+1}, z_n) \\ &= d(w_{n+1}, w_n) - d(z_{n+1}, z_n) + d(z_n, v_n). \end{aligned} \tag{2.7}$$

Since $\lim_n d(z_n, v_n) = 0$, we have

$$\limsup_{n\to\infty} (d(w_{n+1}, z_{n+1}) - d(w_n, z_n)) \leq \limsup_{n\to\infty} (d(w_{n+1}, w_n) - d(z_{n+1}, z_n)). \tag{2.8}$$

By using this fact, we have, for $j \in \mathbb{N}$,

$$\begin{aligned} \limsup_{n\to\infty} \left(d\left(w_{n+j}, z_{n+j}\right) - d(w_n, z_n)\right) &= \limsup_{n\to\infty} \sum_{i=0}^{j-1} (d(w_{n+i+1}, z_{n+i+1}) - d(w_{n+i}, z_{n+i})) \\ &\leq \sum_{i=0}^{j-1} \limsup_{n\to\infty} (d(w_{n+i+1}, z_{n+i+1}) - d(w_{n+i}, z_{n+i})) \\ &\leq \sum_{i=0}^{j-1} \limsup_{n\to\infty} (d(w_{n+i+1}, w_{n+i}) - d(z_{n+i+1}, z_{n+i})) \\ &\leq 0. \end{aligned} \tag{2.9}$$

Put $a = (1 - \limsup_n \alpha_n)/2$. We note that $0 < a \leq 1/2$. Fix $k, l \in \mathbb{N}$ and $\varepsilon > 0$, then there exists $m' \geq l$ such that $a \leq 1 - \alpha_n$, $d(z_n, v_n) \leq \varepsilon/2$, $d(w_{n+1}, w_n) - d(z_{n+1}, z_n) \leq \varepsilon/2$, and

$d(w_{n+j}, z_{n+j}) - d(w_n, z_n) \leq \varepsilon/4$, for all $n \geq m'$ and $j = 1, 2, \ldots, k$. In the case of $r = \limsup_n d(w_n, z_n)$, we choose $m \geq m'$ satisfying

$$d(w_{m+k}, z_{m+k}) \geq r - \frac{\varepsilon}{4}, \tag{2.10}$$

and $d(w_n, z_n) \leq r + (\varepsilon/2)$ for all $n \geq m$. We note that

$$d(w_{m+j}, z_{m+j}) \geq d(w_{m+k}, z_{m+k}) - \frac{\varepsilon}{4} \geq r - \frac{\varepsilon}{2}, \tag{2.11}$$

for $j = 0, 1, \ldots, k-1$. In the case of $r = \liminf_n d(w_n, z_n)$, we choose $m \geq m'$ satisfying

$$d(w_m, z_m) \leq r + \frac{\varepsilon}{4}, \tag{2.12}$$

and $d(w_n, z_n) \geq r - (\varepsilon/2)$ for all $n \geq m$. We note that

$$d(w_{m+j}, z_{m+j}) \leq d(w_m, z_m) + \frac{\varepsilon}{4} \leq r + \frac{\varepsilon}{2}, \tag{2.13}$$

for $j = 1, 2, \ldots, k$. In both cases, such m satisfies that $m \geq l$, $a \leq 1 - \alpha_n \leq 1$, $d(z_n, v_n) \leq \varepsilon/2$,$d(w_{n+1}, w_n) - d(z_{n+1}, z_n) \leq \varepsilon/2$ for all $n \geq m$, and

$$r - \frac{\varepsilon}{2} \leq d(w_{m+j}, z_{m+j}) \leq r + \frac{\varepsilon}{2}, \tag{2.14}$$

for $j = 0, 1, \ldots, k$. We next show that

$$d(w_{m+k}, z_{m+j}) \geq (1 + \alpha_{m+j} + \alpha_{m+j+1} + \cdots + \alpha_{m+k-1})r - \frac{(k-j)(2k+2)}{a^{k-j}}\varepsilon, \tag{2.15}$$

for $j = 0, 1, \ldots, k-1$. Since

$$\begin{aligned}
r - \frac{\varepsilon}{2} &\leq d(w_{m+k}, z_{m+k}) \\
&= d(w_{m+k}, \alpha_{m+k-1} w_{m+k-1} \oplus (1 - \alpha_{m+k-1}) v_{m+k-1}) \\
&\leq \alpha_{m+k-1} d(w_{m+k}, w_{m+k-1}) + (1 - \alpha_{m+k-1}) d(w_{m+k}, v_{m+k-1}) \\
&\leq \alpha_{m+k-1} d(z_{m+k}, z_{m+k-1}) + \frac{\varepsilon}{2} \\
&\quad + (1 - \alpha_{m+k-1}) d(w_{m+k}, z_{m+k-1}) + (1 - \alpha_{m+k-1}) d(z_{m+k-1}, v_{m+k-1}) \\
&\leq \alpha_{m+k-1} (d(z_{m+k}, u_{m+k-1}) + d(u_{m+k-1}, z_{m+k-1})) + \frac{\varepsilon}{2} \\
&\quad + (1 - \alpha_{m+k-1}) d(w_{m+k}, z_{m+k-1}) + (1 - \alpha_{m+k-1}) d(z_{m+k-1}, v_{m+k-1})
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_{m+k-1}(1-\alpha_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) + \alpha_{m+k-1}^2 d(w_{m+k-1}, z_{m+k-1}) + \frac{\varepsilon}{2} \\
&\quad + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + (1-\alpha_{m+k-1})d(z_{m+k-1}, v_{m+k-1}) \\
&\le \alpha_{m+k-1}^2\left(r + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + \left(1-\alpha_{m+k-1}^2\right)d(z_{m+k-1}, v_{m+k-1}) \\
&\le \alpha_{m+k-1}^2 r + \varepsilon + (1-\alpha_{m+k-1})d(w_{m+k}, z_{m+k-1}) + \left(1-\alpha_{m+k-1}^2\right)\frac{\varepsilon}{2}
\end{aligned} \tag{2.16}
$$

and $a \le 1-\alpha_{m+k-1}$, we have

$$
\begin{aligned}
d(w_{m+k}, z_{m+k-1}) &\ge \frac{(1-\alpha_{m+k-1}^2)r - (3/2)\varepsilon - (1-\alpha_{m+k-1}^2)\varepsilon/2}{1-\alpha_{m+k-1}} \\
&\ge (1+\alpha_{m+k-1})r - \frac{2k+1}{a}\varepsilon - \frac{\varepsilon}{a} \\
&= (1+\alpha_{m+k-1})r - \frac{2k+2}{a}\varepsilon.
\end{aligned} \tag{2.17}
$$

Hence, (2.15) holds for $j = k-1$. We assume that (2.15) holds for some $j \in \{1, 2, \ldots, k-1\}$. Then, since

$$
\begin{aligned}
&\left(1 + \sum_{i=j}^{k-1}\alpha_{m+i}\right)r - \frac{(k-j)(2k+2)}{a^{k-j}}\varepsilon \\
&\quad \le d(w_{m+k}, z_{m+j}) \\
&\quad = d(w_{m+k}, \alpha_{m+j-1}w_{m+j-1} \oplus (1-\alpha_{m+j-1})v_{m+j-1}) \\
&\quad \le \alpha_{m+j-1}d(w_{m+k}, w_{m+j-1}) + (1-\alpha_{m+j-1})d(w_{m+k}, v_{m+j-1}) \\
&\quad \le \alpha_{m+j-1}\sum_{i=j-1}^{k-1} d(w_{m+i+1}, w_{m+i}) + (1-\alpha_{m+j-1})d(w_{m+k}, v_{m+j-1}) \\
&\quad \le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}\left(d(z_{m+i+1}, z_{m+i}) + \frac{\varepsilon}{2}\right) + (1-\alpha_{m+j-1})d(w_{m+k}, v_{m+j-1}) \\
&\quad \le \alpha_{m+j-1}\sum_{i=j-1}^{k-1} d(z_{m+i+1}, z_{m+i}) + \frac{k\varepsilon}{2} + (1-\alpha_{m+j-1})d(w_{m+k}, v_{m+j-1}) \\
&\quad \le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}(\alpha_{m+i}d(w_{m+i}, z_{m+i}) + (1-\alpha_{m+i})d(z_{m+i}, v_{m+i})) + \frac{k\varepsilon}{2} \\
&\qquad + (1-\alpha_{m+j-1})d(w_{m+k}, v_{m+j-1}) \\
&\quad \le \alpha_{m+j-1}\sum_{i=j-1}^{k-1}\alpha_{m+i}d(w_{m+i}, z_{m+i}) + \alpha_{m+j-1}\sum_{i=j-1}^{k-1}(1-\alpha_{m+i})d(z_{m+i}, v_{m+i}) + \frac{k\varepsilon}{2} \\
&\qquad + (1-\alpha_{m+j-1})d(w_{m+k}, z_{m+j-1}) + (1-\alpha_{m+j-1})d(z_{m+j-1}, v_{m+j-1})
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \alpha_{m+i}\left(r+\frac{\varepsilon}{2}\right)+(k+1)\frac{\varepsilon}{2}+\frac{k\varepsilon}{2}+\left(1-\alpha_{m+j-1}\right)d\left(w_{m+k}, z_{m+j-1}\right) \\
&\leq \alpha_{m+j-1} \sum_{i=j-1}^{k-1} \alpha_{m+i} r+\frac{(3k+1)\varepsilon}{2}+\left(1-\alpha_{m+j-1}\right)d\left(w_{m+k}, z_{m+j-1}\right),
\end{aligned} \tag{2.18}$$

we obtain

$$\begin{aligned}
&d\left(w_{m+k}, z_{m+j-1}\right) \\
&\quad\geq \frac{1+\sum_{i=j}^{k-1} \alpha_{m+i}-\alpha_{m+j-1} \sum_{i=j-1}^{k-1} \alpha_{m+i}}{1-\alpha_{m+j-1}} r-\frac{(k-j)(2k+2)/a^{k-j}/+(3k+1)/2}{1-\alpha_{m+j-1}} \varepsilon \\
&\quad\geq \left(1+\sum_{i=j-1}^{k-1} \alpha_{m+i}\right) r-\frac{(k-j+1)(2k+2)}{a^{k-j+1}} \varepsilon .
\end{aligned} \tag{2.19}$$

Hence, (2.15) holds for $j := j-1$. Therefore, (2.15) holds for all $j = 0, 1, \ldots, k-1$. Specially, we have

$$d\left(w_{m+k}, z_m\right) \geq \left(1+\alpha_m+\alpha_{m+1}+\cdots+\alpha_{m+k-1}\right) r-\frac{k(2k+2)}{a^k} \varepsilon . \tag{2.20}$$

On the other hand, we have

$$\begin{aligned}
d\left(w_{m+k}, z_m\right) &\leq d\left(w_{m+k}, z_{m+k}\right)+\sum_{i=0}^{k-1} d\left(z_{m+i+1}, z_{m+i}\right) \\
&\leq d\left(w_{m+k}, z_{m+k}\right)+\sum_{i=0}^{k-1} d\left(z_{m+i+1}, u_{m+i}\right)+\sum_{i=0}^{k-1} d\left(u_{m+i}, z_{m+i}\right) \\
&\leq d\left(w_{m+k}, z_{m+k}\right)+\sum_{i=0}^{k-1} d\left(v_{m+i}, z_{m+i}\right)+\sum_{i=0}^{k-1} \alpha_{m+i} d\left(w_{m+i}, z_{m+i}\right) \\
&\leq r+\frac{\varepsilon}{2}+\frac{k\varepsilon}{2}+\sum_{i=0}^{k-1} \alpha_{m+i}\left(r+\frac{\varepsilon}{2}\right) \\
&=\left(1+\sum_{i=0}^{k-1} \alpha_{m+i}\right) r+\left(\frac{2k+1}{2}\right) \varepsilon .
\end{aligned} \tag{2.21}$$

This fact and (2.20) imply

$$\left|d\left(w_{m+k}, z_m\right)-\left(1+\alpha_m+\alpha_{m+1}+\cdots+\alpha_{m+k-1}\right) r\right| \leq \frac{k(2k+2)}{a^k} \varepsilon . \tag{2.22}$$

Since $l \in \mathbb{N}$ and $\varepsilon > 0$ are arbitrary, we obtain the desired result. $\square$

By using Lemma 2.1 together with the argument in the proof of [5, Lemma 2.2], simply replacing $\|\cdot\|$ by $d(\cdot,\cdot)$, we can obtain Lemma 1.1 as desired.

3. Applications

In this section, we apply Lemma 1.1 to prove two strong convergence theorems for the modified Halpern iterations of nonexpansive mappings in CAT(0) spaces. The results we obtain are analogs of the Banach space results of Song and Li [6].

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include pre-Hilbert spaces (see [7]), $\mathbb{R}$–trees (see [8]), Euclidean buildings (see [9]), the complex Hilbert ball with a hyperbolic metric (see [10]), and many others. For a thorough discussion of these spaces and of the fundamental role they play in geometry, we refer the reader to Bridson and Haefliger [7].

Fixed-point theory in CAT(0) spaces was first studied by Kirk (see [2, 11]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed-point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (see, e.g., [12–24] and the references therein). It is worth mentioning that fixed-point theorems in CAT(0) spaces (specially in $\mathbb{R}$–trees) can be applied to graph theory, biology, and computer science (see, e.g., [8, 25–28]).

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t),c(t')) = |t-t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic is denoted by $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle\,(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points x_1,x_2,x_3 in X (the *vertices* of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for geodesic triangle $\triangle\,(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle\,(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom.

CAT(0): Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$, then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$,

$$d(x,y) \le d_{\mathbb{E}^2}(\overline{x},\overline{y}). \tag{3.1}$$

We now collect some elementary facts about CAT(0) spaces.

Lemma 3.1. *Let (X, d) be a CAT(0) space.*

(i) *(see [7, Proposition 2.4]) Let C be a closed-convex subset of X, then, for every $x \in X$, there exists a unique point $Px \in C$ such that $d(x, Px) = \inf\{d(x, y) : y \in C\}$. The mapping $P : X \to C$ is called the nearest point (or metric) projection from X onto C.*

(ii) *(see [15, Lemma 2.5]) For $x, y, z \in X$ and $t \in [0, 1]$, one has*

$$d\big((1-t)x \oplus ty, z\big)^2 \leq (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2. \tag{3.2}$$

Recall that a mapping T on a CAT(0) space (X, d) is called *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y) \quad \forall x, y \in X. \tag{3.3}$$

A point $x \in X$ is called a *fixed point* of T if $x = Tx$. We will denote by $F(T)$ the set of fixed points of T. The following result can be found in [13] (see also [2, Theorem 12]).

Theorem 3.2. *Let C be a convex subset of a CAT(0) space, and let $T : C \to C$ be a nonexpansive mapping whose fixed-point set is nonempty, then $F(T)$ is closed and convex.*

A continuous linear functional μ on ℓ_∞, the Banach space of bounded real sequences, is called a *Banach limit* if $\|\mu\| = \mu(1, 1, \ldots) = 1$ and $\mu_n(a_n) = \mu_n(a_{n+1})$ for all $\{a_n\} \in \ell_\infty$.

Lemma 3.3 (see [29], Proposition 2). *Let $\{a_1, a_2, \cdots\} \in \ell_\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limits μ and* $\limsup_n(a_{n+1} - a_n) \leq 0$, *then* $\limsup_n a_n \leq 0$.

Lemma 3.4 (see [21], Lemma 2.1). *Let C be a closed-convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping. Let $u \in C$ be fixed. For each $t \in (0, 1)$, the mapping $S_t : C \to C$ defined by*

$$S_t z = tu \oplus (1-t)Tz \quad \textit{for } z \in C \tag{3.4}$$

has a unique fixed-point $z_t \in C$, that is,

$$z_t = S_t(z_t) = tu \oplus (1-t)T(z_t). \tag{3.5}$$

Lemma 3.5 (see [21], Lemma 2.2). *Let C and T be as the preceding lemma, then $F(T) \neq \emptyset$ if and only if $\{z_t\}$ given by the formula* (3.5) *remains bounded as $t \to 0$. In this case, the following statements hold:*

(1) *$\{z_t\}$ converges to the unique fixed-point z of T which is the nearest u,*

(2) *$d^2(u, z) \leq \mu_n d^2(u, x_n)$ for all Banach limits μ and all bounded sequences $\{x_n\}$ with* $\lim_n d(x_n, Tx_n) = 0$.

Lemma 3.6 (see [30], Lemma 2.1). *Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, \quad n \geq 1, \tag{3.6}$$

where $\{\gamma_n\}$ *and* $\{\sigma_n\}$ *are sequences of real numbers, such that*

(i) $\{\gamma_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \sigma_n| < \infty$,

then $\lim_{n\to\infty} \alpha_n = 0$.

The following result is an analog of [6, Theorem 3.1].

Theorem 3.7. *Let C be a nonempty closed-convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$.

Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ arbitrarily, and

$$x_{n+1} = \lambda_n x_n \oplus (1-\lambda_n) T(\alpha_n u \oplus (1-\alpha_n) x_n), \quad \forall n \geq 1, \tag{3.7}$$

then $\{x_n\}$ converges to a fixed-point Pu of T, where P is the nearest point projection from C onto $F(T)$.

Proof. For each $n \geq 1$, we let $y_n = T(\alpha_n u \oplus (1-\alpha_n)x_n)$. We divide the proof into 3 steps. (i) We show that $\{x_n\}$ and $\{y_n\}$ are bounded sequences. (ii) We show that $\lim_n d(x_n, Tx_n) = 0$. (iii) We show that $\{x_n\}$ converges to a fixed-point $z \in F(T)$ which is the nearest u.

(i) Let $p \in F(T)$, then we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(\lambda_n x_n \oplus (1-\lambda_n) y_n), p) \\
&\leq \lambda_n d(x_n, p) + (1-\lambda_n) d(T(\alpha_n u \oplus (1-\alpha_n)x_n), p) \\
&\leq \lambda_n d(x_n, p) + (1-\lambda_n)\alpha_n d(u,p) + (1-\lambda_n)(1-\alpha_n) d(x_n, p) \\
&\leq (\lambda_n + (1-\lambda_n)(1-\alpha_n)) d(x_n, p) + (1-\lambda_n)\alpha_n d(u,p) \\
&= [1-(1-\lambda_n)\alpha_n] d(x_n,p) + (1-\lambda_n)\alpha_n d(u,p) \\
&\leq \max\{d(x_n,p), d(u,p)\}.
\end{aligned} \tag{3.8}$$

Now, an induction yields

$$d(x_n, p) \leq \max\{d(x_1,p), d(u,p)\}, \quad n \geq 1. \tag{3.9}$$

Hence, $\{x_n\}$ is bounded and so is $\{y_n\}$.

(ii) First, we show that $\lim_n d(x_n, y_n) = 0$. Consider

$$\begin{aligned} d(y_{n+1}, y_n) &= d(T(\alpha_{n+1}u \oplus (1-\alpha_{n+1})x_{n+1}), T(\alpha_n u \oplus (1-\alpha_n)x_n)) \\ &\le d(\alpha_{n+1}u \oplus (1-\alpha_{n+1})x_{n+1}, \alpha_n u \oplus (1-\alpha_n)x_n) \\ &\le \alpha_{n+1} d(u, \alpha_n u \oplus (1-\alpha_n)x_n) + (1-\alpha_{n+1}) d(x_{n+1}, \alpha_n u \oplus (1-\alpha_n)x_n) \\ &\le \alpha_{n+1}(1-\alpha_n)d(u, x_n) + (1-\alpha_{n+1})\alpha_n d(u, x_{n+1}) + (1-\alpha_{n+1})(1-\alpha_n)d(x_{n+1}, x_n). \end{aligned} \tag{3.10}$$

This implies

$$d(y_{n+1}, y_n) - d(x_{n+1}, x_n) \le \alpha_{n+1}(1-\alpha_n)d(u, x_n) + (1-\alpha_{n+1})\alpha_n d(u, x_{n+1}). \tag{3.11}$$

By the condition (C1), we have

$$\limsup_{n\to\infty} (d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0. \tag{3.12}$$

It follows from Lemma 1.1 that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Now,

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, y_n) + d(y_n, Tx_n) \\ &\le d(x_n, y_n) + d(T(\alpha_n u \oplus (1-\alpha_n)x_n), Tx_n) \\ &\le d(x_n, y_n) + d(\alpha_n u \oplus (1-\alpha_n)x_n, x_n) \\ &\le d(x_n, y_n) + \alpha_n d(u, x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.13}$$

(iii) From Lemma 3.4, let $z = \lim_{t\to 0} z_t$ where z_t is given by the formula (3.5). Then z is the point of $F(T)$ which is the nearest u. By applying Lemma 3.1, we have

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2(\lambda_n x_n \oplus (1-\lambda_n)y_n, z) \\ &\le \lambda_n d^2(x_n, z) + (1-\lambda_n)d^2(y_n, z) - \lambda_n(1-\lambda_n)d^2(x_n, y_n) \\ &= \lambda_n d^2(x_n, z) + (1-\lambda_n)d^2(\alpha_n u \oplus (1-\alpha_n)x_n, z) - \lambda_n(1-\lambda_n)d^2(x_n, y_n) \\ &\le \lambda_n d^2(x_n, z) + (1-\lambda_n)\left[\alpha_n d^2(u, z) + (1-\alpha_n)d^2(x_n, z) - \alpha_n(1-\alpha_n)d^2(u, x_n)\right] \\ &\le [\lambda_{\mathrm{n}} + (1-\lambda_n)(1-\alpha_n)]d^2(x_n, z) + \alpha_n(1-\lambda_n)\left[d^2(u, z) - (1-\alpha_n)d^2(u, x_n)\right] \\ &= (1-(1-\lambda_n)\alpha_n)d^2(x_n, z) + (1-\lambda_n)\alpha_n\left[d^2(u, z) - (1-\alpha_n)d^2(u, x_n)\right]. \end{aligned} \tag{3.14}$$

By Lemma 3.5, we have $\mu_n(d^2(u,z) - d^2(u,x_n)) \leq 0$ for all Banach limit μ. Moreover, since

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big(\lambda_n x_n \oplus (1-\lambda_n) y_n, x_n\big) \\ &\leq (1-\lambda_n) d\big(y_n, x_n\big) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \\ \limsup_{n\to\infty}\Big[\Big(d^2(u,z) - d^2(u,x_{n+1})\Big) &- \Big(d^2(u,z) - d^2(u,x_n)\Big)\Big] = 0. \end{aligned} \tag{3.15}$$

It follows from condition (C1) and Lemma 3.3 that

$$\limsup_{n\to\infty}\Big(d^2(u,z) - (1-\alpha_n) d^2(u,x_n)\Big) = \limsup_{n\to\infty}\Big(d^2(u,z) - d^2(u,x_n)\Big) \leq 0. \tag{3.16}$$

Hence, the conclusion follows from Lemma 3.6. □

Remark 3.8. In the proof of Theorem 3.7, one may observe that it is not necessary to use Lemma 1.1 because Suzuki's original lemma is sufficient. However, in [6], there is a strong convergence theorem for another type of modified Halpern iteration (see [6, Theorem 3.2]). We show that the proof is quite easy when we use Lemma 1.1.

Theorem 3.9. *Let C be a nonempty closed-convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$ and sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ in $[0,1]$, the following conditions are satisfied:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C3) $0 < \liminf_{n\to\infty}\lambda_n \leq \limsup_{n\to\infty}\lambda_n < 1$.

Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ arbitrarily, and

$$x_{n+1} = \lambda_n(\alpha_n u \oplus (1-\alpha_n)x_n) \oplus (1-\lambda_n)Tx_n, \quad \forall n \geq 1, \tag{3.17}$$

Then $\{x_n\}$ converges to a fixed-point Pu of T, where P is the nearest point projection from C onto $F(T)$.

Proof. Using the same technique as in the proof of Theorem 3.7, we easily obtain that both $\{x_n\}$ and $\{Tx_n\}$ are bounded. Let $y_n = \alpha_n u \oplus (1-\alpha_n)x_n$, then $x_{n+1} = \lambda_n y_n \oplus (1-\lambda_n)Tx_n$. By the condition (C1), we have

$$d\big(x_n, y_n\big) = d(x_n, \alpha_n u \oplus (1-\alpha_n)x_n) \leq \alpha_n d(x_n, u) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

It follows from the nonexpansiveness of T that

$$\limsup_{n\to\infty}(d(Tx_{n+1}, Tx_n) - d(x_{n+1}, x_n)) \leq 0. \tag{3.19}$$

By Lemma 1.1, we have

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0. \tag{3.20}$$

From (3.18) and (3.20), we get that

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big(\lambda_n y_n \oplus (1-\lambda_n)Tx_n, x_n\big) \\ &\le \lambda_n d\big(y_n, x_n\big) + (1-\lambda_n) d(Tx_n, x_n) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.21}$$

Let $z = \lim_{t\to 0} z_t$ where z_t is given by (3.5), then z is the point of $\in F(T)$ which is the nearest u. Consider

$$\begin{aligned} d^2(x_{n+1}, z) &= d^2\big(\lambda_n y_n \oplus (1-\lambda_n)Tx_n, z\big) \\ &\le \lambda_n d^2\big(y_n, z\big) + (1-\lambda_n) d^2(Tx_n, z) - \lambda_n(1-\lambda_n) d^2\big(y_n, Tx_n\big) \\ &\le \lambda_n d^2(\alpha_n u \oplus (1-\alpha_n)x_n, z) + (1-\lambda_n) d^2(Tx_n, z) \\ &\le \lambda_n \Big(\alpha_n d^2(u,z) + (1-\alpha_n) d^2(x_n, z) - \alpha_n(1-\alpha_n) d^2(u, x_n)\Big) + (1-\lambda_n) d^2(x_n, z) \\ &\le (\lambda_n(1-\alpha_n) + (1-\lambda_n)) d^2(x_n, z) + \lambda_n\alpha_n d^2(u,z) - \lambda_n\alpha_n(1-\alpha_n) d^2(u, x_n) \\ &= (1-\lambda_n\alpha_n) d^2(x_n, z) + \lambda_n\alpha_n\Big(d^2(u,z) - (1-\alpha_n) d^2(u, x_n)\Big). \end{aligned} \tag{3.22}$$

By Lemma 3.5, we have $\mu_n(d^2(u,z) - d^2(u,x_n)) \le 0$ for all Banach limit μ. Moreover, since $d(x_{n+1}, x_n) \to 0$, then

$$\limsup_{n\to\infty}\Big[\Big(d^2(u,z) - d^2(u, x_{n+1})\Big) - \Big(d^2(u,z) - d^2(u,x_n)\Big)\Big] = 0. \tag{3.23}$$

It follows from condition (C1) and Lemma 3.3 that

$$\limsup_{n\to\infty}\Big(d^2(u,z) - (1-\alpha_n) d^2(u,x_n)\Big) = \limsup_{n\to\infty}\Big(d^2(u,z) - d^2(u,x_n)\Big) \le 0. \tag{3.24}$$

Hence, the conclusion follows from Lemma 3.6. □

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