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Research Article

A Fixed Point Approach to the Stability of the Cauchy Additive and Quadratic Type Functional Equation

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We investigate the stability of the functional equation $2f(x+y)+f(x-y)+f(y-x)-3f(x)-f(-x)-3f(y)-f(-y)=0$ by using the fixed point theory in the sense of Cădariu and Radu.

1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of homomorphisms as follow. Given a group G_1, a metric group G_2 with the metric $d(\cdot,\cdot)$, and a positive number ε, does there exist a $\delta>0$ such that if a mapping $f: G_1 \to G_2$ satisfies the inequality

$$d\big(f(xy), f(x)f(y)\big) < \delta, \tag{1.1}$$

for all $x, y \in G_1$ then there exists a homomorphism $F: G_1 \to G_2$ with

$$d\big(f(x), F(x)\big) < \varepsilon, \tag{1.2}$$

for all $x \in G_1$? When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are *stable.* In the next year, Hyers [2] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Hyers' result was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering the stability problem with unbounded Cauchy's differences.

The paper of Rassias had much influence in the development of stability problems. The terminology Hyers-Ulam-Rassias stability originated from this historical background. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [5–12].

Almost all subsequent proofs, in this very active area, have used Hyers' method of [2]. Namely, the mapping F, which is the solution of a functional equation, is explicitly constructed, starting from the given mapping f, by the formulae $F(x) = \lim_{n\to\infty}(1/2^n)f(2^n x)$ or $F(x) = \lim_{n\to\infty} 2^n f(x/2^n)$. We call it *a direct method.* In 2003, Cădariu and Radu [13] observed that the existence of the solution F for a functional equation and the estimation of the difference with the given mapping f can be obtained from the fixed point theory alternative. This method is called *a fixed point method.* In 2004, they applied this method [14] to prove stability theorems of *the Cauchy functional equation:*

$$f(x+y) - f(x) - f(y) = 0. \tag{1.3}$$

In 2003, they [15] obtained the stability of *the quadratic functional equation*:

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0, \tag{1.4}$$

by using the fixed point method. Notice that if we consider the functions $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = ax$ and $f_2(x) = ax^2$, where a is a real constant, then f_1 satisfies (1.3), and f_2 holds (1.4), respectively. We call a solution of (1.3) *an additive map,* and a mapping satisfying (1.4) is called *a quadratic map.* Now we consider the functional equation:

$$2f(x+y) + f(x-y) + f(y-x) - 3f(x) - f(-x) - 3f(y) - f(-y) = 0, \tag{1.5}$$

which is called *the Cauchy additive and quadratic-type functional equation.* The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = ax^2 + bx$ satisfies this functional equation, where a, b are real constants. We call a solution of (1.5) *a quadratic-additive mapping.*

In this paper, we will prove the stability of the functional equation (1.5) by using the fixed point theory. Precisely, we introduce a strictly contractive mapping with the Lipschitz constant $0 < L < 1$. Using the fixed point theory in the sense of Cădariu and Radu, together with suitable conditions, we can show that the contractive mapping has the fixed point. Actually the fixed point F becomes the precise solution of (1.5). In Section 2, we prove several stability results of the functional equation (1.5) using the fixed point theory, see Theorems 2.3, 2.4, and 2.5. In Section 3, we use the results in the previous sections to get a stability of the Cauchy functional equation (1.3) and that of the quadratic functional equation (1.4), respectively.

2. Main Results

We recall the following result of the fixed point theory by Margolis and Diaz.

Theorem 2.1 (see [16] or [17]). *Suppose that a complete generalized metric space (X, d), which means that the metric d may assume infinite values, and a strictly contractive mapping $J : X \to X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = +\infty, \quad \forall n \in \mathbb{N} \cup \{0\} \tag{2.1}$$

or there exists a nonnegative integer k such that

(1) $d(J^n x, J^{n+1} x) < +\infty$ *for all* $n \geq k$,

(2) *the sequence* $\{J^n x\}$ *is convergent to a fixed point* y^* *of* J,

(3) y^* *is the unique fixed point of* J *in* $Y := \{y \in X, d(J^k x, y) < +\infty\}$,

(4) $d(y, y^*) \leq (1/(1 - L)) d(y, Jy)$ *for all* $y \in Y$.

Throughout this paper, let V be a (real or complex) linear space, and let Y be a Banach space. For a given mapping $f : V \to Y$, we use the following abbreviation:

$$Df(x,y) := 2f(x+y) + f(x-y) + f(y-x) - 3f(x) - f(-x) - 3f(y) - f(-y), \tag{2.2}$$

for all $x, y \in V$. If f is a solution of the functional equation $Df \equiv 0$, see (1.5), we call it a *quadratic-additive mapping*. We first prove the following lemma.

Lemma 2.2. *If $f : V \to Y$ is a mapping such that $Df(x,y) = 0$ for all $x, y \in V \setminus \{0\}$, then f is a quadratic-additive mapping.*

Proof. By choosing $x \in V \setminus \{0\}$, we get

$$\begin{aligned} 8f(0) = {} & Df(3x, 2x) - Df(4x, x) - 2Df(3x, x) - Df(2x, x) + Df(-3x, -2x) \\ & - Df(-4x, -x) - 2Df(-3x, -x) - Df(-2x, -x) + 4Df(x, -x) = 0. \end{aligned} \tag{2.3}$$

Since $f(0) = 0$, we easily obtain $Df(x,y) = 0$ for all $x, y \in V$. □

Now we can prove some stability results of the functional equation (1.5).

Theorem 2.3. *Let $\varphi : (V \setminus \{0\})^2 \to [0, \infty)$ be a given function. Suppose that the mapping $f : V \to Y$ satisfies*

$$\|Df(x,y)\| \leq \varphi(x,y), \tag{2.4}$$

for all $x, y \in V \setminus \{0\}$. If there exist constants $0 < L, L' < 1$ such that φ has the property

$$\frac{1}{L'}\varphi(x,y) \leq \varphi(2x, 2y) \leq 2L\varphi(x,y), \tag{2.5}$$

for all $x, y \in V \setminus \{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{3(\varphi(x,x) + \varphi(-x,-x))}{16(1-L)}, \tag{2.6}$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty}\left(\frac{f(2^n x) + f(-2^n x)}{2\cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}\right), \tag{2.7}$$

for all $x \in V$.

Proof. It follows from (2.5) that

$$\lim_{n\to\infty}\frac{\varphi(2^n x, 2^n y)}{2^n} = 0, \tag{2.8}$$

for all $x, y \in V \setminus \{0\}$, and

$$\begin{aligned}
\|8f(0)\| &= \lim_{n\to\infty}\left\| Df\left(\frac{3x}{2^n},\frac{2x}{2^n}\right) - Df\left(\frac{4x}{2^n},\frac{x}{2^n}\right) - 2Df\left(\frac{3x}{2^n},\frac{x}{2^n}\right)\right. \\
&\qquad - Df\left(\frac{2x}{2^n},\frac{x}{2^n}\right) + Df\left(-\frac{3x}{2^n},-\frac{2x}{2^n}\right) - Df\left(-\frac{4x}{2^n},-\frac{x}{2^n}\right) \\
&\qquad \left. -2Df\left(-\frac{3x}{2^n},-\frac{x}{2^n}\right) - Df\left(-\frac{2x}{2^n},-\frac{x}{2^n}\right) + 4Df\left(\frac{x}{2^n},-\frac{x}{2^n}\right)\right\| \\
&\le \lim_{n\to\infty}\left(\varphi\left(\frac{3x}{2^n},\frac{2x}{2^n}\right) + \varphi\left(\frac{4x}{2^n},\frac{x}{2^n}\right) + 2\varphi\left(\frac{3x}{2^n},\frac{x}{2^n}\right)\right. \\
&\qquad + \varphi\left(\frac{2x}{2^n},\frac{x}{2^n}\right) + \varphi\left(-\frac{3x}{2^n},-\frac{2x}{2^n}\right) + \varphi\left(-\frac{4x}{2^n},-\frac{x}{2^n}\right) \\
&\qquad \left. + 2\varphi\left(-\frac{3x}{2^n},-\frac{x}{2^n}\right) + \varphi\left(-\frac{2x}{2^n},-\frac{x}{2^n}\right) + 4\varphi\left(\frac{x}{2^n},-\frac{x}{2^n}\right)\right) \\
&\le \lim_{n\to\infty} L'^n(\varphi(3x,2x) + \varphi(4x,x) + 2\varphi(3x,x) + \varphi(2x,x) + \varphi(-3x,-2x) \\
&\qquad + \varphi(-4x,-x) + 2\varphi(-3x,-x) + \varphi(-2x,-x) + 4\varphi(x,-x)) \\
&= 0,
\end{aligned} \tag{2.9}$$

for all $x \in V \setminus \{0\}$. From this, we know that $f(0) = 0$. Let S be the set of all mappings $g : V \to Y$ with $g(0) = 0$. We introduce a generalized metric on S by

$$d(g,h) = \inf\{K \in \mathbb{R}^+ \mid \|g(x) - h(x)\| \le K(\varphi(x,x) + \varphi(-x,-x)) \ \forall\, x \in V \setminus \{0\}\}. \tag{2.10}$$

It is easy to show that (S, d) is a generalized complete metric space. Now we consider the mapping $J : S \to S$, which is defined by

$$Jg(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}, \tag{2.11}$$

for all $x \in V$. Notice

$$J^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}, \tag{2.12}$$

for all $n \in \mathbb{N}$ and $x \in V$. Let $g, h \in S$, and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d, we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \frac{3}{8}\|g(2x) - h(2x)\| + \frac{1}{8}\|g(-2x) - h(-2x)\| \\ &\leq \frac{1}{2}K\big(\varphi(2x, 2x) + \varphi(-2x, -2x)\big) \\ &\leq KL\big(\varphi(x, x) + \varphi(-x, -x)\big), \end{aligned} \tag{2.13}$$

for all $x \in V \setminus \{0\}$, which implies that

$$d(Jg, Jh) \leq Ld(g, h), \tag{2.14}$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Moreover, by (2.4), we see that

$$\|f(x) - Jf(x)\| = \frac{1}{16}\|-3Df(x, x) + Df(-x, -x)\| \leq \frac{3}{16}\big(\varphi(x, x) + \varphi(-x, -x)\big), \tag{2.15}$$

for all $x \in V \setminus \{0\}$. It means that $d(f, Jf) \leq 3/16 < \infty$ by the definition of d. Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point $F : V \to Y$ of J in the set $T = \{g \in S \mid d(f, g) < \infty\}$, which is represented by (2.7) for all $x \in V$. By the definition of F, together with (2.4) and (2.7), it follows that

$$\begin{aligned} \|DF(x, y)\| &= \lim_{n \to \infty} \left\| \frac{Df(2^n x, 2^n y) - Df(-2^n x, -2^n y)}{2^{n+1}} + \frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2 \cdot 4^n} \right\| \\ &\leq \lim_{n \to \infty} \frac{2^n + 1}{2 \cdot 4^n}\big(\varphi(2^n x, 2^n y) + \varphi(-2^n x, -2^n y)\big) \\ &= 0, \end{aligned} \tag{2.16}$$

for all $x, y \in V \setminus \{0\}$. By Lemma 2.2, we have proved that

$$DF(x, y) = 0, \tag{2.17}$$

for all $x, y \in V$. □

Theorem 2.4. *Let $\varphi : (V \setminus \{0\})^2 \to [0, \infty)$ be a given function. Suppose that the mapping $f : V \to Y$ satisfies (2.5) for all $x, y \in V \setminus \{0\}$. If there exists a constant $0 < L < 1/2$ such that φ has the property*

$$\varphi(2x, 2y) \leq 2L\varphi(x, y) \tag{2.18}$$

for all $x, y \in V \setminus \{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying (2.6) for all $x \in V \setminus \{0\}$. Moreover, if $\varphi(x, y)$ is continuous, then f itself is a quadratic-additive mapping.

Proof. It follows from (2.18) that

$$\lim_{n\to\infty} \varphi(2^n x, 2^n y) = \lim_{n\to\infty} (2L)^n \varphi(x, y) = 0, \tag{2.19}$$

for all $x, y \in V \setminus \{0\}$, and

$$\begin{aligned}
\|8f(0)\| = \lim_{n\to\infty} \Big\| & Df\left(3\cdot 2^n x, 2^{n+1}x\right) - Df\left(2^{n+2}x, 2^n x\right) - 2Df(3\cdot 2^n x, 2^n x) \\
& - Df\left(2^{n+1}x, 2^n x\right) + Df\left(-3\cdot 2^n x, -2^{n+1}x\right) \\
& - Df\left(-2^{n+2}x, -2^n x\right) - 2Df(-3\cdot 2^n x, -2^n x) \\
& - Df\left(-2^{n+1}x, -2^n x\right) + 4Df(2^n x, -2^n x)\Big\| \\
\leq \lim_{n\to\infty} \Big(& \varphi\left(3\cdot 2^n x, 2^{n+1}x\right) + \varphi\left(2^{n+2}x, 2^n x\right) + 2\varphi(3\cdot 2^n x, 2^n x) \\
& + \varphi\left(2^{n+1}x, 2^n x\right) + \varphi\left(-3\cdot 2^n x, -2^{n+1}x\right) + \varphi\left(-2^{n+2}x, -2^n x\right) \\
& + 2\varphi(-3\cdot 2^n x, -2^n x) + \varphi\left(-2^{n+1}x, -2^n x\right) + 4\varphi(2^n x, -2^n x)\Big) \\
\leq \lim_{n\to\infty} & (2L)^n (\varphi(3x, 2x) + \varphi(4x, x) + 2\varphi(3x, x) + \varphi(2x, x) + \varphi(-3x, -2x) \\
& + \varphi(-4x, -x) + 2\varphi(-3x, -x) + \varphi(-2x, -x) + 4\varphi(x, -x)) \\
= 0, &
\end{aligned} \tag{2.20}$$

for all $x \in V \setminus \{0\}$. By the same method used in Theorem 2.3, we know that there exists a unique quadratic-additive mapping $F : V \to Y$ satisfying (2.6) for all $x \in V \setminus \{0\}$. Since φ is continuous, we get

$$\begin{aligned}\lim_{n\to\infty} \varphi((a_1 \cdot 2^n + a_2)x, (b_1 \cdot 2^n + b_2)y) &\le \lim_{n\to\infty} (2L)^n \varphi\left(\left(a_1 + \frac{a_2}{2^n}\right)x, \left(b_1 + \frac{b_2}{2^n}\right)y\right) \\ &= 0 \cdot \varphi(a_1 x, b_1 y) = 0,\end{aligned} \tag{2.21}$$

for all $x, y \in V \setminus \{0\}$ and for any fixed integers a_1, a_2, b_1, b_2 with $a_1, b_1 \neq 0$. Therefore, we obtain

$$\begin{aligned}2\|f(x) - F(x)\| \le \lim_{n\to\infty} \Big(&\|Df((2^n + 1)x, -2^n x) - DF((2^n + 1)x, -2^n x)\| \\ &+ \left\|(F - f)\left(\left(2^{n+1} + 1\right)x\right)\right\| + \left\|(F - f)\left(-\left(2^{n+1} + 1\right)x\right)\right\| \\ &+ 3\|(f - F)((2^n + 1)x)\| + \|(f - F)(-(2^n + 1)x)\| \\ &+3\|(f - F)(-2^n x)\| + \|(f - F)(2^n x)\|\Big) \\ \le{}& \lim_{n\to\infty} \varphi\left((2^n + 1)x, -2^n x\right) + \frac{3}{8(1 - L)} \lim_{n\to\infty} \left(\psi\left(\left(2^{n+1} + 1\right)x\right)\right. \\ &\left. + 2\psi((2^n + 1)x) + 2\psi(2^n x)\right) \\ ={}& 0,\end{aligned} \tag{2.22}$$

for all $x \in V \setminus \{0\}$, where $\psi(x)$ is defined by $\psi(x) = \varphi(x, x) + \varphi(-x, -x)$. Since $f(0) = 0 = F(0)$, we have shown that $f \equiv F$. This completes the proof of this theorem. □

We continue our investigation with the next result.

Theorem 2.5. *Let $\varphi : (V \setminus \{0\})^2 \to [0, \infty)$. Suppose that $f : V \to Y$ satisfies the inequality $\|Df(x, y)\| \le \varphi(x, y)$ for all $x, y \in V \setminus \{0\}$. If there exists $0 < L < 1$ such that the mapping φ has the property*

$$L\varphi(2x, 2y) \ge 4\varphi(x, y) \tag{2.23}$$

for all $x, y \in V \setminus \{0\}$, then there exists a unique quadratic-additive mapping $F : V \to Y$ such that

$$\|f(x) - F(x)\| \le \frac{L}{8(1 - L)} (\varphi(x, x) + \varphi(-x, -x)), \tag{2.24}$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$F(x) = \lim_{n\to\infty} \left(2^{n-1}\left(f\left(\frac{x}{2^n}\right) - f\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(f\left(\frac{x}{2^n}\right) + f\left(-\frac{x}{2^n}\right)\right)\right), \tag{2.25}$$

for all $x \in V$.

Proof. By the similar method used to prove $f(0) = 0$ in the proof of Theorem 2.3, we can easily show that $f(0) = 0$. Let the set (S, d) be as in the proof of Theorem 2.3. Now we consider the mapping $J : S \to S$ defined by

$$Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right), \tag{2.26}$$

for all $g \in S$ and $x \in V$. Notice that

$$J^n g(x) = 2^{n-1}\left(g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right)\right) + \frac{4^n}{2}\left(g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right)\right) \tag{2.27}$$

and $J^0 g(x) = g(x)$, for all $x \in V$. Let $g, h \in S$, and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le K$. From the definition of d, we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= 3\left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| + \left\|g\left(-\frac{x}{2}\right) - h\left(-\frac{x}{2}\right)\right\| \\ &\le 4K\left(\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(-\frac{x}{2}, -\frac{x}{2}\right)\right) \\ &\le LK(\varphi(x, x) + \varphi(-x, -x)), \end{aligned} \tag{2.28}$$

for all $x \in V \setminus \{0\}$. So

$$d(Jg, Jh) \le Ld(g, h), \tag{2.29}$$

for any $g, h \in S$. That is, J is a strictly contractive self-mapping of S with the Lipschitz constant L. Also we see that

$$\begin{aligned} \|f(x) - Jf(x)\| &= \frac{1}{2}\left\|Df\left(\frac{x}{2}, \frac{x}{2}\right)\right\| \\ &\le \frac{1}{2}\left(\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(-\frac{x}{2}, -\frac{x}{2}\right)\right) \\ &\le \frac{L}{8}(\varphi(x, x) + \varphi(-x, -x)), \end{aligned} \tag{2.30}$$

for all $x \in V \setminus \{0\}$, which implies that $d(f, Jf) \le L/8 < \infty$. Therefore, according to Theorem 2.1, the sequence $\{J^n f\}$ converges to the unique fixed point F of J in the set $T := \{g \in S \mid d(f, g) < \infty\}$, which is represented by (2.25). Since

$$d(f, F) \le \frac{1}{1 - L} d(f, Jf) \le \frac{L}{8(1 - L)}, \tag{2.31}$$

the inequality (2.24) holds. From the definition of $F(x)$, (2.4), and (2.23), we have

$$\begin{aligned}
\|DF(x,y)\| &= \lim_{n\to\infty}\Big\|2^{n-1}\Big(Df\Big(\frac{x}{2^n},\frac{y}{2^n}\Big) - Df\Big(-\frac{x}{2^n},-\frac{y}{2^n}\Big)\Big) \\
&\qquad + \frac{4^n}{2}\Big(Df\Big(\frac{x}{2^n},\frac{y}{2^n}\Big) + Df\Big(-\frac{x}{2^n},-\frac{y}{2^n}\Big)\Big)\Big\| \\
&\leq \lim_{n\to\infty}\frac{2^n+4^n}{2}\Big(\varphi\Big(\frac{x}{2^n},\frac{y}{2^n}\Big) + \varphi\Big(-\frac{x}{2^n},-\frac{y}{2^n}\Big)\Big) \\
&= 0,
\end{aligned} \tag{2.32}$$

for all $x, y \in V \setminus \{0\}$. By Lemma 2.2, F is quadratic additive. □

Remark 2.6. If φ satisfies the equality $\varphi(x,y) = \varphi(-x,-y)$ for all $x, y \in V \setminus \{0\}$ in Theorems 2.3, 2.4, and 2.5, then the inequalities (2.6) and (2.24) can be replaced by

$$\begin{aligned}
\|f(x) - F(x)\| &\leq \frac{\varphi(x,x)}{4(1-L)}, \\
\|f(x) - F(x)\| &\leq \frac{L\varphi(x,x)}{8(1-L)},
\end{aligned} \tag{2.33}$$

for all $x \in V \setminus \{0\}$, respectively.

3. Applications

For a given mapping $f : V \to Y$, we use the following abbreviations:

$$\begin{aligned}
Af(x,y) &:= f(x+y) - f(x) - f(y), \\
Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y),
\end{aligned} \tag{3.1}$$

for all $x, y \in V$. Using Theorems 2.3, 2.4, and 2.5 we will show the stability results of the additive functional equation $Af \equiv 0$ and the quadratic functional equation $Qf \equiv 0$ in the following corollaries.

Corollary 3.1. *Let $f_i : V \to Y$, $i = 1, 2, 3$, be mappings for which there exist functions $\phi_i : (V \setminus \{0\})^2 \to [0, \infty)$, $i = 1, 2, 3$, such that*

$$\|Af_i(x,y)\| \leq \phi_i(x,y), \tag{3.2}$$

for all $x, y \in V \setminus \{0\}$. *If there exists* $0 < L < 1$ *such that*

$$\frac{1}{L}\phi_1(x,y) \le \phi_1(2x,2y) \le 2L\phi_1(x,y), \tag{3.3}$$

$$\phi_2(2x,2y) \le L\phi_2(x,y), \tag{3.4}$$

$$4\phi_3(x,y) \le L\phi_3(2x,2y) \tag{3.5}$$

for all $x, y \in V \setminus \{0\}$, *then there exist unique additive mappings* $F_i : V \to Y,\ i = 1,2,3$, *such that*

$$\|f_1(x) - F_1(x)\| \le \frac{3(\phi_1(x,x) + \phi_1(x,-x) + \phi_1(-x,x) + \phi_1(-x,-x))}{8(1-L)}, \tag{3.6}$$

$$\|f_2(x) - F_2(x)\| \le \frac{3(\phi_2(x,x) + \phi_2(x,-x) + \phi_2(-x,x) + \phi_2(-x,-x))}{8(1-L/2)}, \tag{3.7}$$

$$\|f_3(x) - F_3(x)\| \le \frac{L(\phi_3(x,x) + \phi_3(x,-x) + \phi_3(-x,x) + \phi_3(-x,-x))}{4(1-L)}, \tag{3.8}$$

for all $x \in V \setminus \{0\}$. *In particular, the mappings* $F_i,\ i = 1,2,3$, *are represented by*

$$F_1(x) = \lim_{n\to\infty} \frac{f_1(2^n x)}{2^n}, \tag{3.9}$$

$$F_2(x) = \lim_{n\to\infty} \frac{f_2(2^n x)}{2^n}, \tag{3.10}$$

$$F_3(x) = \lim_{n\to\infty} 2^n f_3\left(\frac{x}{2^n}\right), \tag{3.11}$$

for all $x \in V$. *Moreover, if* $\phi_2(x,y)$ *is continuous, then* f_2 *is itself an additive mapping.*

Proof. Notice that

$$Df_i(x,y) = 2Af_i(x,y) + Af_i(x,-y) + Af_i(y,-x), \tag{3.12}$$

for all $x, y \in V$ and $i = 1,2,3$. Put

$$\varphi_i(x,y) := 2\phi_i(x,y) + \phi_i(x,-y) + \phi_i(y,-x), \tag{3.13}$$

for all $x, y \in V$ and $i = 1,2,3$, then φ_1 satisfies (2.5), φ_2 satisfies (2.18), and φ_3 satisfies (2.23). Therefore, $\|Df_i(x,y)\| \le \varphi_i(x,y)$, for all $x, y \in V \setminus \{0\}$ and $i = 1,2,3$. According to

Theorem 2.3, there exists a unique mapping $F_1 : V \to Y$ satisfying (3.6), which is represented by (2.7). Observe that, by (3.2) and (3.3),

$$\begin{aligned}
\lim_{n\to\infty}\left\|\frac{f_1(2^n x)+f_1(-2^n x)}{2^{n+1}}\right\| &= \lim_{n\to\infty}\left\|\frac{f_1(2^n x)+f_1(-2^n x)-f_1(0)}{2^{n+1}}\right\| \\
&= \lim_{n\to\infty}\frac{1}{2^{n+1}}\left\|Af_1(2^n x,-2^n x)\right\| \\
&\leq \lim_{n\to\infty}\frac{1}{2^{n+1}}\phi_1(2^n x,-2^n x) \\
&\leq \lim_{n\to\infty}\frac{L^n}{2}\phi_1(x,-x) \\
&= 0,
\end{aligned} \tag{3.14}$$

as well as

$$\lim_{n\to\infty}\left\|\frac{f_1(2^n x)+f_1(-2^n x)}{2\cdot 4^n}\right\| \leq \lim_{n\to\infty}\frac{2^n L^n}{2\cdot 4^n}\phi_1(x,-x)=0, \tag{3.15}$$

for all $x \in V \setminus \{0\}$. From this and (2.7), we get (3.9). Moreover, we have

$$\left\|\frac{Af_1(2^n x,2^n y)}{2^n}\right\| \leq \frac{\phi_1(2^n x,2^n y)}{2^n} \leq L^n\phi_1(x,y), \tag{3.16}$$

for all $x, y \in V \setminus \{0\}$. Taking the limit as $n \to \infty$ in the above inequality and using $F_1(0) = 0$, we get

$$AF_1(x,y)=0, \tag{3.17}$$

for all $x, y \in V$. According to Theorem 2.4, there exists a unique mapping $F_2 : V \to Y$ satisfying (3.7), which is represented by (2.7). By using the similar method to prove (3.9), we can show that F_2 is represented by (3.10). In particular, if $\phi_2(x, y)$ is continuous, then φ_2 is continuous on $(V \setminus \{0\})^2$, and we can say that f_2 is an additive map by Theorem 2.4. On the other hand, according to Theorem 2.5, there exists a unique mapping $F_3 : V \to Y$ satisfying (3.8) which is represented by (2.25). Observe that, by (3.2) and (3.5),

$$\begin{aligned}
\lim_{n\to\infty}2^{2n-1}\left\|f_3\left(\frac{x}{2^n}\right)+f_3\left(\frac{-x}{2^n}\right)\right\| &= \lim_{n\to\infty}2^{2n-1}\left\|Af_3\left(\frac{x}{2^n},-\frac{x}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty}2^{2n-1}\phi_3\left(\frac{x}{2^n},-\frac{x}{2^n}\right) \\
&\leq \lim_{n\to\infty}\frac{L^n}{2}\phi_3(x,-x)=0,
\end{aligned} \tag{3.18}$$

as well as

$$\lim_{n\to\infty} 2^{n-1}\left\| f_3\left(\frac{x}{2^n}\right) + f_3\left(\frac{-x}{2^n}\right)\right\| \le \lim_{n\to\infty}\frac{L^n}{2^{n+1}}\phi_3(x,-x) = 0, \tag{3.19}$$

for all $x \in V \setminus \{0\}$. From these and (2.25), we get (3.11). Moreover, we have

$$\left\|2^n Af_3\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \le 2^n\phi_3\left(\frac{x}{2^n},\frac{y}{2^n}\right) \le \frac{L^n}{2^n}\phi_3(x,y), \tag{3.20}$$

for all $x, y \in V \setminus \{0\}$. Taking the limit as $n \to \infty$ in the above inequality and using $F_3(0) = 0$, we get

$$AF_3(x,y) = 0, \tag{3.21}$$

for all $x, y \in V$. □

Corollary 3.2. *Let $f_i : V \to Y$, $i = 1,2,3$, be mappings for which there exist functions $\phi_i : (V \setminus \{0\})^2 \to [0,\infty)$, $i = 1,2,3$, such that*

$$\|Qf_i(x,y)\| \le \phi_i(x,y), \tag{3.22}$$

for all $x, y \in V \setminus \{0\}$. If there exists $0 < L < 1$ such that the mapping ϕ_1 satisfies (3.3), ϕ_2 satisfies (3.4), and ϕ_3 satisfies (3.5) for all $x, y \in V \setminus \{0\}$, then there exist unique quadratic mappings $F_i : V \to Y$, $i = 1,2,3$, such that

$$\|f_1(x) - F_1(x)\| \le \frac{3(\phi_1(x,x) + \phi_1(x,-x) + \phi_1(-x,x) + \phi_1(-x,-x))}{16(1-L)}, \tag{3.23}$$

$$\|f_2(x) - F_2(x)\| \le \frac{3(\phi_2(x,x) + \phi_2(x,-x) + \phi_2(-x,x) + \phi_2(-x,-x))}{16(1-L/2)}, \tag{3.24}$$

$$\|f_3(x) - F_3(x)\| \le \frac{L(\phi_3(x,x) + \phi_3(x,-x) + \phi_3(-x,x) + \phi_3(-x,-x))}{8(1-L)}, \tag{3.25}$$

for all $x \in V \setminus \{0\}$. In particular, the mappings F_i, $i = 1,2,3$, are represented by

$$F_1(x) = \lim_{n\to\infty}\frac{f_1(2^n x)}{4^n}, \tag{3.26}$$

$$F_2(x) = \lim_{n\to\infty}\frac{f_2(2^n x)}{4^n}, \tag{3.27}$$

$$F_3(x) = \lim_{n\to\infty} 4^n f_3\left(\frac{x}{2^n}\right), \tag{3.28}$$

for all $x \in V$. Moreover, if $\phi_2(x,y)$ is continuous, then f_2 itself is a quadratic mapping.

Proof. Notice that

$$Df_i(x,y) = Qf_i(x,y) + \frac{1}{2}\left(Qf_i(x,-y) + Qf_i(y,-x)\right), \tag{3.29}$$

for all $x,y \in V$ and $i = 1,2,3$. Put $\varphi_i(x,y) := \phi_i(x,y) + (1/2)(\phi_i(x,-y) + \phi_i(y,-x))$, for all $x,y \in V$ and $i = 1,2,3$, then φ_1 satisfies (2.5), φ_2 satisfies (2.18), and φ_3 satisfies (2.23). Moreover,

$$\|Df_i(x,y)\| \le \varphi_i(x,y), \tag{3.30}$$

for all $x,y \in V \setminus \{0\}$ and $i = 1,2,3$. According to Theorem 2.3, there exists a unique mapping $F_1 : V \to Y$ satisfying (3.23) which is represented by (2.7). Observe that

$$\begin{aligned}
\lim_{n\to\infty}\left\|\frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}}\right\| &= \lim_{n\to\infty}\frac{1}{2^{n+1}}\left\|Qf_1\left(2^{n-1}x,-2^{n-1}x\right) - Qf_1\left(-2^{n-1}x,2^{n-1}x\right)\right\| \\
&\le \lim_{n\to\infty}\frac{1}{2^{n+1}}\left(\phi_1\left(2^{n-1}x,-2^{n-1}x\right) + \phi_1\left(-2^{n-1}x,2^{n-1}x\right)\right) \\
&\le \lim_{n\to\infty}\frac{L^n}{2}\left(\phi_1\left(\frac{x}{2},-\frac{x}{2}\right) + \phi_1\left(-\frac{x}{2},\frac{x}{2}\right)\right) \\
&= 0,
\end{aligned} \tag{3.31}$$

as well as

$$\lim_{n\to\infty}\left\|\frac{f_1(2^n x) - f_1(-2^n x)}{2\cdot 4^n}\right\| \le \lim_{n\to\infty}\frac{L^n}{2^{n+1}}\left(\phi_1\left(\frac{x}{2},-\frac{x}{2}\right) + \phi_1\left(-\frac{x}{2},\frac{x}{2}\right)\right) = 0, \tag{3.32}$$

for all $x \in V \setminus \{0\}$. From this and (2.7), we get (3.26) for all $x \in V$. Moreover, we have

$$\left\|\frac{Qf_1(2^n x,2^n y)}{4^n}\right\| \le \frac{\phi_1(2^n x,2^n y)}{4^n} \le \frac{L^n}{2^n}\phi_1(x,y), \tag{3.33}$$

for all $x,y \in V \setminus \{0\}$. Taking the limit as $n \to \infty$ in the above inequality, we get

$$QF_1(x,y) = 0 \tag{3.34}$$

for all $x,y \in V \setminus \{0\}$. Using $F_1(0) = 0$, we have

$$\begin{gathered}
QF_1(x,0) = 0, \\
QF_1(0,y) = -F_1\left(\frac{y}{2},-\frac{y}{2}\right) + F_1\left(-\frac{y}{2},\frac{y}{2}\right) = 0,
\end{gathered} \tag{3.35}$$

for all $x,y \in V \setminus \{0\}$. Therefore, $QF_1(x,y) = 0$ for all $x,y \in V$.

Next, by Theorem 2.4, there exists a unique mapping $F_2 : V \to Y$ satisfying (3.24), which is represented by (2.7). By using the similar method to prove (3.26), we can show that F_2 is represented by (3.27). In particular, $\phi_2(x,y)$ is continuous, then φ_2 is continuous on $(V \setminus \{0\})^2$, and we can say that f_2 is a quadratic map by Theorem 2.4. On the other hand, according to Theorem 2.5, there exists a unique mapping $F_3 : V \to Y$ satisfying (3.25) which is represented by (2.25). Observe that

$$\begin{aligned}4^n\left\|f_3\left(\frac{x}{2^n}\right)-f_3\left(-\frac{x}{2^n}\right)\right\| &= 4^n\left\|Qf_3\left(\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\right)-Qf_3\left(-\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\right\| \\ &\leq 4^n\left(\phi_3\left(\frac{x}{2^{n+1}},-\frac{x}{2^{n+1}}\right)+\phi_3\left(-\frac{x}{2^{n+1}},\frac{x}{2^{n+1}}\right)\right) \\ &\leq L^n\left(\phi_3\left(\frac{x}{2},-\frac{x}{2}\right)+\phi_3\left(-\frac{x}{2},\frac{x}{2}\right)\right),\end{aligned} \tag{3.36}$$

for all $x \in V \setminus \{0\}$. It leads us to get

$$\lim_{n\to\infty} 4^n\left(f_3\left(\frac{x}{2^n}\right)-f_3\left(-\frac{x}{2^n}\right)\right)=0, \qquad \lim_{n\to\infty} 2^n\left(f_3\left(\frac{x}{2^n}\right)-f_3\left(-\frac{x}{2^n}\right)\right)=0, \tag{3.37}$$

for all $x \in V \setminus \{0\}$. From these and (2.25), we obtain (3.28). Moreover, we have

$$\left\|4^n Qf_3\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right\| \leq 4^n\phi_3\left(\frac{x}{2^n},\frac{y}{2^n}\right) \leq L^n\phi_3(x,y), \tag{3.38}$$

for all $x, y \in V \setminus \{0\}$. Taking the limit as $n \to \infty$ in the above inequality and using $F_3(0) = 0$, we get

$$QF_3(x,y)=0, \tag{3.39}$$

for all $x, y \in V$. □

Now, we obtain Hyers-Ulam-Rassias stability results in the framework of normed spaces using Theorems 2.3 and 2.4.

Corollary 3.3. *Let X be a normed space, and let Y be a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \theta\left(\|x\|^p+\|y\|^p\right) \tag{3.40}$$

for all $x, y \in X \setminus \{0\}$, where $\theta \geq 0$ and $p \in (-\infty, 0) \cup (0, 1) \cup (2, \infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x)-F(x)\| \leq \begin{cases}\dfrac{\theta}{2^p-4}\|x\|^p & \text{if } p>2,\\ \dfrac{\theta}{2-2^p}\|x\|^p & \text{if } 0<p<1\end{cases} \tag{3.41}$$

for all $x \in X \setminus \{0\}$. Moreover if $p < 0$, then f is itself a quadratic-additive mapping.

Proof. This corollary follows from Theorems 2.3, 2.4, and 2.5, and Remark 2.6, by putting

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p), \tag{3.42}$$

for all $x, y \in X \setminus \{0\}$ with $L = 2^{p-1} < 1$ if $p < 1$, $L = 2^{2-p} < 1$ if $p > 2$, and $L' = 2^{-p} < 1$ if $p > 0$. □

Corollary 3.4. *Let X be a normal space let and Y be a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \le \theta\|x\|^p\|y\|^q, \tag{3.43}$$

for all $x, y \in X \setminus \{0\}$, where $\theta \ge 0$ and $p + q \in (-\infty, 0) \cup (0, 1) \cup (2, \infty)$. Then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \le \begin{cases} \dfrac{\theta\|x\|^{p+q}}{2(2^{p+q} - 4)} & \text{if } p + q > 2, \\ \dfrac{\theta\|x\|^{p+q}}{2(2 - 2^{p+q})} & \text{if } 0 < p + q < 1, \end{cases} \tag{3.44}$$

for all $x \in X \setminus \{0\}$. Moreover, if $p + q < 0$, then f is itself a quadratic-additive mapping.

Proof. This corollary follows from Theorems 2.3, 2.4, 2.5 and Remark 2.6, by putting

$$\varphi(x,y) := \theta\|x\|^p\|y\|^q, \tag{3.45}$$

for all $x, y \in X \setminus \{0\}$ with $L = 2^{p+q-1} < 1$ if $p+q < 1$, $L = 2^{2-p-q} < 1$ if $p+q > 2$, and $L' = 2^{-p-q} < 1$ if $p + q > 0$. □

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