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Research Article

On the Difference Equation $x_{n+1} = x_n x_{n-2} - 1$

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The long-term behavior of solutions of the following difference equation: $x_{n+1} = x_n x_{n-2} - 1$, $n \in \mathbb{N}_0$, where the initial values x_{-2}, x_{-1}, x_0 are real numbers, is investigated in the paper.

1. Introduction

Recently there has been a huge interest in studying nonlinear difference equations which do not stem from differential equations (see, e.g., [1–36] and the references therein). Usual properties which have been studied are the boundedness character [8, 13, 15, 28–30, 33, 35, 36], the periodicity [8, 13], asymptotic periodicity [16–19, 21], local and global stability [1, 8, 13, 15, 16, 28–34], as well as the existence of specific solutions such as monotone or nontrivial [2, 3, 5, 9, 10, 15, 18, 20, 22–27].

In this paper we will study solutions of the following difference equation:

$$x_{n+1} = x_n x_{n-2} - 1, \quad n \in \mathbb{N}_0. \tag{1.1}$$

The difference equation (1.1) belongs to the class of equations of the form

$$x_{n+1} = x_{n-k} x_{n-l} - 1, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with particular choices of k and l, where $k, l \in \mathbb{N}_0$. Although (1.2) looks simple, it is fascinating how its behavior changes for different choices of k and l. The cases $k = 0$ and $l = 1$, $k = 1$ and

$l = 2$ have been correspondingly investigated in papers [11, 12]. This paper can be regarded as a continuation of our systematic investigation of (1.2).

Note that (1.1) has two equilibria:

$$\overline{x}_1 = \frac{1-\sqrt{5}}{2}, \qquad \overline{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{1.3}$$

2. Periodic Solutions

In this section we prove some results regarding periodicity of solutions of (1.1). The first result concerns periodic solutions with prime period two which will play an important role in studying the equation.

Theorem 2.1. *Equation* (1.1) *has prime period-two solutions if and only if the initial conditions are* $x_{-2} = 0$, $x_{-1} = -1$, $x_0 = 0$ *or* $x_{-2} = -1$, $x_{-1} = 0$, $x_0 = -1$.

Proof. Let $(x_n)_{n=-2}^{\infty}$ be a prime period-two solution of (1.1). Then $x_{2n-2} = a$ and $x_{2n-1} = b$, for every $n \in \mathbb{N}_0$ and for some $a, b \in \mathbb{R}$ such that $a \neq b$. We have $x_1 = x_0 x_{-2} - 1 = a^2 - 1 = b$ and $x_2 = x_1 x_{-1} - 1 = b^2 - 1 = a$. From these two equations we obtain $(a^2-1)^2 - 1 = a$ or equivalently

$$a(a+1)\left(a^2 - a - 1\right) = 0. \tag{2.1}$$

We have four cases to be considered.

Case 1. If $a = 0$, then $b = -1$, and we obtain the first prime period-two solution.

Case 2. If $a = -1$, then $b = 0$, and we obtain the second prime period-two solution.

Case 3. If $a = \overline{x}_1$, then $b = a^2 - 1 = \overline{x}_1$, which is an equilibrium solution.

Case 4. If $a = \overline{x}_2$, then $b = a^2 - 1 = \overline{x}_2$, which is the second equilibrium solution. Thus, the result holds. □

Theorem 2.2. *Equation* (1.1) *has no prime period-three solutions.*

Proof. Let $(x_n)_{n=-2}^{\infty}$ be a prime period-three solution of (1.1). Then $x_{3n-2} = a$, $x_{3n-1} = b$, $x_{3n} = c$, for every $n \in \mathbb{N}_0$ and some $a, b, c \in \mathbb{R}$ such that at least two of them are different. We have

$$\begin{aligned} x_1 &= x_0 x_{-2} - 1 = ac - 1 = a, \\ x_2 &= x_1 x_{-1} - 1 = ab - 1 = b, \\ x_3 &= x_2 x_0 - 1 = bc - 1 = c. \end{aligned} \tag{2.2}$$

From (2.2) we easily see that $a \neq 0$, $b \neq 0$, and $c \neq 0$, so that

$$c = 1 + \frac{1}{a}, \qquad a = 1 + \frac{1}{b}, \qquad b = 1 + \frac{1}{c}. \tag{2.3}$$

From (2.3) we obtain

$$c = 1 + \frac{b}{b+1} = \frac{2b+1}{b+1} \Longrightarrow b = 1 + \frac{b+1}{2b+1} = \frac{3b+2}{2b+1}, \tag{2.4}$$

which implies $b^2 - b - 1 = 0$. Hence $b = \overline{x}_1$ or $b = \overline{x}_2$. From this and (2.3) it follows that $a = b = c = \overline{x}_1$ or $a = b = c = \overline{x}_2$, from which the result follows. □

Theorem 2.3. *Equation* (1.1) *has no prime period-four solutions.*

Proof. Let $(x_n)_{n=-2}^{\infty}$ be a prime period-four solution of (1.1) and $x_{-2} = a,\ x_{-1} = b,\ x_0 = c$. Then we have

$$x_1 = x_0 x_{-2} - 1 = ac - 1, \tag{2.5}$$

$$x_2 = x_1 x_{-1} - 1 = (ac-1)b - 1 = a, \tag{2.6}$$

$$x_3 = x_2 x_0 - 1 = ac - 1 = b, \tag{2.7}$$

$$x_4 = x_3 x_1 - 1 = b(ac-1) - 1 = c. \tag{2.8}$$

Thus, from (2.6) and (2.8), we have that $a = c$. This along with (2.7) gives

$$a^2 - 1 = b, \tag{2.9}$$

while from (2.8) we get $b(a^2-1) - 1 = a$ or equivalently

$$(a+1)(b(a-1)-1) = 0. \tag{2.10}$$

Case 1. Suppose $a = -1$. Then $b = a^2 - 1 = 0$ and $c = a = -1$, which, by Theorem 2.1, yields a period-two solution.

Suppose $a \neq -1$. If $a = 1$, then from (2.10) we get a contradiction. If $a \neq 1$, then $a^2 - 1 = b = 1/(a-1)$, so that $a(a^2 - a - 1) = 0$. Hence $a = 0$, $a = \overline{x}_1$, or $a = \overline{x}_2$.

Case 2. Suppose $a = 0$. Then $b = a^2 - 1 = -1$ and $c = a = 0$ which results in a period-two solution as proved in Theorem 2.1.

Case 3. Suppose $a = \overline{x}_1$. Then $b = a^2 - 1 = \overline{x}_1$ and $c = \overline{x}_1$ which is an equilibrium solution.

Case 4. Suppose $a = \overline{x}_2$. Then $b = a^2 - 1 = \overline{x}_2$ and $c = \overline{x}_2$ which is the second equilibrium solution. Proof is complete. □

3. Local Stability

Here we study the local stability at the equilibrium points $\overline{x}_1$ and $\overline{x}_2$.

Theorem 3.1. *The negative equilibrium of* (1.1), $\overline{x}_1$, *is unstable. Moreover, it is a hyperbolic equilibrium.*

Proof. The linearized equation associated with the equilibrium $\overline{x}_1 = (1-\sqrt{5})/2 \in (-1,0)$ is

$$x_{n+1} - \overline{x}_1 x_n - \overline{x}_1 x_{n-2} = 0. \tag{3.1}$$

Its characteristic polynomial is

$$P_{\overline{x}_1}(\lambda) = \lambda^3 - \overline{x}_1\lambda^2 - \overline{x}_1. \tag{3.2}$$

Hence

$$P'_{\overline{x}_1}(\lambda) = 3\lambda^2 - 2\overline{x}_1\lambda = \lambda(3\lambda - 2\overline{x}_1). \tag{3.3}$$

Since

$$P_{\overline{x}_1}(-2) = -8 - 5\overline{x}_1 < 0, \qquad P_{\overline{x}_1}(-1) = -1 - 2\overline{x}_1 = \sqrt{5} - 2 > 0, \tag{3.4}$$

there is a zero $\lambda_1 \in (-2,-1)$ of $P_{\overline{x}_1}$.

On the other hand, from $P_{\overline{x}_1}(0) = -\overline{x}_1 > 0$ and (3.3), it follows that λ_1 is a unique real zero of $P_{\overline{x}_1}$. Hence, the other two roots $\lambda_{2,3}$ are conjugate complex.

Since

$$\lambda_1|\lambda_2|^2 = \overline{x}_1, \tag{3.5}$$

we obtain $|\lambda_2| = |\lambda_3| < 1$. From this, the theorem follows. □

Theorem 3.2. *The positive equilibrium of* (1.1), $\overline{x}_2$, *is unstable. Moreover, it is also a hyperbolic equilibrium.*

Proof. The linearized equation associated with the equilibrium $\overline{x}_2 = (1+\sqrt{5})/2 \in (1,2)$ is

$$x_{n+1} - \overline{x}_2 x_n - \overline{x}_2 x_{n-2} = 0. \tag{3.6}$$

Its characteristic polynomial is

$$P_{\overline{x}_2}(\lambda) = \lambda^3 - \overline{x}_2\lambda^2 - \overline{x}_2. \tag{3.7}$$

We have

$$P'_{\overline{x}_2}(\lambda) = 3\lambda^2 - 2\overline{x}_2\lambda = \lambda(3\lambda - 2\overline{x}_2). \tag{3.8}$$

Since

$$P_{\overline{x}_2}(2) = 8 - 5\overline{x}_2 = \frac{\left(11 - 5\sqrt{5}\right)}{2} < 0, \qquad P_{\overline{x}_2}(3) = 27 - 10\overline{x}_2 = 22 - 5\sqrt{5} > 0, \tag{3.9}$$

there is a zero $\lambda_1 \in (2,3)$ of $P_{\overline{x}_2}$.

From this and since $P_{\overline{x}_2}(0) = -\overline{x}_2 < 0$, we have that λ_1 is a unique real zero of $P_{\overline{x}_2}$. Thus, the other two roots $\lambda_{2,3}$ are conjugate complex.

Since

$$\lambda_1|\lambda_2|^2 = \overline{x}_2, \tag{3.10}$$

we obtain $|\lambda_2| = |\lambda_3| < 1$. From this, the theorem follows. □

4. Case $x_{-2}, x_{-1}, x_0 \in (-1, 0)$

This section considers the solutions of (1.1) with $x_{-2}, x_{-1}, x_0 \in (-1, 0)$. Before we formulate the main result in this section we need some auxiliary results.

Lemma 4.1. *Suppose that* $x_{-2}, x_{-1}, x_0 \in (-1, 0)$. *Then the solution* $(x_n)_{n=-2}^{\infty}$ *of* (1.1) *is such that* $x_n \in (-1, 0)$ *for* $n \geq -2$.

Proof. We have $x_{-2}, x_0 \in (-1, 0)$ which implies $x_1 = x_0 x_{-2} - 1 \in (-1, 0)$. Assume that we have proved $x_n \in (-1, 0)$ for $-2 \leq n \leq k$, for some $k \geq 2$. Then we have $x_{k+1} = x_k x_{k-2} - 1 \in (-1, 0)$, finishing an inductive proof of the lemma. □

Remark 4.2. We would like to say here that a similar argument gives the following extension of Lemma 4.1.

Suppose $k_i \in \mathbb{N}$, $1 \leq i \leq 2m$, $x_{-s}, \ldots, x_{-1}, x_0 \in (-1, 0)$, where $s = \max\{k_1, \ldots, k_{2m}\}$. Then the solution $(x_n)_{n=-s}^{\infty}$ of the difference equation

$$x_{n+1} = \prod_{i=1}^{2m} x_{n-k_i} - 1, \quad n \in \mathbb{N}_0, \tag{4.1}$$

is such that $x_n \in (-1, 0)$ for $n \geq -s$.

We now find an equation which is satisfied for the even terms of a solution of (1.1) as well as for the odd terms of the solution.

From (1.1) we have

$$x_{2n+3} = x_{2n+2} x_{2n} - 1, \qquad x_{2n+2} = x_{2n+1} x_{2n-1} - 1, \quad n \in \mathbb{N}_0. \tag{4.2}$$

Then we have the following:

$$\begin{aligned} x_{2n+3} &= (x_{2n+1} x_{2n-1} - 1)(x_{2n-1} x_{2n-3} - 1) - 1 \\ &= x_{2n-1}(x_{2n+1} x_{2n-1} x_{2n-3} - x_{2n+1} - x_{2n-3}), \quad n \in \mathbb{N}, \end{aligned} \tag{4.3}$$

and similarly

$$x_{2n+4} = x_{2n}(x_{2n+2} x_{2n} x_{2n-2} - x_{2n+2} - x_{2n-2}), \quad n \in \mathbb{N}. \tag{4.4}$$

Hence, the subsequences $y_n = x_{2n+1}$ and $z_n = x_{2n+2}$ satisfy the difference equation

$$u_{n+1} = u_{n-1}(u_n u_{n-1} u_{n-2} - u_n - u_{n-2}), \quad n \in \mathbb{N}, \tag{4.5}$$

and $u_n \in (-1, 0)$.

For convenience, we make another change of variable $v_n = -u_n$. Then (4.5) becomes

$$v_{n+1} = v_{n-1}(v_n + v_{n-2} - v_n v_{n-1} v_{n-2}), \quad n \in \mathbb{N}. \tag{4.6}$$

Note also that $v_n \in (0, 1)$.

It is easy to see that (4.6) has the following four equilibria:

$$\overline{v}_0 = -\frac{1+\sqrt{5}}{2}, \qquad \overline{v}_1 = 0, \qquad \overline{v}_2 = \frac{\sqrt{5}-1}{2}, \qquad \overline{v}_3 = 1. \tag{4.7}$$

If we let

$$f(u, v, w) = v(u + w - uvw), \tag{4.8}$$

where $u, v, w \in (0, 1)$, then we find the following:

(1) $f_u = v - v^2 w = v(1 - vw) > 0$,

(2) $f_v = u + w - 2vuw = u(1 - vw) + w(1 - vu) > 0$,

(3) $f_w = v - v^2 u = v(1 - vu) > 0$.

Thus, the function $f(u, v, w)$ is strictly increasing in each argument.

Lemma 4.3. *Let $(x_n)_{n=-2}^{\infty}$ be a solution of (1.1) which is not equal to the equilibrium solution*

$$\overline{x}_1 = \frac{1-\sqrt{5}}{2} \tag{4.9}$$

of the equation. Suppose that

$$x_{-2}, x_{-1}, x_0 \in (-1, 0) \tag{4.10}$$

and that one of the following conditions holds:

(H1) $x_{-2} \leq \overline{x}_1$, $x_{-1} \geq \overline{x}_1$, $x_0 \leq \overline{x}_1$ with at least one of the inequalities strict,

(H2) $x_{-2} \geq \overline{x}_1$, $x_{-1} \leq \overline{x}_1$, $x_0 \geq \overline{x}_1$ with at least one of the inequalities strict.

Then $x_n \in (-1,0)$*, for every* $n \geq -2$ *and*

(a) if (H1) holds, then there is an $N \in \mathbb{N}_0$ *such that*

$$x_{2n+1} > \overline{x}_1, \qquad x_{2n+2} < \overline{x}_1, \quad n \geq N, \tag{4.11}$$

(b) if (H2) holds, then there is an $N \in \mathbb{N}_0$ *such that*

$$x_{2n+1} < \overline{x}_1, \qquad x_{2n+2} > \overline{x}_1, \quad n \geq N. \tag{4.12}$$

Proof. By Lemma 4.1 we have that $x_n \in (-1,0)$, for $n \geq -2$. We will prove only *(a)*. The proof of *(b)* is dual and is omitted. Since $-1 < x_0, x_{-2} \leq \overline{x}_1$, we have

$$x_1 = x_0 x_{-2} - 1 \geq \overline{x}_1^2 - 1 = \overline{x}_1. \tag{4.13}$$

From this and since $\overline{x}_1 \leq x_{-1} < 0$, we have

$$x_2 = x_1 x_{-1} - 1 \leq \overline{x}_1^2 - 1 = \overline{x}_1. \tag{4.14}$$

If $x_{-2} < \overline{x}_1$ or $x_0 < \overline{x}_1$, then inequality (4.13) is strict and, consequently, inequality (4.14) is strict too. If $x_{-2} = x_0 = \overline{x}_1$, then $x_{-1} > \overline{x}_1$, from which it follows that inequality (4.14) is strict. In this case we have

$$x_3 = x_2 x_0 - 1 > \overline{x}_1^2 - 1 = \overline{x}_1, \tag{4.15}$$

which is a strict inequality. Hence $N = 0$ and $N = 1$ are the obvious candidates, depending on which of the two cases, just described, holds.

Assume that we have proved (4.11) for $N \leq n \leq k$ and that $N = 0$. The case $N = 1$ is proved similarly and so is omitted. Then we have

$$x_{2k+3} = x_{2k+2} x_{2k} - 1 > \overline{x}_1^2 - 1 = \overline{x}_1. \tag{4.16}$$

From this and since $\overline{x}_1 < x_{2k+1} < 0$, we have

$$x_{2k+4} = x_{2k+3} x_{2k+1} - 1 < \overline{x}_1^2 - 1 = \overline{x}_1. \tag{4.17}$$

Hence by induction the lemma follows. □

Theorem 4.4. *Let* $(x_n)_{n=-2}^{\infty}$ *be a solution of* (1.1) *which is not equal to the equilibrium solution*

$$\overline{x}_1 = \frac{1 - \sqrt{5}}{2} \tag{4.18}$$

of the equation. Suppose that

$$x_{-2}, x_{-1}, x_0 \in (-1, 0) \tag{4.19}$$

and that one of the conditions, (H1) or (H2), holds.

Then $x_n \in (-1, 0)$, for every $n \geq -2$, and $(x_n)_{n=-2}^{\infty}$ converges to a two-cycle $\{-1, 0\}$.

Proof. First of all $x_n \in (-1, 0)$, for $n \geq -2$, by Lemma 4.1. We next show that $(x_n)_{n=-2}^{\infty}$ converges to a two-cycle $\{-1, 0\}$. To this end we show that one of the subsequences, $(x_{2n})_{n=-1}^{\infty}$ or $(x_{2n+1})_{n=-1}^{\infty}$, converges to 0 and the other one to -1. Showing this, in turn, is equivalent to showing the following:

(a) the corresponding solution $(v_n)_{n=-1}^{\infty}$ of (4.6) converges to $\overline{v}_3$ if for some $N \geq -1$, $v_n > \overline{v}_2$, for $n \geq N$, where $v_n \in (0, 1) = (\overline{v}_1, \overline{v}_3)$ for $n \geq -1$,

(b) the corresponding solution $(v_n)_{n=-1}^{\infty}$ of (4.6) converges to $\overline{v}_1$ if for some $N \geq -1, v_n < \overline{v}_2$, for $n \geq N$, where $v_n \in (0, 1) = (\overline{v}_1, \overline{v}_3)$ for $n \geq -1$.

We prove (a). The proof of (b) is similar and will be omitted. We have

$$v_n \in (\overline{v}_2, \overline{v}_3), \quad n \geq N. \tag{4.20}$$

Let

$$I = \liminf_{n \to \infty} v_n, \qquad S = \limsup_{n \to \infty} v_n. \tag{4.21}$$

Then we have

$$\overline{v}_2 \leq I \leq S \leq \overline{v}_3. \tag{4.22}$$

First assume $I = \overline{v}_2$. From (4.20) it follows that there is an $\varepsilon > 0$ such that $I + \varepsilon < v_N, v_{N+1}, v_{N+2} < \overline{v}_3$. By the monotonicity of f and since

$$f(x, x, x) > x \quad \text{for } x \in (\overline{v}_2, \overline{v}_3), \tag{4.23}$$

we have

$$v_{N+3} = f(v_{N+2}, v_{N+1}, v_N) > f(I + \varepsilon, I + \varepsilon, I + \varepsilon) > I + \varepsilon. \tag{4.24}$$

From this and by induction we obtain $v_n > I + \varepsilon$, $n \geq N$, which implies $\liminf_{n \to \infty} v_n \geq I + \varepsilon$, which is a contradiction.

Now assume $I \in (\overline{v}_2, \overline{v}_3)$. Let $(v_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(v_n)_{n=-1}^{\infty}$ such that $\lim_{k \to \infty} v_{n_k} = I$. Then there is a subsequence of $(v_{n_k})_{k \in \mathbb{N}}$, which we may denote the same, such that there are the following limits: $\lim_{k \to \infty} v_{n_k - 1}$, $\lim_{k \to \infty} v_{n_k - 2}$, and $\lim_{k \to \infty} v_{n_k - 3}$, which we denote, respectively, by K_{-i}, $i = 1, 2, 3$. From this and by (4.23) we have that

$$f(K_{-1}, K_{-2}, K_{-3}) = I < f(I, I, I). \tag{4.25}$$

Hence there is an $i_0 \in \{1,2,3\}$ such that $K_{-i_0} < I$. Otherwise, $K_{-i} \geq I$ for $i = 1,2,3$ and by the monotonicity of f we would get

$$f(I,I,I) \leq f(K_{-1},K_{-2},K_{-3}) = I < f(I,I,I), \tag{4.26}$$

which is a contradiction. On the other hand, $K_{-i_0} < I$ contradicts the choice of I. Hence I cannot be in the interval $(\overline{v}_2, \overline{v}_3)$.

From all of the above we have that $\overline{v}_3 = I \leq S \leq \overline{v}_3$. Therefore, $\lim_{n\to\infty} v_n = \overline{v}_3$, as desired. □

Theorem 4.5. *Assume that for a solution $(x_n)_{n=-2}^{\infty}$ of (1.1) there is an $N \geq -1$ such that*

$$-1 < x_N < x_{N+2} < 0, \qquad 0 > x_{N-1} > x_{N+1} > x_{N+3} > -1. \tag{4.27}$$

Then the solution converges to a two-cycle $\{-1,0\}$ or to the equilibrium $\overline{x}_1$.

Proof. First note that by Lemma 4.1 we have $x_n \in (-1,0)$, $n \geq N$. From (1.1) we obtain the identity

$$x_{n+4} - x_{n+2} = x_{n+1}(x_{n+3} - x_{n-1}). \tag{4.28}$$

Applying (4.28) for $n = N$ and using the fact $x_{N+1} \in (-1,0)$, we get $0 > x_{N+4} > x_{N+2}$. Hence

$$x_N < x_{N+2} < x_{N+4} < 0, \qquad x_{N-1} > x_{N+1} > x_{N+3} > -1. \tag{4.29}$$

Using induction along with identity (4.28) it is shown that

$$x_N < x_{N+2} < \cdots < x_{N+2k} < 0, \qquad x_{N-1} > x_{N+1} > \cdots > x_{N+2k+1} > -1, \tag{4.30}$$

for every $k \in \mathbb{N}$. Hence, there are finite limits $\lim_{k\to\infty} x_{N+2k}$ and $\lim_{k\to\infty} x_{N+2k+1}$, say l_1 and l_2. Letting $k \to \infty$ in the relations

$$x_{N+2k+2} = x_{N+2k+1}x_{N+2k-1} - 1, \qquad x_{N+2k+3} = x_{N+2k+2}x_{N+2k} - 1, \tag{4.31}$$

we get $l_1 = l_2^2 - 1$ and $l_2 = l_1^2 - 1$. Hence

$$(l_1 - l_2)(l_1 + l_2 + 1) = 0. \tag{4.32}$$

From this we have $l_1 = l_2 = \overline{x}_1$, or if $l_1 \neq l_2$, then $l_1 + l_2 = -1$ so that $l_1 = 0$ and $l_2 = -1$. □

Remark 4.6. Let $x_{-2} = a$, $x_{-1} = b$ and $x_0 = c$ with $a, b, c \in (-1, 0)$. For $N = -1$, (4.27) will be

$$x_{-2} > x_0 \Longleftrightarrow a > c,$$
$$x_1 - x_{-1} = x_0 x_{-2} - 1 - x_{-1} = ac - 1 - b > 0, \tag{4.33}$$
$$x_2 - x_0 = x_1 x_{-1} - 1 - x_0 = x_0 x_{-1} x_{-2} - x_{-1} - x_0 - 1 = abc - b - c - 1 < 0.$$

Hence, under the conditions

$$a > c, \qquad ac > b + 1, \qquad abc < b + c + 1, \tag{4.34}$$

we have that (4.27) is satisfied for $N = -1$. It is easy to show that there are some $a, b, c \in (-1, 0)$ such that the set in (4.34) is nonempty.

Note also that in the proof of Theorem 4.5 the relation (4.28) plays an important role. Relations of this type have been successfully used also in [17, 21].

It is a natural question if there are nontrivial solutions of (1.1) converging to the negative equilibrium $\overline{x}_1$. The next theorem, which is a product of an E-mail communication between Stević and Professor Berg [6], gives a positive answer to the question. In the proof of the result we use an asymptotic method from Proposition 3.3 in [3]. Some asymptotic methods for solving similar problems have been also used, for example, in the following papers: [2–5, 20, 22–27]. For related results, see also [9, 10, 15, 18] and the references therein.

Theorem 4.7. *There are nontrivial solutions of* (1.1) *converging to the negative equilibrium* $\overline{x}_1$.

Proof. In order to find a solution tending to $\overline{x}_1$, we make the substitution $x_n = y_n + \overline{x}_1$, yielding the equation

$$y_{n+3} - \overline{x}_1 (y_{n+2} + y_n) = y_{n+2} y_n, \quad n \geq -2, \tag{4.35}$$

and for $n \in \mathbb{N}_0$ we make the ansatz

$$y_n = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} a_{kl} p^{nk} q^{nl}, \tag{4.36}$$

with $a_{00} = 0$, where p and q are the conjugate complex zeros of the characteristic polynomial

$$P(\lambda) = \lambda^3 - \overline{x}_1 \left(\lambda^2 + 1\right). \tag{4.37}$$

Note that $|p| = |q| = r \approx 0.74448$.

Replacing (4.36) into (4.35) and comparing the coefficients, we find

$$d_{kl} a_{kl} = \sum_{i=0}^{k} \sum_{j=0,\ j+i \neq 0}^{l} a_{ij} p^{2i} q^{2j} a_{k-i, l-j}, \tag{4.38}$$

with

$$d_{kl} = p^{3k}q^{3l} - \overline{x}_1\left(p^{2k}q^{2l} + 1\right). \tag{4.39}$$

Equation (4.38) is satisfied for $k + l \leq 1$, where a_{10} and a_{01} are arbitrary, so that it suffices to consider (4.38) for k and l such that $k + l > 1$. If a_{10} and a_{01} are chosen to be conjugate complex numbers, then according to (4.38) all a_{kl} are conjugate complex numbers to a_{lk} and consequently series (4.36) is real. We look for a solution (4.36) with $a_{10} = \overline{a_{01}}$, $|a_{10}| = 1$, and determine a positive constant λ such that

$$|a_{kl}| \leq \lambda^{k+l-1}. \tag{4.40}$$

Since the inequality is valid for $k + l \leq 1$, by induction, we get from (4.38)

$$|a_{kl}| \leq \lambda^{k+l-2}\frac{1}{|d_{kl}|}\sum_{i=0}^{k}\sum_{j=0,\ j+i\neq 0}^{l} r^{2(i+j)}. \tag{4.41}$$

Note that

$$\sum_{i=0}^{k}\sum_{j=0,\ j+i\neq 0}^{l} r^{2(i+j)} < \frac{1}{(1-r^2)^2} - 1 = \frac{2r^2 - r^4}{(1-r^2)^2}. \tag{4.42}$$

It is not difficult to check that

$$D := \sup_{k+l\geq 2}\frac{1}{|d_{kl}|} = \frac{1}{|d_{21}|} \approx 2.095. \tag{4.43}$$

Hence (4.40) holds with

$$\lambda = D\frac{2r^2 - r^4}{(1-r^2)^2}. \tag{4.44}$$

For such chosen λ the series in (4.36) converges if $\lambda r^n < 1$, which implies $n > \ln\lambda/\ln(1/r)$. We have $\lambda \approx 8.450$, so that $\ln\lambda/\ln(1/r) \approx 7.233$, and therefore we have the convergence of the series for $n > 7$. In this way for $n > 7$, we obtain a solution of (4.35) converging to a real solution of (4.35) as $n \to \infty$, that is, a solution of (1.1) converging to $\overline{x}_1$, as desired. □

Remark 4.8. For $a_{10} = a_{01} = 1$, the first coefficients in (4.38) are

$$\begin{gathered} a_{20} = \frac{p^2}{d_{20}}, \qquad a_{11} = \frac{p^2 + q^2}{d_{11}}, \qquad a_{02} = \frac{q^2}{d_{02}}, \\ a_{30} = \frac{p^2(p^2+1)a_{20}}{d_{30}}, \qquad a_{21} = \frac{(p^4 + q^2)a_{20} + p^2(q^2+1)a_{11}}{d_{21}}. \end{gathered} \tag{4.45}$$

Remark 4.9. If we replace n by $n+c$ in (4.35) with an arbitrary $c \in \mathbb{R}$, then we can choose c in such a way that we get arbitrary first coefficients (not only of modulus 1).

5. Unbounded Solutions of (1.1)

In this section we find sets of initial values of (1.1) for which unbounded solutions exist. For related results, see, for example, [8, 13, 15, 28–30, 33, 35, 36] and the references therein.

The next theorem shows the existence of unbounded solutions of (1.1).

Theorem 5.1. *Assume that*

$$\min\{|x_{-2}|, |x_{-1}|, |x_0|\} > \overline{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{5.1}$$

Then

$$\overline{x}_2 < |x_0| < |x_1| < |x_2| < \cdots < |x_n| < \cdots. \tag{5.2}$$

Proof. From the hypothesis we have that $|x_{-2}|-1 > \overline{x}_2-1$, and so $|x_0|(|x_{-2}|-1) > \overline{x}_2(\overline{x}_2-1) = 1$. Therefore, $|x_0||x_{-2}| - |x_0| > 1$, and so $|x_0||x_{-2}| - 1 > |x_0|$. On the other hand, we have

$$|x_1| = |x_0 x_{-2} - 1| > |x_0||x_{-2}| - 1. \tag{5.3}$$

Combining the last two inequalities, we have that $|x_1| > |x_0| > \overline{x}_2$. Assume that we have proved

$$\overline{x}_2 < |x_0| < |x_1| < |x_2| < \cdots < |x_k|, \tag{5.4}$$

for some $k \in \mathbb{N}$. We have $|x_{k-2}| - 1 > \overline{x}_2 - 1$, which implies $|x_k|(|x_{k-2}| - 1) > \overline{x}_2(\overline{x}_2 - 1) = 1$, or equivalently $|x_k||x_{k-2}| - 1 > |x_k|$. From this and (1.1), we get

$$|x_{k+1}| = |x_k x_{k-2} - 1| > |x_k||x_{k-2}| - 1 > |x_k| > \overline{x}_2, \tag{5.5}$$

finishing the inductive proof of the theorem. □

Corollary 5.2. *Assume that the initial values of a solution $(x_n)_{n=-2}^{\infty}$ of (1.1) satisfy the condition*

$$\min\{x_{-2}, x_{-1}, x_0\} > \overline{x}_2 = \frac{1+\sqrt{5}}{2}. \tag{5.6}$$

Then the solution tends to $+\infty$.

Proof. Assume to the contrary that the sequence does not tend to plus infinity. Since the sequence is increasing and bounded, then it must converge. But (1.1) has only two equilibria, and they are both less than x_0. We have a contradiction. The proof is complete. □

For our next result, we need to introduce the following definition.

Definition 5.3. *Let* $(x_n)_{n=-2}^{\infty}$ *be a solution of* (1.1), *and let* $i \in \{1,2\}$. *Then we say that the solution has the* eventual semicycle pattern k^+, l^- (or k^-, l^+) *if there exists* $N \in \mathbb{N}$ *such that, for* $n \in \mathbb{N}_0$, $x_{N+n(k+l)+1}, \ldots, x_{N+n(k+l)+k} \geq \overline{x}_i$ *and* $x_{N+n(k+l)+k+1}, \ldots, x_{N+n(k+l)+k+l} < \overline{x}_i$ *(or, resp.,* $x_{N+n(k+l)+1}, \ldots, x_{N+n(k+l)+k} < \overline{x}_i$ *and* $x_{N+n(k+l)+k+1}, \ldots, x_{N+n(k+l)+k+l} \geq \overline{x}_i$*).*

Remark 5.4. Note that the eventual semicycle pattern can be extended to $k_1^{\pm}, k_2^{\mp}, k_3^{\pm}, \ldots, k_M^{\mp}$ for $M > 2$.

Theorem 5.5. *Assume that* $(x_n)_{n=-2}^{\infty}$ *is a solution of* (1.1) *such that*

$$\min\{|x_{-2}|, |x_{-1}|, |x_0|\} > \overline{x}_2 = \frac{1+\sqrt{5}}{2} \tag{5.7}$$

and that at least one of x_{-2}, x_{-1}, x_0 *is negative. Then*

$$|x_n| \geq \overline{x}_2, \qquad n \geq -2, \tag{5.8}$$

and the solution is separated into seven unbounded eventually increasing subsequences such that the solution has the eventual semicycle pattern

$$1^+, 1^-, 2^+, 3^-. \tag{5.9}$$

Proof. Assume that $x_{-2}, x_{-1}, x_0 < -\overline{x}_2$. Then we have

$$\begin{aligned}
x_1 &= x_0 x_{-2} - 1 > \overline{x}_2^2 - 1 = \overline{x}_2, \\
x_2 &= x_1 x_{-1} - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2, \\
x_3 &= x_2 x_0 - 1 > \overline{x}_2^2 - 1 = \overline{x}_2, \\
x_4 &= x_3 x_1 - 1 > \overline{x}_2^2 - 1 = \overline{x}_2, \\
x_5 &= x_4 x_2 - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2, \\
x_6 &= x_5 x_3 - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2, \\
x_7 &= x_6 x_4 - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2.
\end{aligned} \tag{5.10}$$

Hence $|x_i| > \overline{x}_2$, for $-2 \leq i \leq 7$ and $x_5, x_6, x_7 < -\overline{x}_2 - 2 < -\overline{x}_2 < 0$. An inductive argument shows that

$$\begin{aligned}
x_{7k+1} &= x_{7k} x_{7k-2} - 1 > \overline{x}_2^2 - 1 = \overline{x}_2, \\
x_{7k+2} &= x_{7k+1} x_{7k-1} - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2, \\
x_{7k+3} &= x_{7k+2} x_{7k} - 1 > \overline{x}_2^2 - 1 = \overline{x}_2,
\end{aligned}$$

$$
\begin{aligned}
x_{7k+4} &= x_{7k+3}x_{7k+1} - 1 > \overline{x}_2^2 - 1 = \overline{x}_2, \\
x_{7k+5} &= x_{7k+4}x_{7k+2} - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2, \\
x_{7k+6} &= x_{7k+5}x_{7k+3} - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2, \\
x_{7k+7} &= x_{7k+6}x_{7k+4} - 1 < -\overline{x}_2^2 - 1 = -\overline{x}_2 - 2 < -\overline{x}_2,
\end{aligned}
\tag{5.11}
$$

for each $k \in \mathbb{N}_0$, from which the first part of the result follows in this case. The other six cases follow from the above case by shifting indices for 1, 2, 3, 4, 5, or 6 places forward.

From this and Theorem 5.1, we see that the sequences $(x_{7k+i})_{k=0}^{\infty}$ monotonically tend to $-\infty$ or $+\infty$ with the aforementioned eventual semicycle pattern. □

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