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Research Article

Existence of Solutions for a Class of Variable Exponent Integrodifferential System Boundary Value Problems

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This paper investigates the existence of solutions for a class of variable exponent integrodifferential system with multipoint and integral boundary value condition in half line. When the nonlinearity term f satisfies sub-$(p^- - 1)$ growth condition or general growth condition, we give the existence of solutions and nonnegative solutions via Leray-Schauder degree at nonresonance, respectively. Moreover, the existence of solutions for the problem at resonance has been discussed.

1. Introduction

In this paper, we consider the existence of solutions for the following variable exponent integrodifferential system

$$-\Delta_{p(t)}u + \delta f\Big(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u)\Big) = 0, \quad t \in (0, +\infty), \tag{1.1}$$

with the following nonlinear multipoint and integral boundary value condition

$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1, \qquad \lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + e_2, \tag{1.2}$$

where $u : [0,+\infty) \to \mathbb{R}^N$; S and T are linear operators defined by

$$S(u)(t) = \int_0^t \varphi(s,t)u(s)ds, \quad T(u)(t) = \int_0^{+\infty} \chi(s,t)u(s)ds, \tag{1.3}$$

where $\varphi \in C(D,\mathbb{R})$, $\chi \in C(D,\mathbb{R})$, $D = \{(s,t) \in [0,+\infty) \times [0,+\infty)\}$; $\int_0^{+\infty} |\varphi(s,t)|ds$ and $\int_0^{+\infty} |\chi(s,t)|ds$ are uniformly bounded with t; $p \in C([0,+\infty),\mathbb{R})$, $p(t) > 1$, $\lim_{t\to+\infty} p(t)$ exists and $\lim_{t\to+\infty} p(t) > 1$; $-\Delta_{p(t)} u := -(w(t)|u'|^{p(t)-2}u')'$ is called the weighted $p(t)$-Laplacian; $w \in C([0,+\infty),\mathbb{R})$ satisfies $0 < w(t)$, for all $t \in (0,+\infty)$, and $(w(t))^{-1/(p(t)-1)} \in L^1(0,+\infty)$; $0 < \xi_1 < \cdots < \xi_{m-2} < +\infty$, $\alpha_i \geq 0$, $(i = 1,\ldots,m-2)$ and $0 \leq \sum_{i=1}^{m-2} \alpha_i \leq 1$; $e \in L^1(0,+\infty)$ is nonnegative, $\sigma = \int_0^{+\infty} e(t)dt$ and $\sigma \in [0,1]$; $e_1, e_2 \in \mathbb{R}^N$; δ is a positive parameter.

If $\sum_{i=1}^{m-2} \alpha_i < 1$ and $\sigma < 1$, we say the problem is nonresonant; but if $\sum_{i=1}^{m-2} \alpha_i \in [0,1]$ and $\sigma = 1$, we say the problem is resonant.

The study of differential equations and variational problems with variable exponent growth conditions is a new and interesting topic. Many results have been obtained on these problems, for example, [1–23]. We refer to [3, 19, 23], for the applied background on these problems. If $w(t) \equiv 1$ and $p(t) \equiv p$ (a constant), $-\Delta_{p(t)}$ becomes the well-known p-Laplacian. If $p(t)$ is a general function, $-\Delta_{p(t)}$ represents a nonhomogeneity and possesses more nonlinearity, and thus $-\Delta_{p(t)}$ is more complicated than $-\Delta_p$. For example, if $\Omega \subset \mathbb{R}^n$ is a bounded domain, the Rayleigh quotient

$$\lambda_{p(x)} = \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega (1/p(x))|\nabla u|^{p(x)} dx}{\int_\Omega (1/p(x))|u|^{p(x)} dx} \tag{1.4}$$

is zero in general, and only under some special conditions $\lambda_{p(x)} > 0$ (see [9, 16–18]), but the fact that $\lambda_p > 0$ is very important in the study of p-Laplacian problems.

Integral boundary conditions for evolution problems have been applied variously in chemical engineering, thermoelasticity, underground water flow, and population dynamics. There are many papers on the differential equations with integral boundary value conditions, for example, [24–29]. On the existence of solutions for $p(x)$-Laplacian systems boundary value problems, we refer to [2, 4, 7, 8, 10–12, 20–22]. In [20], the present author deals with the existence and asymptotic behavior of solutions for (1.1) with the following linear boundary value conditions

$$u(0) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i) + e_0, \qquad \lim_{t\to+\infty} u(t) = \int_0^{+\infty} e(t)u(t)dt, \tag{1.5}$$

when $0 \leq \sum_{i=1}^{m-2} \alpha_i < 1$ and $0 \leq \int_0^{+\infty} e(t) \leq 1$. But results on the existence of solutions for variable exponent integrodifferential systems with nonlinear boundary value conditions are rare. In this paper, when $p(t)$ is a general function, we investigate the existence of solutions and nonnegative solutions for variable exponent integrodifferential systems with nonlinear multipoint and integral boundary value conditions, when the problem is at nonresonance. Moreover, we discuss the existence of solutions for the problem at resonance. Since the nonlinear multipoint boundary value condition is on the derivative of solution u, we meet more difficulties than [20].

Let $N \geq 1$ and $J = [0, +\infty)$, the function $f = (f^1, \dots, f^N) : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be Caratheodory, by this we mean,

(i) for almost every $t \in J$, the function $f(t, \cdot, \cdot, \cdot, \cdot)$ is continuous;

(ii) for each $(x, y, z, w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, the function $f(\cdot, x, y, z, w)$ is measurable on J;

(iii) for each $R > 0$, there is a $\beta_R \in L^1(J, \mathbb{R})$ such that, for almost every $t \in J$ and every $(x, y, z, w) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x| \leq R,\ |y| \leq R,\ |z| \leq R,\ |w| \leq R$, one has

$$\left| f(t, x, y, z, w) \right| \leq \beta_R(t). \tag{1.6}$$

Throughout the paper, we denote

$$\begin{gathered} w(0)\left|u'\right|^{p(0)-2}u'(0) = \lim_{t \to 0^+} w(t)\left|u'\right|^{p(t)-2}u'(t), \\ w(+\infty)\left|u'\right|^{p(+\infty)-2}u'(+\infty) = \lim_{t \to +\infty} w(t)\left|u'\right|^{p(t)-2}u'(t). \end{gathered} \tag{1.7}$$

The inner product in $\mathbb{R}^N$ will be denoted by $\langle \cdot, \cdot \rangle$, $|\cdot|$ will denote the absolute value and the Euclidean norm on $\mathbb{R}^N$. Let $AC(0, +\infty)$ denote the space of absolutely continuous functions on the interval $(0, +\infty)$. For $N \geq 1$, we set $C = C(J, \mathbb{R}^N)$, $C^1 = \{u \in C \mid u' \in C((0, +\infty), \mathbb{R}^N),\ \lim_{t \to 0^+} w(t)^{1/(p(t)-1)} u'(t) \text{ exists}\}$. For any $u(t) = (u^1(t), \dots, u^N(t)) \in C$, we denote $|u^i|_0 = \sup_{t \in (0, +\infty)} |u^i(t)|$, $\|u\|_0 = (\sum_{i=1}^N |u^i|_0^2)^{1/2}$, and $\|u\|_1 = \|u\|_0 + \|(w(t))^{1/(p(t)-1)} u'\|_0$. Spaces C and C^1 will be equipped with the norm $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively. Then, $(C, \|\cdot\|_0)$ and $(C^1, \|\cdot\|_1)$ are Banach spaces. Denote $L^1 = L^1(J, \mathbb{R}^N)$ with the norm $\|u\|_{L^1} = [\sum_{i=1}^N (\int_0^\infty |u^i| dt)^2]^{1/2}$.

We say a function $u : J \to \mathbb{R}^N$ is a solution of (1.1) if $u \in C^1$ with $w(t)|u'|^{p(t)-2}u'$ absolutely continuous on $(0, +\infty)$, which satisfies (1.1) a.e. on J.

In this paper, we always use C_i to denote positive constants, if it cannot lead to confusion. Denote

$$z^- = \inf_{t \in J} z(t), \quad z^+ = \sup_{t \in J} z(t), \quad \text{for any } z \in C(J, \mathbb{R}). \tag{1.8}$$

We say f satisfies sub-$(p^- - 1)$ growth condition, if f satisfies

$$\lim_{|x|+|y|+|z|+|w| \to +\infty} \frac{f(t, x, y, z, w)}{(|x| + |y| + |z| + |w|)^{q(t)-1}} = 0, \quad \text{for } t \in J \text{ uniformly}, \tag{1.9}$$

where $q(t) \in C(J, \mathbb{R})$ and $1 < q^- \leq q^+ < p^-$. We say f satisfies general growth condition, if f does not satisfy sub-$(p^- - 1)$ growth condition.

We will discuss the existence of solutions of (1.1)-(1.2) in the following three cases.

Case (i): $\sum_{i=1}^{m-2} \alpha_i \in [0, 1),\ \sigma \in [0, 1)$;

Case (ii): $\sum_{i=1}^{m-2} \alpha_i \in [0, 1),\ \sigma = 1$;

Case (iii): $\sum_{i=1}^{m-2} \alpha_i = 1,\ \sigma = 1$.

This paper is divided into five sections. In the second section, we present some preliminary and give the operator equations which have the same solutions of (1.1)-(1.2) in the three cases, respectively. In the third section, we will discuss the existence of solutions of (1.1)-(1.2) when $\sum_{i=1}^{m-2}\alpha_i \in [0,1),\ \sigma \in [0,1)$, and we give the existence of nonnegative solutions. In the fourth section, we will discuss the existence of solutions of (1.1)-(1.2) when $\sum_{i=1}^{m-2}\alpha_i \in [0,1),\ \sigma = 1$. In the fifth section, we will discuss the existence of solutions of (1.1)-(1.2) when $\sum_{i=1}^{m-2}\alpha_i = 1,\ \sigma = 1$.

2. Preliminary

For any $(t,x) \in J \times \mathbb{R}^N$, denote $\varphi(t,x) = |x|^{p(t)-2}x$. Obviously, φ has the following properties.

Lemma 2.1 (see [7]). *φ is a continuous function and satisfies the following.*

(i) *For any $t \in [0,+\infty)$, $\varphi(t,\cdot)$ is strictly monotone, that is*

$$\langle \varphi(t,x_1) - \varphi(t,x_2), x_1 - x_2 \rangle > 0, \quad \text{for any } x_1, x_2 \in \mathbb{R}^N,\ x_1 \neq x_2. \tag{2.1}$$

(ii) *There exists a function $\beta : [0,+\infty) \to [0,+\infty)$, $\beta(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \varphi(t,x), x \rangle \geq \beta(|x|)|x|, \quad \forall x \in \mathbb{R}^N. \tag{2.2}$$

It is well known that $\varphi(t,\cdot)$ is a homeomorphism from $\mathbb{R}^N$ to $\mathbb{R}^N$ for any fixed $t \in [0,+\infty)$. For any $t \in J$, denote by $\varphi^{-1}(t,\cdot)$ the inverse operator of $\varphi(t,\cdot)$, then

$$\varphi^{-1}(t,x) = |x|^{(2-p(t))/(p(t)-1)}x, \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}, \qquad \varphi^{-1}(t,0) = 0. \tag{2.3}$$

It is clear that $\varphi^{-1}(t,\cdot)$ is continuous and sends bounded sets into bounded sets.

Let us now consider the following problem with boundary value condition (1.2)

$$(w(t)\varphi(t,u'(t)))' = g(t), \quad t \in (0,+\infty), \tag{2.4}$$

where $g \in L^1$.

If u is a solution of (2.4) with (1.2), by integrating (2.4) from 0 to t, we find that

$$w(t)\varphi(t,u'(t)) = w(0)\varphi(0,u'(0)) + \int_0^t g(s)ds. \tag{2.5}$$

Define operator $F : L^1 \to C$ as

$$F(g)(t) = \int_0^t g(s)ds, \quad \forall t \in J,\ \forall g \in L^1. \tag{2.6}$$

By solving for u' in (2.5) and integrating, we find that

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(w(0)\varphi\left(0, u'(0)\right) + F(g)\right)\right]\right\}(t), \quad t \in J. \tag{2.7}$$

In the following, we will give the operator equations which have the same solutions of (1.1)-(1.2) in three cases, respectively.

2.1. Case (i): $\sum_{i=1}^{m-2}\alpha_i \in [0,1)$, $\sigma \in [0,1)$

We denote $\rho = w(0)\varphi(0, u'(0))$ in (2.7). It is easy to see that ρ is dependent on $g(\cdot)$, then we find that

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho + F(g)\right)\right]\right\}(t), \quad t \in J. \tag{2.8}$$

The boundary value condition (1.2) implies that

$$\begin{aligned} u(0) = {} & \frac{\int_0^{+\infty}\left\{e(t)\int_0^t \varphi^{-1}\left[r, (w(r))^{-1}\left(\rho + F(g)(r)\right)\right]dr\right\}dt + e_1}{1-\sigma} \\ & - \frac{\int_0^{+\infty}\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho + F(g)(r)\right)\right]dr}{1-\sigma}, \\ \rho = {} & \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} g(s)ds - \int_0^{+\infty} g(s)ds + e_2\right). \end{aligned} \tag{2.9}$$

For fixed $h \in L^1$, we define $\rho : L^1 \to \mathbb{R}^N$ as

$$\rho(h) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} h(t)dt - \int_0^{+\infty} h(t)dt + e_2\right). \tag{2.10}$$

Lemma 2.2. *$\rho : L^1 \to \mathbb{R}^N$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$. Moreover,*

$$\left|\rho(h)\right| \le \frac{2N}{1-\sum_{i=1}^{m-2}\alpha_i}\cdot\left(\|h\|_{L^1} + |e_2|\right). \tag{2.11}$$

Proof. Since $\rho(\cdot)$ consists of continuous operators, it is continuous. It is easy to see that

$$\left|\rho(h)\right| \le \frac{2N}{1-\sum_{i=1}^{m-2}\alpha_i}\cdot\left(\|h\|_{L^1} + |e_2|\right). \tag{2.12}$$

This completes the proof. □

It is clear that $\rho(\cdot)$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$, and hence it is a compact continuous mapping.

If u is a solution of (2.4) with (1.2), we find that

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}(\rho + F(g))\right]\right\}(t), \quad t \in J,$$
$$u(0) = \frac{\int_0^{+\infty}\left\{e(t)\int_0^t \varphi^{-1}\left[r, (w(r))^{-1}(\rho + F(g)(r))\right]dr\right\}dt + e_1}{1-\sigma} - \frac{\int_0^{+\infty}\varphi^{-1}\left[r, (w(r))^{-1}(\rho + F(g)(r))\right]dr}{1-\sigma}. \tag{2.13}$$

We denote

$$K(h)(t) := (K \circ h)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}(\rho(h) + F(h))\right]\right\}(t), \quad \forall t \in (0, +\infty). \tag{2.14}$$

We say a set $U \subset L^1$ be equi-integrable, if there exists a nonnegative $\rho_* \in L^1(J, \mathbb{R})$, such that

$$|h(t)| \le \rho_*(t) \quad \text{a.e. in } J, \text{ for any } h \in U. \tag{2.15}$$

Lemma 2.3. *The operator K is continuous and sends equi-integrable sets in L^1 to relatively compact sets in C^1.*

Proof. It is easy to check that $K(h)(t) \in C^1$, for all $h \in L^1$. Since $(w(t))^{-1/(p(t)-1)} \in L^1$ and

$$K(h)'(t) = \varphi^{-1}\left[t, (w(t))^{-1}(\rho(h) + F(h))\right], \quad \forall t \in [0, +\infty), \tag{2.16}$$

it is easy to check that K is a continuous operator from L^1 to C^1.

Let now U be an equi-integrable set in L^1, then there exists a nonnegative $\rho_* \in L^1(J, \mathbb{R})$, such that

$$|h(t)| \le \rho_*(t) \quad \text{a.e. in } J, \text{ for any } h \in U. \tag{2.17}$$

We want to show that $\overline{K(U)} \subset C^1$ is a compact set.

Let $\{u_n\}$ be a sequence in $K(U)$, then there exists a sequence $\{h_n\} \subset U$ such that $u_n = K(h_n)$. Since $h_n(t) = (h_n^1(t), \ldots, h_n^N(t))$, where $h_n^i(t) \in L^1(J, \mathbb{R})$ $(i = 1, \ldots, N)$, we have

$$\left|h_n^i(t)\right| \le |h_n(t)|, \quad \forall i = 1, \ldots, N, \tag{2.18}$$

then for any $t_1, t_2 \in J$ with $t_1 < t_2$, we have

$$\begin{aligned}\left|\int_{t_1}^{t_2} h_n(t)dt\right| &= \sqrt{\sum_{i=1}^{N}\left(\int_{t_1}^{t_2} h_n^i(t)dt\right)^2} \le \sqrt{\sum_{i=1}^{N}\left(\int_{t_1}^{t_2} \left|h_n^i(t)\right|dt\right)^2} \\ &\le \sqrt{\sum_{i=1}^{N}\left(\int_{t_1}^{t_2} |h_n(t)|dt\right)^2} \le N\int_{t_1}^{t_2} |h_n(t)|dt,\end{aligned} \tag{2.19}$$

which together with (2.17) implies

$$\begin{aligned}|F(h_n)(t_1) - F(h_n)(t_2)| &= \left|\int_0^{t_1} h_n(t)dt - \int_0^{t_2} h_n(t)dt\right| \\ &= \left|\int_{t_1}^{t_2} h_n(t)dt\right| \le N\int_{t_1}^{t_2} |h_n(t)|dt \le N\int_{t_1}^{t_2} \rho_*(t)dt.\end{aligned} \tag{2.20}$$

Hence, the sequence $\{F(h_n)\}$ is uniformly bounded. According to the absolute continuity of Lebesgue integral, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $0 \le t_1 - t_2 < \delta$, then we have $0 \le N\int_{t_1}^{t_2} \rho_*(t)dt < \varepsilon$. Thus, (2.20) means that $\{F(h_n)\}$ is equicontinuous.

Denote $\Omega_m = [0, m]$. Obviously, $\{F(h_n)\}$ is uniformly bounded and equicontinuous on Ω_m for $m = 1, 2, \ldots$. By Ascoli-Arzela Theorem, there exists a subsequence $\{F(h_n^{(1)})\}$ of $\{F(h_n)\}$ being convergent in $C(\Omega_1)$, we may assume $F(h_n^{(1)}) \to v_1(\cdot)$ in $C(\Omega_1)$. Since $\{F(h_n^{(1)})\}$ is uniformly bounded and equicontinuous on Ω_2, there exists a subsequence $\{F(h_n^{(2)})\}$ of $\{F(h_n^{(1)})\}$ such that $\{F(h_n^{(2)})\}$ is convergent in $C(\Omega_2)$, we may assume $F(h_n^{(2)}) \to v_2(\cdot)$ in $C(\Omega_2)$. Obviously, $v_2(t) = v_1(t)$, for any $t \in \Omega_1$. Repeating the process, we get a subsequence $\{F(h_n^{(m+1)})\}$ of $\{F(h_n^{(m)})\}$ such that $\{F(h_n^{(m+1)})\}$ is convergent in $C(\Omega_{m+1})$, we may assume $F(h_n^{(m+1)}) \to v_{m+1}(\cdot)$ in $C(\Omega_{m+1})$. Obviously, $v_{m+1}(t) = v_m(t)$ for any $t \in \Omega_m$. Select the diagonal element, we can see that $\{F(h_n^{(n)})\}$ is a subsequence of $\{F(h_n)\}$ which satisfies that $\{F(h_n^{(n)})\}$ is convergent in $C(\Omega_m)$ $(m = 1, 2, \ldots)$ and $F(h_n^{(n)}) \to v_m(\cdot)$ in $C(\Omega_m)$ $(m = 1, 2, \ldots)$. Thus, we get a function v which is defined on $[0, +\infty)$ such that $v(t) = v_m(t)$ for any $t \in \Omega_m$, and $F(h_n^{(n)}) \to v(\cdot)$ in $C(\Omega_m)$ $(m = 1, 2, \ldots)$.

From (2.20), it is easy to see that for any $n = 1, 2, \ldots$, $\lim_{t\to\infty} F(h_n)(t)$ exists (we denote the limit by $F(h_n^{(n)})(+\infty)$), and, for any $\varepsilon > 0$, there exists an integer $R_\varepsilon > 0$ such that $\int_{R_\varepsilon}^{+\infty} \rho_*(t)dt < \varepsilon/N$, and then

$$\left|F\left(h_n^{(n)}\right)(+\infty) - F(h_n)(t)\right| \le N\int_{R_\varepsilon}^{+\infty} \rho_*(t)dt < \varepsilon, \quad \forall t \ge R_\varepsilon,\ \forall n = 1, 2, \ldots. \tag{2.21}$$

Since $\{F(h_n)\}$ is uniformly bounded, then $\{F(h_n^{(n)})(+\infty)\}$ is bounded. By choosing a subsequence, we may assume that

$$\lim_{n\to\infty} F\left(h_n^{(n)}\right)(+\infty) = b. \tag{2.22}$$

We claim that $\lim_{t\to\infty} v(t) = b$. In fact, for any $t \ge R_\varepsilon$, from (2.21), we have

$$\begin{aligned} |v(t) - b| &\le \left| v(t) - F\left(h_n^{(n)}\right)(t) \right| + \left| F\left(h_n^{(n)}\right)(t) - F\left(h_n^{(n)}\right)(+\infty) \right| + \left| F\left(h_n^{(n)}\right)(+\infty) - b \right| \\ &\le \left| v(t) - F\left(h_n^{(n)}\right)(t) \right| + \varepsilon + \left| F\left(h_n^{(n)}\right)(+\infty) - b \right|. \end{aligned} \tag{2.23}$$

Since $\lim_{n\to\infty} F(h_n^{(n)})(t) = v(t)$ and $\lim_{n\to\infty} F(h_n^{(n)})(+\infty) = b$, letting $n \to \infty$, the above inequality implies

$$|v(t) - b| \le \varepsilon, \quad \forall t \ge R_\varepsilon. \tag{2.24}$$

Thus,

$$\lim_{t\to\infty} v(t) = b = \lim_{n\to\infty} F\left(h_n^{(n)}\right)(+\infty). \tag{2.25}$$

Next, we will prove that $F(h_n^{(n)})$ tend to v uniformly.

Suppose $t \ge R_\varepsilon$. From (2.21) and (2.24), we have

$$\begin{aligned} \left| F\left(h_n^{(n)}\right)(t) - v(t) \right| &\le \left| F\left(h_n^{(n)}\right)(t) - F\left(h_n^{(n)}\right)(+\infty) \right| + \left| F\left(h_n^{(n)}\right)(+\infty) - b \right| + |b - v(t)| \\ &\le \varepsilon + \left| F\left(h_n^{(n)}\right)(+\infty) - b \right| + \varepsilon = 2\varepsilon + \left| F\left(h_n^{(n)}\right)(+\infty) - b \right|. \end{aligned} \tag{2.26}$$

From (2.22), there exists a $N_1 > 0$ such that $|F(h_n^{(n)})(+\infty) - b| \le \varepsilon$ for $n \ge N_1$. Thus, for any $t \ge R_\varepsilon$,

$$\left| F\left(h_n^{(n)}\right)(t) - v(t) \right| \le 3\varepsilon, \quad \forall n \ge N_1. \tag{2.27}$$

Suppose $t \in [0, R_\varepsilon]$. Since $F(h_n^{(n)}) \to v$ in $C([0, R_\varepsilon])$, there exists a $N_2 > 0$ such that

$$\left| F\left(h_n^{(n)}\right)(t) - v(t) \right| \le \varepsilon, \quad \forall n \ge N_2. \tag{2.28}$$

Thus,

$$\left| F\left(h_n^{(n)}\right)(t) - v(t) \right| \le 3\varepsilon, \quad \forall t \in [0, +\infty),\ \forall n \ge N_1 + N_2. \tag{2.29}$$

This means that $F(h_n^{(n)})$ tend to v uniformly, that is, $F(h_n^{(n)})$ tend to v in C.

According to the bounded continuous of the operator ρ, we can choose a subsequence of $\{\rho(h_n) + F(h_n)\}$ (which we still denote $\{\rho(h_n) + F(h_n)\}$) which is convergent in C, then $w(t)\varphi(t, K(h_n)'(t)) = \rho(h_n) + F(h_n)$ is convergent in C.

Since

$$K(h_n)(t) = F\left\{\varphi^{-1}\left[t, (w(t))^{-1}(\rho(h_n) + F(h_n))\right]\right\}(t), \quad \forall t \in [0, +\infty), \tag{2.30}$$

it follows from the continuity of φ^{-1} and the integrability of $w(t)^{-1/(p(t)-1)}$ in L^1 that $K(h_n)$ is convergent in C. Thus, $\{u_n\}$ is convergent in C^1. This completes the proof. □

Let us define $P : C^1 \to C^1$ as

$$P(h) = \frac{1}{1-\sigma}\left(\int_0^{+\infty} e(t)K(h)(t)dt - K(h)(+\infty) + e_1\right). \tag{2.31}$$

It is easy to see that P is compact continuous.

Throughout the paper, we denote $N_f(u) : [0, +\infty) \times C^1 \to L^1$ the Nemytskii operator associated to f defined by

$$N_f(u)(t) = f\left(t, u(t), (w(t))^{1/(p(t)-1)}u'(t), S(u)(t), T(u)(t)\right), \quad \text{a.e. on } J. \tag{2.32}$$

Lemma 2.4. *In the Case (i), u is a solution of* (1.1)-(1.2) *if and only if u is a solution of the following abstract equation:*

$$u = P(\delta N_f(u)) + K(\delta N_f(u)). \tag{2.33}$$

Proof. If u is a solution of (1.1)-(1.2), by integrating (1.1) from 0 to t, we find that

$$w(t)\varphi(t, u'(t)) = \rho(\delta N_f(u)) + F(\delta N_f(u))(t), \quad \forall t \in (0, +\infty), \tag{2.34}$$

which implies that

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho(\delta N_f(u)) + F(\delta N_f(u))(t)\right)\right]\right\}(t), \quad \forall t \in [0, +\infty). \tag{2.35}$$

From $u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1$, we have

$$\begin{aligned} u(0) = & \frac{\int_0^{+\infty}\left\{e(t)\int_0^t \varphi^{-1}\left[r, (w(r))^{-1}(\rho + F(\delta N_f(u)))\right]dr\right\}dt + e_1}{1-\sigma} \\ & - \frac{\int_0^{+\infty} \varphi^{-1}\left[r, (w(r))^{-1}(\rho + F(\delta N_f(u)))\right]dr}{1-\sigma} = P(\delta N_f(u)). \end{aligned} \tag{2.36}$$

So we have

$$u = P(\delta N_f(u)) + K(\delta N_f(u)). \tag{2.37}$$

Conversely, if u is a solution of (2.33), we have

$$\begin{aligned} u(0) &= P(\delta N_f(u)) + K(\delta N_f(u))(0) \\ &= P(\delta N_f(u)) \\ &= \frac{1}{1-\sigma}\left(\int_0^{+\infty} e(t)K(h)(t)dt - K(h)(+\infty) + e_1\right), \end{aligned} \tag{2.38}$$

which implies that

$$(1-\sigma)u(0) + K(h)(+\infty) = \int_0^{+\infty} e(t)(u(t)-u(0))dt + e_1, \tag{2.39}$$

then

$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1. \tag{2.40}$$

From (2.33), we can obtain

$$w(t)\varphi(t, u'(t)) = \rho(\delta N_f(u)) + F(\delta N_f(u))(t). \tag{2.41}$$

It follows from the condition of the mapping ρ that

$$\begin{aligned} w(+\infty)\varphi(+\infty, u'(+\infty)) &= \rho(\delta N_f(u)) + F(\delta N_f(u))(+\infty) \\ &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \delta N_f(u)(t)dt - \int_0^{+\infty} \delta N_f(u)(t)dt + e_2\right) \\ &\quad + \int_0^{+\infty} \delta N_f(u)(t)dt, \end{aligned} \tag{2.42}$$

and then

$$\begin{aligned} &\left(1-\sum_{i=1}^{m-2}\alpha_i\right) w(+\infty)\varphi(+\infty, u'(+\infty)) \\ &\quad = \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \delta N_f(u)(t)dt - \int_0^{+\infty} \delta N_f(u)(t)dt + e_2 + \left(1-\sum_{i=1}^{m-2}\alpha_i\right)\int_0^{+\infty} \delta N_f(u)(t)dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} \delta N_f(u)(t)dt - \sum_{i=1}^{m-2}\alpha_i \int_0^{+\infty} \delta N_f(u)(t)dt + e_2 \\
&= \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\varphi\left(\xi_i, u'(\xi_i)\right) - \sum_{i=1}^{m-2}\alpha_i w(+\infty)\varphi\left(+\infty, u'(+\infty)\right) + e_2,
\end{aligned}
\tag{2.43}
$$

thus

$$
\lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + e_2. \tag{2.44}
$$

From (2.40) and (2.44), we obtain (1.2).
From (2.41), we have

$$
\left(w(t)\varphi(t,u')\right)' = \delta N_f(u)(t). \tag{2.45}
$$

Hence, u is a solution of (1.1)-(1.2). This completes the proof. □

2.2. *Case (ii):* $\sum_{i=1}^{m-2}\alpha_i \in [0,1)$, $\sigma = 1$

We denote $\rho_1 = w(0)\varphi(0, u'(0))$ in (2.7). It is easy to see that ρ_1 is dependent on $g(\cdot)$, and we have

$$
u(t) = u(0) + F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_1 + F(g)\right)\right]\right\}(t), \quad t \in J. \tag{2.46}
$$

The boundary value condition (1.2) implies that

$$
\begin{gathered}
\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r, (w(r))^{-1}\left(\rho_1 + F(g)(r)\right)\right]dr\right\}dt - e_1 = 0, \\
\rho_1 = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} g(t)dt - \int_0^{+\infty} g(t)dt + e_2\right).
\end{gathered}
\tag{2.47}
$$

For fixed $h \in L^1$, we define $\rho_1 : L^1 \to \mathbb{R}^N$ as

$$
\rho_1(h) = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} h(t)dt - \int_0^{+\infty} h(t)dt + e_2\right). \tag{2.48}
$$

Similar to Lemma 2.2, we have the following.

Lemma 2.5. *$\rho_1 : L^1 \to \mathbb{R}^N$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$. Moreover,*

$$\left|\rho_1(h)\right| \le \frac{2N}{1-\sum_{i=1}^{m-2}\alpha_i}\cdot\left(\|h\|_{L^1}+|e_2|\right). \tag{2.49}$$

It is clear that $\rho_1(\cdot)$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$, and, hence, it is a compact continuous mapping.

Let us define

$$\begin{gathered} P_1 : C^1 \longrightarrow C^1, \qquad u \longmapsto u(0), \\ \Theta : L^1 \longrightarrow \mathbb{R}^N, \qquad h \longmapsto \int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}\left(\rho_1(h)+F(h)(r)\right)\right]dr\right\}dt - e_1, \\ K_1(h)(t) := (K_1\circ h)(t) = F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1(h)+F(h)\right)\right]\right\}(t), \quad \forall t\in[0,+\infty). \end{gathered} \tag{2.50}$$

Lemma 2.6. *The operator K_1 is continuous and sends equi-integrable sets in L^1 to relatively compact sets in C^1.*

Proof. Similar to the proof of Lemma 2.3, we omit it here. □

Lemma 2.7. *In Case (ii), u is a solution of* (1.1)-(1.2) *if and only if u is a solution of the following abstract equation:*

$$u = P_1u + \Theta\left(\delta N_f(u)\right) + K_1\left(\delta N_f(u)\right). \tag{2.51}$$

Proof. If u is a solution of (1.1)-(1.2), by integrating (1.1) from 0 to t, we find that

$$w(t)\varphi\left(t,u'(t)\right) = \rho_1\left(\delta N_f(u)\right) + F\left(\delta N_f(u)\right)(t), \quad \forall t\in(0,+\infty), \tag{2.52}$$

which implies that

$$u(t) = u(0) + F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1\left(\delta N_f(u)\right)+F\left(\delta N_f(u)\right)(t)\right)\right]\right\}(t), \quad \forall t\in[0,+\infty). \tag{2.53}$$

From $\sigma = 1$ and $u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1$, we obtain

$$\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}\left(\rho_1\left(\delta N_f(u)\right)+F\left(\delta N_f(u)\right)(t)\right)\right]dr\right\}dt - e_1 = \Theta\left(\delta N_f(u)\right) = 0, \tag{2.54}$$

then

$$u = P_1u + \Theta\left(\delta N_f(u)\right) + K_1\left(\delta N_f(u)\right). \tag{2.55}$$

Conversely, if u is a solution of (2.51), then

$$u(0) = P_1 u + \Theta\big(\delta N_f(u)\big) + K_1\big(\delta N_f(u)\big)(0) = u(0) + \Theta\big(\delta N_f(u)\big). \tag{2.56}$$

Thus, $\Theta(\delta N_f(u)) = 0$, and we have

$$\begin{aligned} &\int_0^{+\infty}\left\{e(t)\int_0^{+\infty}\varphi^{-1}\Big[r,(w(r))^{-1}\big(\rho_1\big(\delta N_f(u)\big)+F\big(\delta N_f(u)\big)(r)\big)\Big]dr\right\}dt \\ &\quad = \int_0^{+\infty}\left\{e(t)\int_0^{t}\varphi^{-1}\Big[r,(w(r))^{-1}\big(\rho_1\big(\delta N_f(u)\big)+F\big(\delta N_f(u)\big)(r)\big)\Big]dr\right\}dt + e_1, \end{aligned} \tag{2.57}$$

then

$$\int_0^{+\infty}\{e(t)(u(+\infty)-u(0))\}dt = \int_0^{+\infty}\{e(t)(u(t)-u(0))\}dt + e_1. \tag{2.58}$$

Thus,

$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1. \tag{2.59}$$

Similar to the proof of Lemma 2.4, we can have

$$\lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + e_2. \tag{2.60}$$

From (2.59) and (2.60), we obtain (1.2).
From (2.51), we have

$$w(t)\varphi\big(t,u'(t)\big) = \rho_1\big(\delta N_f(u)\big) + F\big(\delta N_f(u)\big)(t), \tag{2.61}$$

then

$$\big(w(t)\varphi(t,u')\big)' = \delta N_f(u)(t). \tag{2.62}$$

Hence, u is a solution of (1.1)-(1.2). This completes the proof. □

2.3. *Case (iii):* $\sum_{i=1}^{m-2}\alpha_i = 1,\ \sigma = 1$

We denote $\rho_2 = w(0)\varphi(0,u'(0))$ in (2.7). It is easy to see that ρ_2 is dependent on $g(\cdot)$, then we find that

$$u(t) = u(0) + F\Big\{\varphi^{-1}\Big[t,(w(t))^{-1}\big(\rho_2 + F(g)\big)\Big]\Big\}(t), \quad t \in J. \tag{2.63}$$

The boundary value condition (1.2) implies that

$$\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}(\rho_2+F(g)(r))\right]dr\right\}dt-e_1=0,$$
$$\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}g(t)dt-e_2=0. \tag{2.64}$$

For fixed $h\in L^1$, we denote

$$\Lambda_h(\rho_2)=\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}(\rho_2+F(h)(r))\right]dr\right\}dt-e_1. \tag{2.65}$$

Throughout the paper, we denote

$$E_\#=\int_0^{+\infty}e(t)\int_t^{+\infty}(w(r))^{-1/(p(r)-1)}dr\,dt. \tag{2.66}$$

Lemma 2.8. *The function $\Lambda_h(\cdot)$ has the following properties.*

(i) *For any fixed $h\in L^1$, the equation*

$$\Lambda_h(\rho_2)=0 \tag{2.67}$$

has a unique solution $\rho_2(h)\in\mathbb{R}^N$.

(ii) *The function $\rho_2:L^1\to\mathbb{R}^N$, defined in* (i), *is continuous and sends bounded sets to bounded sets. Moreover,*

$$|\rho_2(h)|\le 3N\left(\frac{E_\#+1}{E_\#}\right)^{p^+}\left[\|h\|_0+2N|e_1|^{p^\#-1}\right], \tag{2.68}$$

where the notation $M^{p^\#-1}$ means

$$M^{p^\#-1}=\begin{cases}M^{p^+-1}, & M>1,\\ M^{p^--1}, & M\le 1.\end{cases} \tag{2.69}$$

Proof. (i) From Lemma 2.1, it is immediate that

$$\langle\Lambda_h(x)-\Lambda_h(y),x-y\rangle>0,\quad \text{for } x\ne y, \tag{2.70}$$

and, hence, if (2.67) has a solution, then it is unique.

Let $t_0=3N((E_\#+1)/E_\#)^{p^+}[\|h\|_0+2N|e_1|^{p^\#-1}]$. Since $(w(t))^{-1/(p(t)-1)}\in L^1(0,+\infty)$ and $h\in L^1$, if $|\rho_2|>t_0$, it is easy to see that there exists an $i\in\{1,\dots,N\}$ such that the

ith component ρ_2^i of ρ_2 satisfies $|\rho_2^i| \geq |\rho_2|/N > 3((E_\# + 1)/E_\#)^{p^+}[\|h\|_0 + 2N|e_1|^{p^\#-1}]$. Thus, $(\rho_2^i + h^i(t))$ keeps sign on J and

$$\left|\rho_2^i + h^i(t)\right| \geq \left|\rho_2^i\right| - \|h\|_0 \geq \frac{2|\rho_2|}{3N} > 2\left(\frac{E_\# + 1}{E_\#}\right)^{p^+}\left[\|h\|_0 + 2N|e_1|^{p^\#-1}\right], \quad \forall t \in J. \tag{2.71}$$

Obviously, $|\rho_2 + h(t)| \leq 4|\rho_2|/3 \leq 2N|\rho_2^i + h^i(t)|$, then

$$\left|\rho_2 + h(t)\right|^{(2-p(t))/(p(t)-1)}\left|\rho_2^i + h^i(t)\right| > \frac{1}{2N}\left|\rho_2^i + h^i(t)\right|^{1/(p(t)-1)} > \frac{E_\# + 1}{2NE_\#}|e_1|, \quad \forall t \in J. \tag{2.72}$$

Thus, the ith component $\Lambda_h^i(\rho_2)$ of $\Lambda_h(\rho_2)$ is nonzero and keeps sign, and then we have

$$\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}(\rho_2 + F(h)(r))\right]dr\right\}dt - e_1 \neq 0. \tag{2.73}$$

Let us consider the equation

$$\lambda\Lambda_h(\rho_2) + (1-\lambda)\rho_2 = 0, \quad \lambda \in [0,1]. \tag{2.74}$$

It is easy to see that all the solutions of (2.74) belong to $b(t_0+1) = \{x \in \mathbb{R}^N \mid |x| < t_0+1\}$. So, we have

$$d_B\left[\Lambda_h(\rho_2), b(t_0+1), 0\right] = d_B[I, b(t_0+1), 0] \neq 0, \tag{2.75}$$

and it means the existence of solutions of $\Lambda_h(\rho_2) = 0$.

In this way, we define a function $\rho_2(h) : L^1 \to \mathbb{R}^N$, which satisfies

$$\Lambda_h(\rho_2(h)) = 0. \tag{2.76}$$

(ii) By the proof of (i), we also obtain ρ_2 sends bounded sets to bounded sets, and

$$\left|\rho_2(h)\right| \leq 3N\left(\frac{E_\# + 1}{E_\#}\right)^{p^+}\left[\|h\|_0 + 2N|e_1|^{p^\#-1}\right]. \tag{2.77}$$

It only remains to prove the continuity of ρ_2. Let $\{u_n\}$ be a convergent sequence in L^1 and $u_n \to u$ as $n \to +\infty$. Since $\{\rho_2(u_n)\}$ is a bounded sequence, then it contains a convergent subsequence $\{\rho_2(u_{n_j})\}$. Let $\rho_2(u_{n_j}) \to \rho_0$ as $j \to +\infty$. Since $\Lambda_{u_{n_j}}(\rho_2(u_{n_j})) = 0$, letting $j \to +\infty$, we have $\Lambda_u(\rho_0) = 0$. From (i), we get $\rho_0 = \rho_2(u)$, it means that ρ_2 is continuous. This completes the proof. □

It is clear that $\rho_2(\cdot)$ is continuous and sends bounded sets of L^1 to bounded sets of $\mathbb{R}^N$, and, hence, it is a compact continuous mapping.

Let us define

$$P_2 : C^1 \longrightarrow C^1, \qquad u \longmapsto u(0), \tag{2.78}$$

$$Q : L^1 \longrightarrow \mathbb{R}^N, \qquad h \longmapsto \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{+\infty} h(t)dt - e_2, \tag{2.79}$$

$$Q^* : L^1 \longrightarrow L^1, \qquad h \longmapsto \tau(t)\left(\sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{+\infty} h(t)dt - e_2\right), \tag{2.80}$$

where $\tau \in ([0,+\infty),\mathbb{R})$ and satisfies $0 < \tau(t) < 1$, $t \in J$, $\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{+\infty} \tau(t)dt = 1$. We denote $K_2 : L^1 \to C^1$ as

$$K_2(h)(t) := F\left\{\varphi^{-1}\left[t, (w(t))^{-1}\left(\rho_2((I-Q^*)h) + F((I-Q^*)h)\right)\right]\right\}(t), \quad \forall t \in [0,+\infty). \tag{2.81}$$

Similar to Lemmas 2.3 and 2.7, we have the following

Lemma 2.9. *The operator K_2 is continuous and sends equi-integrable sets in L^1 to relatively compact sets in C^1.*

Lemma 2.10. *In Case (iii), u is a solution of* (1.1)-(1.2) *if and only if u is a solution of the following abstract equation:*

$$u = P_2 u + Q(\delta N_f(u)) + K_2(\delta N_f(u)). \tag{2.82}$$

3. Existence of Solutions in Case (i)

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1)-(1.2) when $\sum_{i=1}^{m-2}\alpha_i \in [0,1)$, $\sigma \in [0,1)$. Moreover, we give the existence of nonnegative solutions.

Theorem 3.1. *In Case (i), if f satisfies sub-($p^- - 1$) growth condition, then problem* (1.1)-(1.2) *has at least a solution for any fixed parameter δ.*

Proof. Denote $\Psi_f(u,\lambda) := P(\lambda\delta N_f(u)) + K(\lambda\delta N_f(u))$, where $N_f(u)$ is defined in (2.32). We know that (1.1)-(1.2) has the same solution of

$$u = \Psi_f(u,\lambda), \tag{3.1}$$

when $\lambda = 1$.

It is easy to see that the operator P is compact continuous. According to Lemmas 2.2 and 2.3, we can see that $\Psi_f(\cdot,\cdot)$ is compact continuous from $C^1 \times [0,1]$ to C^1.

We claim that all the solutions of (3.1) are uniformly bounded for $\lambda \in [0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n,\lambda_n)\}$ for (3.1) such that $\|u_n\|_1 \to +\infty$ as $n \to +\infty$ and $\|u_n\|_1 > 1$ for any $n = 1,2,\ldots$.

From Lemma 2.2, we have

$$\begin{aligned} \left|\rho\left(\lambda_n \delta N_f(u_n)\right)\right| &\leq C_1\left(\left\|N_f(u_n)\right\|_{L^1} + |e_2|\right) \\ &\leq C_2 \|u_n\|_1^{q^+-1}, \end{aligned} \tag{3.2}$$

then we have

$$\left|\rho\left(\lambda_n \delta N_f(u_n)\right) + F\left(\lambda_n \delta N_f(u_n)\right)\right| \leq \left|\rho\left(\lambda_n \delta N_f(u_n)\right)\right| + \left|F\left(\lambda_n \delta N_f(u_n)\right)\right| \leq C_3 \|u_n\|_1^{q^+-1}. \tag{3.3}$$

From (3.1), we have

$$w(t)\left|u_n'(t)\right|^{p(t)-2} u_n'(t) = \rho\left(\lambda_n \delta N_f(u_n)\right) + F\left(\lambda_n \delta N_f(u_n)\right), \quad \forall t \in J, \tag{3.4}$$

then

$$w(t)\left|u_n'(t)\right|^{p(t)-1} \leq \left|\rho\left(\lambda_n \delta N_f(u_n)\right)\right| + \left|F\left(\lambda_n \delta N_f(u_n)\right)\right| \leq C_3 \|u_n\|_1^{q^+-1}. \tag{3.5}$$

Denote $\alpha = (q^+ - 1)/(p^- - 1)$, from the above inequality, we have

$$\left\|(w(t))^{1/(p(t)-1)} u_n'(t)\right\|_0 \leq C_4 \|u_n\|_1^{\alpha}. \tag{3.6}$$

It follows from (2.36) and (3.3) that

$$|u_n(0)| \leq C_5 \|u_n\|_1^{\alpha}, \quad \text{where } \alpha = \frac{q^+ - 1}{p^- - 1}. \tag{3.7}$$

For any $j = 1, \ldots, N$, we have

$$\begin{aligned} \left|u_n^j(t)\right| &= \left|u_n^j(0) + \int_0^t \left(u_n^j\right)'(r)\,dr\right| \\ &\leq \left|u_n^j(0)\right| + \left|\int_0^t (w(r))^{-1/(p(r)-1)} \sup_{t\in(0,+\infty)} \left|(w(t))^{1/(p(t)-1)} \left(u_n^j\right)'(t)\right| dr\right| \\ &\leq [C_6 + C_4 E] \|u_n\|_1^{\alpha} \leq C_7 \|u_n\|_1^{\alpha}, \end{aligned} \tag{3.8}$$

which implies that

$$\left|u_n^j\right|_0 \leq C_8 \|u_n\|_1^{\alpha}, \quad j = 1, \ldots, N, \ n = 1, 2, \ldots. \tag{3.9}$$

Thus,

$$\|u_n\|_0 \leq C_9 \|u_n\|_1^{\alpha}, \quad n = 1, 2, \ldots. \tag{3.10}$$

It follows from (3.6) and (3.10) that $\{\|u_n\|_1\}$ is bounded.

Thus, we can choose a large enough $R_0 > 0$ such that all the solutions of (3.1) belong to $B(R_0) = \{u \in C^1 \mid \|u\|_1 < R_0\}$. Thus, the Leray-Schauder degree $d_{LS}[I - \Psi_f(\cdot,\lambda), B(R_0), 0]$ is well defined for each $\lambda \in [0,1]$, and

$$d_{LS}\left[I - \Psi_f(\cdot,1), B(R_0), 0\right] = d_{LS}\left[I - \Psi_f(\cdot,0), B(R_0), 0\right]. \tag{3.11}$$

Let

$$\begin{aligned} u_0 = {} & \frac{\int_0^{+\infty}\left\{e(t)\int_0^t \varphi^{-1}\left[r,(w(r))^{-1}\rho(0)\right]dr\right\}dt - \int_0^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}\rho(0)\right]dr + e_1}{1-\sigma} \\ & + \int_0^r \varphi^{-1}\left[t,(w(t))^{-1}\rho(0)\right]dt, \end{aligned} \tag{3.12}$$

where $\rho(0)$ is defined in (2.10), thus u_0 is the unique solution of $u = \Psi_f(u,0)$.

It is easy to see that u is a solution of $u = \Psi_f(u,0)$ if and only if u is a solution of the following system

$$\begin{gathered} -\Delta_{p(t)}u = 0, \quad t \in (0,+\infty), \\ u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1, \\ \lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + e_2. \end{gathered} \tag{I}$$

Obviously, system (I) possesses a unique solution u_0. Note that $u_0 \in B(R_0)$, we have

$$d_{LS}\left[I - \Psi_f(\cdot,1), B(R_0), 0\right] = d_{LS}\left[I - \Psi_f(\cdot,0), B(R_0), 0\right] \neq 0. \tag{3.13}$$

Therefore, (1.1)-(1.2) has at least one solution when $\sum_{i=1}^{m-2}\alpha_i \in [0,1)$, $\sigma \in [0,1)$. This completes the proof. □

Denote

$$\Omega_\varepsilon = \left\{u \in C^1 \mid \max_{1\le i\le N}\left(\left|u^i\right|_0 + \left|(w(t))^{1/(p(t)-1)}\left(u^i\right)'\right|_0\right) < \varepsilon\right\}, \qquad \theta = \frac{\varepsilon}{2+(1/E)}. \tag{3.14}$$

Assume the following

(A_1) Let positive constant ε such that $u_0 \in \Omega_\varepsilon$, $|P(0)| < \theta$, and $|\rho(0)| < (1/N(2E+2)) \inf_{t\in J}|\varepsilon/2(E+1)|^{p(t)-1}$, where u_0 is defined in (3.12) and $\rho(\cdot)$ is defined in (2.10).

It is easy to see that Ω_ε is an open bounded domain in C^1. We have the following.

Theorem 3.2. *In the Case (i), assume that f satisfies general growth condition and* (A_1) *is satisfied, then the problem* (1.1)-(1.2) *has at least one solution on $\overline{\Omega_\varepsilon}$ when the positive parameter δ is small enough.*

Proof. Denote $\Psi_f(u,\lambda) = P(\lambda\delta N_f(u)) + K(\lambda\delta N_f(u))$. According to Lemma 2.4, u is a solution of

$$-\Delta_{p(t)}u + \lambda\delta f\left(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u)\right) = 0, \quad t \in (0,+\infty), \tag{3.15}$$

with (1.2) if and only if u is a solution of the following abstract equation

$$u = \Psi_f(u,\lambda). \tag{3.16}$$

From Lemmas 2.2 and 2.3, we can see that $\Psi_f(\cdot,\cdot)$ is compact continuous from $C^1 \times [0,1]$ to C^1. According to Leray-Schauder's degree theory, we only need to prove that

(1°) $u = \Psi_f(u,\lambda)$ has no solution on $\partial\Omega_\varepsilon$ for any $\lambda \in [0,1)$,

(2°) $d_{\mathrm{LS}}[I - \Psi_f(\cdot,0), \Omega_\varepsilon, 0] \neq 0$,

then we can conclude that the system (1.1)-(1.2) has a solution on $\overline{\Omega_\varepsilon}$.

(1°) If there exists a $\lambda \in [0,1)$ and $u \in \partial\Omega_\varepsilon$ is a solution of (3.15) with (1.2), then (λ,u) satisfies

$$w(t)\varphi(t,u'(t)) = \rho(\lambda\delta N_f(u)) + \lambda\delta F(N_f(u))(t), \quad \forall t \in (0,+\infty). \tag{3.17}$$

Since $u \in \partial\Omega_\varepsilon$, there exists an i such that $|u^i|_0 + |(w(t))^{1/(p(t)-1)}(u^i)'|_0 = \varepsilon$.

(i) Suppose that $|u^i|_0 > 2\theta$, then $|(w(t))^{1/(p(t)-1)}(u^i)'|_0 < \varepsilon - 2\theta = \theta/E$. On the other hand, for any $t, t' \in J$, we have

$$\left|u^i(t) - u^i(t')\right| = \left|\int_{t'}^{t}\left(u^i\right)'(r)dr\right| \le \int_0^{+\infty}(w(r))^{-1/(p(r)-1)}\left|(w(r))^{1/(p(r)-1)}\left(u^i\right)'(r)\right|dr < \theta. \tag{3.18}$$

This implies that $|u^i(t)| > \theta$ for each $t \in J$.

Note that $u \in \overline{\Omega_\varepsilon}$, then $|f(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u))| \le \beta_{C_*\varepsilon}(t)$ (where $C_* := N + \sup_{t\in J}\int_0^{+\infty}|\psi(s,t)|ds + \sup_{t\in J}\int_0^{+\infty}|\chi(s,t)|ds$), holding $|F(N_f(u))| \le \int_0^{+\infty}\beta_{C_*\varepsilon}(t)dt$. Since $P(\cdot)$ is continuous, when $0 < \delta$ is small enough, from (A_1), we have

$$|u(0)| = \left|P(\lambda\delta N_f(u))\right| < \theta. \tag{3.19}$$

It is a contradiction to $|u^i(t)| > \theta$ for any $t \in J$.

(ii) Suppose that $|u^i|_0 \le 2\theta$, then $\theta/E \le |(w(t))^{1/(p(t)-1)}(u^i)'|_0 \le \varepsilon$. This implies that

$$\left|(w(t_2))^{1/(p(t_2)-1)}\left(u^i\right)'(t_2)\right| > \frac{\varepsilon}{2(E+1)} \quad \text{for some } t_2 \in J. \tag{3.20}$$

Since $u \in \overline{\Omega_\varepsilon}$, it is easy to see that

$$\left|(w(t_2))^{1/(p(t_2)-1)}\left(u^i\right)'(t_2)\right| > \frac{\varepsilon}{2(E+1)} = \frac{N\varepsilon}{N(2E+2)} \geq \frac{\left|(w(t_2))^{1/(p(t_2)-1)}u'(t_2)\right|}{N(2E+2)}. \tag{3.21}$$

Combining (3.17) and (3.21), we have

$$\begin{aligned}\frac{|\varepsilon/2(E+1)|^{p(t_2)-1}}{N(2E+2)} &< \frac{1}{N(2E+2)}w(t_2)\left|\left(u^i\right)'(t_2)\right|^{p(t_2)-1} \leq \frac{1}{N(2E+2)}w(t_2)\left|u'(t_2)\right|^{p(t_2)-1} \\ &\leq w(t_2)\left|u'(t_2)\right|^{p(t_2)-2}\left|\left(u^i\right)'(t_2)\right| \leq \left|\rho\left(\lambda\delta N_f\right)\right| + \lambda\left|\delta F\left(N_f\right)(t_2)\right|.\end{aligned} \tag{3.22}$$

Since $u \in \overline{\Omega_\varepsilon}$ and f is Caratheodory, it is easy to see that

$$\left|f\left(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u)\right)\right| \leq \beta_{C_*\varepsilon}(t), \tag{3.23}$$

thus

$$\left|\delta F\left(N_f(u)\right)\right| \leq \delta\int_0^{+\infty}\beta_{C_*\varepsilon}(t)dt. \tag{3.24}$$

From Lemma 2.2, $\rho(\cdot)$ is continuous, then we have

$$\left|\rho\left(\lambda\delta N_f(u)\right)\right| \longrightarrow \left|\rho(0)\right| \quad \text{as } \delta \longrightarrow 0. \tag{3.25}$$

When $0 < \delta$ is small enough, from (A_1) and (3.22), we can conclude that

$$\frac{|\varepsilon/2(E+1)|^{p(t_2)-1}}{N(2E+2)} < \left|\rho\left(\lambda\delta N_f(u)\right)\right| + \lambda\left|\delta F\left(N_f(u)\right)(t)\right| < \frac{1}{N(2E+2)}\inf_{t\in J}\left|\frac{\varepsilon}{2(E+1)}\right|^{p(t)-1}. \tag{3.26}$$

It is a contradiction.

Summarizing this argument, for each $\lambda \in [0,1)$, the problem (3.15) with (1.2) has no solution on $\partial\Omega_\varepsilon$.

(2°) Since u_0 (where u_0 is defined in (3.12)) is the unique solution of $u = \Psi_f(u,0)$, and (A_1) holds $u_0 \in \Omega_\varepsilon$, we can see that the Leray-Schauder degree

$$d_{LS}\left[I - \Psi_f(\cdot,0), \Omega_\varepsilon, 0\right] \neq 0. \tag{3.27}$$

This completes the proof. □

In the following, we will deal with the existence of nonnegative solutions of (1.1)-(1.2) when $\sum_{i=1}^{m-2}\alpha_i \in [0,1)$, $\sigma \in [0,1)$. For any $x = (x^1,\ldots,x^N) \in \mathbb{R}^N$, the notation $x \geq 0$ $(x > 0)$

means $x^j \geq 0$ ($x^j > 0$) for any $j = 1, \ldots, N$. For any $x, y \in \mathbb{R}^N$, the notation $x \geq y$ means $x - y \geq 0$ and the notation $x > y$ means $x - y > 0$.

Theorem 3.3. *In Case (i), we assume*

(1^0) $\delta f(t,x,y,z,w) \geq 0,\ \forall (t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$,

(2^0) $e_1 \geq 0$,

(3^0) $e_2 \leq 0$.

Then, all the solutions of (1.1)-(1.2) *are nonnegative.*

Proof. If u is a solution of (1.1)-(1.2), then

$$w(t)\varphi(t,u'(t)) = \rho(\delta N_f(u)) + \int_0^t \delta f\Big(r, u, (w(r))^{1/(p(r)-1)}u', S(u), T(u)\Big)dr, \quad \forall t \in J. \tag{3.28}$$

It follows from (2.10), (1^0), and (3^0) that

$$\begin{aligned} &w(t)\varphi(t,u'(t)) \\ &\quad = \rho(\delta N_f(u)) + F(\delta N_f(u))(t) \\ &\quad = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(\sum_{i=1}^{m-2}\alpha_i F(\delta N_f(u))(\xi_i) - F(\delta N_f(u))(+\infty) + e_2\right) + F(\delta N_f(u))(t) \\ &\quad = \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}\delta N_f(u)(r)dr - \left(1-\sum_{i=1}^{m-2}\alpha_i\right)\int_t^{+\infty}\delta N_f(u)(r)dr + e_2\right) \\ &\quad \leq 0. \end{aligned} \tag{3.29}$$

Thus, $u'(t) \leq 0$ for any $t \in J$. Holding $u(t)$ is decreasing, namely, $u(t_1) \geq u(t_2)$ for any $t_1, t_2 \in J$ with $t_1 < t_2$.

According to the boundary value condition (1.2) and condition (2°), we have

$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + e_1 \geq \int_0^{+\infty} e(t)u(+\infty)dt + e_1 = \sigma u(+\infty) + e_1, \tag{3.30}$$

then

$$u(+\infty) \geq \frac{e_1}{1-\sigma} \geq 0. \tag{3.31}$$

Thus, all the solutions of (1.1)-(1.2) are nonnegative. The proof is completed. □

Corollary 3.4. *In Case (i), we assume*

(1^0) $\delta f(t,x,y,z,w) \geq 0,\ \forall (t,x,y,z,w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ *with* $x, z, w \geq 0$,

(2^0) $\varphi(s,t) \geq 0,\ \chi(s,t) \geq 0,\ \forall (s,t) \in D$,

(3^0) $e_1 \geq 0$,

(4^0) $e_2 \leq 0$.

Then, we have the following.

(a) *On the conditions of Theorem 3.1, then* (1.1)-(1.2) *has at least a nonnegative solution* u.

(b) *On the conditions of Theorem 3.2, then* (1.1)-(1.2) *has at least a nonnegative solution* u.

Proof. (a) Define

$$L(u) = \left(L_*\left(u^1\right), \ldots, L_*\left(u^N\right)\right), \tag{3.32}$$

where

$$L_*(t) = \begin{cases} t, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{3.33}$$

Denote

$$\widetilde{f}(t, u, v, S(u), T(u)) = f(t, L(u), v, S(L(u)), T(L(u))), \quad \forall (t, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N, \tag{3.34}$$

then $\widetilde{f}(t, u, v, S(u), T(u))$ satisfies Caratheodory condition and $\widetilde{f}(t, u, v, S(u), T(u)) \geq 0$ for any $(t, u, v) \in J \times \mathbb{R}^N \times \mathbb{R}^N$.

Obviously, we have

(A_2) $\lim_{|u|+|v| \to +\infty} (\widetilde{f}(t, u, v, S(u), T(u)) / (|u| + |v|)^{q(t)-1}) = 0$, for $t \in J$ uniformly,

where $q(t) \in C(J, \mathbb{R})$, and $1 < q^- \leq q^+ < p^-$.

Then, $\widetilde{f}(t, \cdot, \cdot, \cdot, \cdot)$ satisfies sub-$(p^- - 1)$ growth condition.

Let us consider the existence of solutions of the following system:

$$-\Delta_{p(t)} u + \delta \widetilde{f}\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u)\right) = 0, \quad t \in (0, +\infty), \tag{3.35}$$

with boundary value condition (1.2). According to Theorem 3.1, (3.35) with (1.2) has at least a solution u. From Theorem 3.3, we can see that u is nonnegative. Thus, u is a nonnegative solution of (1.1)-(1.2).

(b) It is similar to the proof of (a).

This completes the proof. □

4. Existence of Solutions in Case (ii)

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1)-(1.2) when $\sum_{i=1}^{m-2} \alpha_i \in [0, 1)$, $\sigma = 1$.

Theorem 4.1. *Assume that* Ω *is an open bounded set in* C^1 *such that the following conditions hold.*

(1^0) *For each $\lambda \in (0,1)$, the problem*

$$\left(w(t)\varphi\left(t,\frac{u'(t)}{\lambda}\right)\right)' = \delta f\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right), \quad t \in (0,+\infty),$$
$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + \lambda^2 e_1, \tag{4.1}$$
$$\lim_{t\to+\infty} w(t)\varphi\left(t,\frac{u'(t)}{\lambda}\right) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\varphi\left(\xi_i,\frac{u'(\xi_i)}{\lambda}\right) + e_2,$$

has no solution on $\partial\Omega$.

(2^0) *The equation*

$$\omega(a) := \int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}\delta f(s,a,0,S(a),T(a))ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\right.\right.$$
$$\left.\left.\left.\times\int_r^{+\infty}\delta f(s,a,0,S(a),T(a))ds+e_2\right)\right]dr\right\}dt=0 \tag{4.2}$$

has no solution on $\partial\Omega\cap\mathbb{R}^N$.

(3^0) *The Brouwer degree $d_B[\omega,\Omega\cap\mathbb{R}^N,0]\neq 0$.*

Then, problems (1.1)-(1.2) *have a solution on $\overline{\Omega}$.*

Proof. For any $\lambda \in (0,1]$, it is easy to have problem (4.1) can be written in the equivalent form

$$u = \Phi_f(u,\lambda) = P_1 u + \Theta_\lambda(\delta N_f(u)) + K_1^\lambda(\delta N_f(u)), \tag{4.3}$$

where

$$P_1 : C^1 \longrightarrow C^1, \qquad u \longmapsto u(0),$$
$$\Theta_\lambda : L^1 \longrightarrow \mathbb{R}^N,$$
$$h \longmapsto \int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}h(s)ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\right.\right. \tag{4.4}$$
$$\left.\left.\left.\times\int_r^{+\infty}h(s)ds+e_2\right)\right]dr\right\}dt-\lambda e_1,$$
$$K_1^\lambda(h)(t) := F\left\{\lambda\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_1^\lambda(h)+F(h)\right)\right]\right\}(t), \quad \forall t\in[0,+\infty),$$

where $\rho_1^\lambda(\cdot) = w(0)\varphi(0,u'(0)/\lambda) = \rho_1(\cdot)$.

It is easy to see that the operator P_1 is compact continuous. According to Lemmas 2.5 and 2.6, we can conclude that Φ_f is continuous and compact from $C^1 \times [0,1]$ to C^1. We assume that for $\lambda = 1$, (4.3) does not have a solution on $\partial\Omega$, otherwise we complete the proof. Now from hypothesis (1^0), it follows that (4.3) has no solutions for $(u,\lambda) \in \partial\Omega \times (0,1]$.

For $\lambda \in [0,1]$, if u is a solution of (4.3), we have

$$\Theta_\lambda\left(\delta N_f(u)\right) = 0. \tag{4.5}$$

Thus, for $\lambda = 0$, it follows from (4.3) and (4.4) that

$$u = \Phi_f(u,0) = P_1 u + \Theta_0\left(\delta N_f(u)\right), \tag{4.6}$$

it holds $u \equiv d$, a constant.

Therefore, when $\lambda = 0$, by (4.5),

$$\begin{aligned}\int_0^{+\infty}\Bigg\{e(t)\int_t^{+\infty}\varphi^{-1}\Bigg[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\Bigg(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}\delta f(s,d,0,S(d),T(d))ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\\ \times\int_r^{+\infty}\delta f(s,d,0,S(d),T(d))ds+e_2\Bigg)\Bigg]dr\Bigg\}dt=0,\end{aligned} \tag{4.7}$$

which together with hypothesis (2^0) implies that $u = d \notin \partial\Omega$. Thus, we have proved that (4.3) has no solution (u,λ) on $\partial\Omega \times [0,1]$, then we get that for each $\lambda \in [0,1]$, the Leray-Schauder degree $d_{\mathrm{LS}}[I-\Phi_f(\cdot,\lambda),\Omega,0]$ is well defined, and, from the properties of that degree, we have

$$d_{\mathrm{LS}}\left[I-\Phi_f(\cdot,1),\Omega,0\right] = d_{\mathrm{LS}}\left[I-\Phi_f(\cdot,0),\Omega,0\right]. \tag{4.8}$$

Now, it is clear that problem

$$u = \Phi_f(u,1) \tag{4.9}$$

is equivalent to problem (1.1)-(1.2), and (4.8) tells us that problem (4.9) will have a solution if we can show that

$$d_{\mathrm{LS}}\left[I-\Phi_f(\cdot,0),\Omega,0\right] \neq 0. \tag{4.10}$$

Since

$$\Phi_f(u,0) = P_1 u + \Theta_0\left(N_{\delta f}(u)\right) + K_1^0\left(N_{\delta f}(u)\right), \tag{4.11}$$

then

$$u - \Phi_f(u,0) = u - P_1 u - \Theta_0\left(N_{\delta f}(u)\right) - K_1^0\left(N_{\delta f}(u)\right). \tag{4.12}$$

From (4.4), we have $K_1^0(N_{\delta f}(u)) \equiv 0$. By the properties of the Leray-Schauder degree, we have

$$d_{\mathrm{LS}}\left[I-\Phi_f(\cdot,0),\Omega,0\right] = (-1)^N d_B\left[\omega,\Omega\cap\mathbb{R}^N,0\right], \tag{4.13}$$

where the function ω is defined in (4.2) and d_B denotes the Brouwer degree. Since by hypothesis (3^0), this last degree is different from zero. This completes the proof. □

Our next theorem is a consequence of Theorem 4.1. As an application of Theorem 4.1, let us consider the following equation with (1.2):

$$\begin{aligned}\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' = {}& g\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right)\\ &+ b\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right),\end{aligned} \tag{4.14}$$

where $b : J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N \rightarrow \mathbb{R}^N$ is Caratheodory, $g = (g^1,\ldots,g^N) : J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous and Caratheodory, and, for any fixed $y_0\in\mathbb{R}^N\setminus\{0\}$, if $y_0^i\neq 0$, then $g^i(t,y_0,0,S(y_0),T(y_0))\neq 0$, for all $t\in J$, for all $i=1,\ldots,N$.

Theorem 4.2. *Assume that the following conditions hold:*

(1^0) $g(t,kx,ky,kz,kw) = k^{q(t)-1}g(t,x,y,z,w)$ *for all* $k>0$ *and all* $(t,x,y,z,w)\in J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$, *where* $q(t)\in C(J,\mathbb{R})$ *satisfies* $1<q^-\leq q^+<p^-$,

(2^0) $\lim_{|x|+|y|+|z|+|w|\to+\infty}(b(t,x,y,z,w)/(|x|+|y|+|z|+|w|)^{q(t)-1}) = 0$, *for* $t\in J$ *uniformly*,

(3^0) *for large enough* $R_0>0$, *the equation*

$$\begin{aligned}\omega_g(a) := \int_0^{+\infty}\Bigg\{e(t)\int_t^{+\infty}\varphi^{-1}\Bigg[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\Bigg(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}g(s,a,0,,S(a),T(a))ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\\ \times\int_r^{+\infty}g(s,a,0,S(a),T(a))ds+e_2\Bigg)\Bigg]dr\Bigg\}dt = 0\end{aligned} \tag{4.15}$$

has no solution on $\partial B(R_0)\cap\mathbb{R}^N$, *where* $B(R_0) = \{u\in C^1\|u\|_1<R_0\}$,

(4^0) *the Brouwer degree* $d_B[\omega_g,b(R_0),0]\neq 0$ *for large enough* $R_0>0$, *where* $b(R_0) = \{x\in\mathbb{R}^N \mid |x|<R_0\}$.

Then, problem (4.14) *with* (1.2) *has at least one solution.*

Proof. Denote

$$\begin{aligned}N_f(u,\lambda) &= f\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u),\lambda\right)\\ &= g\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right)+\lambda b\left(t,u,(w(r))^{1/(p(t)-1)}u',S(u),T(u)\right).\end{aligned} \tag{4.16}$$

At first, we consider the following problem

$$\left(w(t)\varphi\left(t,\frac{u'(t)}{\lambda}\right)\right)' = N_f(u,\lambda), \quad t\in(0,+\infty),$$
$$u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + \lambda^2 e_1, \tag{4.17}$$
$$\lim_{t\to+\infty} w(t)\varphi\left(t,\frac{u'(t)}{\lambda}\right) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\varphi\left(\xi_i,\frac{u'(\xi_i)}{\lambda}\right) + e_2.$$

For any $\lambda\in(0,1]$, it is easy to have problem (4.17) can be written in the equivalent form

$$u = \Phi_f(u,\lambda) = P_1 u + \Theta_\lambda(N_f(u,\lambda)) + K_1^\lambda(N_f(u,\lambda)), \tag{4.18}$$

where Θ_λ and K_1^λ are defined in Theorem 4.1.

We claim that all the solutions of (4.17) are uniformly bounded for $\lambda\in(0,1]$. In fact, if it is false, we can find a sequence of solutions $\{(u_n,\lambda_n)\}$ for (4.17) such that $\|u_n\|_1\to+\infty$ as $n\to+\infty$ and $\|u_n\|_1>1$ for any $n=1,2,\ldots$.

Since (u_n,λ_n) are solutions of (4.17), we have

$$\begin{aligned}\int_0^{+\infty}\Bigg\{e(t)\int_t^{+\infty}\varphi^{-1}\Bigg[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\Bigg(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}N_f(u_n,\lambda_n)ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\\ \times\int_r^{+\infty}N_f(u_n,\lambda_n)ds+e_2\Bigg)\Bigg]dr\Bigg\}dt-\lambda_n e_1\\ =0,\quad\forall t\in(0,+\infty).\end{aligned} \tag{4.19}$$

It follows from Lemma 2.5 that

$$\begin{aligned}\left|\rho_1^{\lambda_n}(N_f(u_n,\lambda_n))\right| &\le \frac{2N}{1-\sum_{i=1}^{m-2}\alpha_i}\cdot\left(\left\|N_f(u_n,\lambda_n)\right\|_{L^1}+|e_2|\right)\\ &\le C_1\|u_n\|_1^{q^+-1}\left\|g\left[t,\frac{u_n}{\|u_n\|_1},\frac{(w(t))^{1/(p(t)-1)}u_n'}{\|u_n\|_1},\frac{S(u_n)}{\|u_n\|_1},\frac{T(u_n)}{\|u_n\|_1}\right]+o(1)\right\|_{L^1},\end{aligned} \tag{4.20}$$

where $o(1)$ means the function which is uniformly convergent to 0 (as $n\to+\infty$). According to the property of g and (4.20), then there exists a positive constant C_2 such that

$$\left|\rho_1^{\lambda_n}(N_f(u_n,\lambda_n))\right| \le C_2\|u_n\|_1^{q^+-1},\quad\forall t\in(0,+\infty), \tag{4.21}$$

then we have

$$\left|\rho_1^{\lambda_n}\left(N_f(u_n,\lambda_n)\right)\right| + \left|F\left(N_f(u_n,\lambda_n)\right)\right| \le C_3\|u_n\|_1^{q^+-1}, \quad \forall t \in (0,+\infty). \tag{4.22}$$

From (4.18), we have

$$w(t)\varphi\left(t, \frac{u_n'(t)}{\lambda_n}\right) = \rho_1^{\lambda_n}\left(N_f(u_n,\lambda_n)\right) + F\left(N_f(u_n,\lambda_n)\right), \quad t \in J, \tag{4.23}$$

then

$$w(t)\left|\frac{u_n'(t)}{\lambda_n}\right|^{p(t)-1} \le \left|\rho_1^{\lambda_n}\left(N_f(u_n,\lambda_n)\right)\right| + \left|F\left(N_f(u_n,\lambda_n)\right)\right| \le C_3\|u_n\|_1^{q^+-1}. \tag{4.24}$$

Denote $\alpha = (q^+ - 1)/(p^- - 1)$, then

$$\left\|(w(t))^{1/(p(t)-1)}u_n'(t)\right\|_0 \le \lambda_n C_4\|u_n\|_1^{\alpha} \le C_5\|u_n\|_1^{\alpha}. \tag{4.25}$$

Since $\alpha \in (0,1)$, from (4.25), we have

$$\lim_{n\to+\infty}\frac{\|u_n\|_0}{\|u_n\|_1} = 1. \tag{4.26}$$

Denote $\mu_n = (|u_n^1|_0/\|u_n\|_0, |u_n^2|_0/\|u_n\|_0, \ldots, |u_n^N|_0/\|u_n\|_0)$, then $\mu_n \in \mathbb{R}^N$ and $|\mu_n| = 1$ $(n = 1,2,\ldots)$, then $\{\mu_n\}$ possesses a convergent subsequence (which denoted by μ_n), and then there exists a vector $\mu_0 = (\mu_0^1, \mu_0^2, \ldots, \mu_0^N) \in \mathbb{R}^N$ such that

$$\left|\mu_0\right| = 1, \qquad \lim_{n\to+\infty}\mu_n = \mu_0. \tag{4.27}$$

Without loss of generality, we assume that $\mu_0^1 > 0$. Since $u_n \in C(J,\mathbb{R})$, there exist $\eta_n^i \in (0,+\infty)$ such that

$$\left|u_n^i\left(\eta_n^i\right)\right| \ge \left(1 - \frac{1}{n}\right)\left|u_n^i\right|_0, \quad i = 1,2,\ldots,N, \ n = 1,2,\ldots, \tag{4.28}$$

and, then, from (4.25), we have

$$0 \le \left|u_n^1(t) - u_n^1\left(\eta_n^1\right)\right| = \left|\int_{\eta_n^1}^t \left(u_n^1\right)'(r)dr\right| \le C_5\|u_n\|_1^{\alpha}\int_0^{+\infty}(w(t))^{-1/(p(t)-1)}dt. \tag{4.29}$$

Since $\|u_n\|_1 \to +\infty$ (as $n \to +\infty$), $\alpha \in (0,1)$, and $\mu_0^1 > 0$, we have

$$\lim_{n\to+\infty}\frac{1}{\left|u_n^1(\eta_n^1)\right|}C_5\|u_n\|_1^{\alpha}\int_0^{+\infty}(w(t))^{-1/(p(t)-1)}dt = 0. \tag{4.30}$$

From (4.26)–(4.30), we have

$$\lim_{n\to+\infty}\frac{u_n^1(t)}{u_n^1(\eta_n^1)}=1,\quad \text{for } t\in J \text{ uniformly}. \tag{4.31}$$

So we get

$$\lim_{n\to+\infty}\frac{u_n(t)}{\|u_n\|_1}=\mu_*,\quad \lim_{n\to+\infty}\frac{(w(t))^{1/(p(t)-1)}u_n'(t)}{\|u_n\|_1}=0,\quad \text{for } t\in J \text{ uniformly}, \tag{4.32}$$

where $\mu_*\in\mathbb{R}^N$ satisfies $|\mu_*|=1$, $|\mu_*^i|=\mu_0^i$.

We denote

$$g_*^1=\|u_n\|_1^{q(s)-1}\left\{g^1\left[s,\mu_*+o(1),o(1),S(\mu_*+o(1)),T(\mu_*+o(1))\right]+o(1)\right\},\quad \text{where } s\in J. \tag{4.33}$$

Since $\mu_0^1\neq 0$, from (4.19) and (4.32), we have

$$\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}g_*^1ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\right.\right.$$
$$\left.\left.\left.\times\int_r^{+\infty}g_*^1ds+e_2\right)\right]dr\right\}dt-\lambda_n e_1=0. \tag{4.34}$$

Since $g^1(s,\mu_*,0,S(\mu_*),T(\mu_*))\neq 0$, according to the continuity of g^1, we have

$$\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}g_*^1ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\right.\right.$$
$$\left.\left.\left.\times\int_r^{+\infty}g_*^1ds+e_2\right)\right]dr\right\}dt-\lambda_n e_1\neq 0, \tag{4.35}$$

and it is a contradiction to (4.34). This implies that there exists a big enough $R_0>0$ such that all the solutions of (4.18) when $\lambda\in(0,1]$ belongs to $B(R_0)$.

For $\lambda\in[0,1]$, if u is a solution of (4.18), we have

$$\Theta_\lambda(N_f(u,\lambda))=0. \tag{4.36}$$

For $\lambda=0$, $f=g$, from (4.18), we have

$$u=\Phi_g(u,0)=P_1u+\Theta_0(g), \tag{4.37}$$

it holds $u\equiv d$, a constant.

Therefore, when $\lambda = 0$, we have

$$\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,\frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty}g(s,d,0,S(d),T(d))ds-\left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\right.\right.$$
$$\left.\left.\left.\times\int_r^{+\infty}g(s,d,0,S(d),T(d))ds+e_2\right)\right]dr\right\}dt=0, \tag{4.38}$$

which together with hypothesis (3^0) implies that $u = d \notin \partial B(R_0)$. Thus, we have proved that (4.18) has no solution (u,λ) on $\partial B(R_0)\times[0,1]$, then we get that the Leray-Schauder degree $d_{\mathrm{LS}}[I-\Phi_f(\cdot,\lambda),B(R_0),0]$ is well defined for each $\lambda\in[0,1]$, which implies that

$$d_{\mathrm{LS}}\left[I-\Phi_f(\cdot,1),B(R_0),0\right]=d_{\mathrm{LS}}\left[I-\Phi_g(\cdot,0),B(R_0),0\right]. \tag{4.39}$$

Now it is clear that problem

$$u=\Phi_f(u,1) \tag{4.40}$$

is equivalent to problem (4.14) with (1.2), and (4.39) tells us that problem (4.40) will have a solution if we can show that

$$d_{\mathrm{LS}}\left[I-\Phi_g(\cdot,0),B(R_0),0\right]\neq 0. \tag{4.41}$$

Since

$$\Phi_g(u,0)=P_1u+\Theta_0(g), \tag{4.42}$$

then

$$u-\Phi_g(u,0)=u-P_1u-\Theta_0(g). \tag{4.43}$$

By the properties of the Leray-Schauder degree, we have

$$d_{\mathrm{LS}}\left[I-\Phi_g(\cdot,0),B(R_0),0\right]=(-1)^N d_B\left[\omega_g,b(R_0),0\right], \tag{4.44}$$

where the function ω_g is defined in (4.15) and d_B denotes the Brouwer degree. By hypothesis (4^0), this last degree is different from zero. This completes the proof. □

Corollary 4.3. *If $b: J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$ is Caratheodory, which satisfies the conditions of Theorem 4.2, $g(t,u,v,S(u),T(u)) = \beta(t)(|u|^{q(t)-2}u+|v|^{q(t)-2}v+|S(u)|^{q(t)-2}S(u)+|T(u)|^{q(t)-2}T(u))$, where $\beta(t)\in L^1(J,\mathbb{R})$, $\beta(t),q(t)\in C(J,\mathbb{R})$ are positive functions, and satisfies $1<q^-\le q^+<p^-$ and $\varphi(s,t)$ and $\chi(s,t)$ are nonnegative, then* (4.14) *with* (1.2) *has at least one solution.*

Proof. Since

$$g(t,u,v,S(u),T(u)) = \beta(t)\Big(|u|^{q(t)-2}u + |v|^{q(t)-2}v + |S(u)|^{q(t)-2}S(u) + |T(u)|^{q(t)-2}T(u)\Big), \quad (4.45)$$

then

$$\begin{aligned}
&\omega_g(a)\\
&= \int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r, \frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty} g(s,a,0,S(a),T(a))ds - \left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\right.\right.\\
&\qquad\left.\left.\left.\times\int_r^{+\infty} g(s,a,0,S(a),T(a))ds + e_2\right)\right]dr\right\}dt\\
&= \int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r, \frac{(w(r))^{-1}}{1-\sum_{i=1}^{m-2}\alpha_i}\left(-\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty} \beta(s)\Big(|a|^{q(s)-2}a + |S(a)|^{q(s)-2}S(a)\right.\right.\right.\\
&\qquad\left. + |T(a)|^{q(s)-2}T(a)\Big)ds - \left(1-\sum_{i=1}^{m-2}\alpha_i\right)\right.\\
&\qquad\times\int_r^{+\infty}\beta(s)\Big(|a|^{q(s)-2}a + |S(a)|^{q(s)-2}S(a)\\
&\qquad\left.\left.\left. + |T(a)|^{q(s)-2}T(a)\Big)ds + e_2\right)\right]dr\right\}dt,
\end{aligned} \quad (4.46)$$

then it is easy to say that $\omega_g(a) = 0$ has only one solution in $\mathbb{R}^N$, and

$$d_B[\omega_g, b(R_0), 0] = d_B[I, b(R_0), 0] \neq 0, \quad (4.47)$$

and, according to Theorem 4.2, we get that (4.14) with (1.2) has at least a solution. This completes the proof. □

In the following, let us consider

$$-\Big(w(t)|u'|^{p(t)-2}u'\Big)' + f\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\Big) = 0, \quad t \in (0,+\infty), \quad (4.48)$$

where

$$\begin{aligned}
&f\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u), \delta\Big)\\
&\quad = g\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\Big) + \delta h\Big(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u)\Big),
\end{aligned} \quad (4.49)$$

where $g, h : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ are Caratheodory.

We have the following.

Theorem 4.4. *We assume that conditions of (1^0), (3^0), and (4^0) of Theorem 4.2 are satisfied, then problem* (4.48) *with* (1.2) *has at least one solution when the parameter δ is small enough.*

Proof. Denote

$$\begin{aligned} &f_{\lambda\delta}\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big) \\ &\quad = g\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big) + \lambda\delta h\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big). \end{aligned} \tag{4.50}$$

Let us consider the existence of solutions of the following

$$-\Big(w(t)|u'|^{p(t)-2}u'\Big)' + f_{\lambda\delta}\Big(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\Big) = 0, \quad t \in (0,+\infty), \tag{4.51}$$

with (1.2).

We know that (4.51) with (1.2) has the same solution of

$$u = \Psi_\delta(u,\lambda) = P_1 u + \Theta\big(N_{f_{\lambda\delta}}(u)\big) + K_1\big(N_{f_{\lambda\delta}}(u)\big), \tag{4.52}$$

where $N_{f_{\lambda\delta}}(u)$ is defined in (2.32).

Obviously, $f_0 = g$. So $\Psi_\delta(u,0) = \Phi_g(u,1)$. From the proof of Theorem 4.2, we can see that all the solutions of $u = \Psi_\delta(u,0)$ are uniformly bounded, then there exists a large enough $R_0 > 0$ such that all the solutions of $u = \Psi_\delta(u,0)$ belong to $B(R_0) = \{u \in C^1 \mid \|u\|_1 < R_0\}$. Since $\Psi_\delta(u,0)$ is compact continuous from C^1 to C^1, we have

$$\inf_{u \in \partial B(R_0)} \|u - \Psi_\delta(u,0)\|_1 > 0. \tag{4.53}$$

Since g, h are Caratheodory, we have

$$\begin{aligned} \big|\rho_1\big(N_{f_{\lambda\delta}}(u)\big) - \rho_1\big(N_{f_0}(u)\big)\big| &\longrightarrow 0, \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\ \big\|F\big(N_{f_{\lambda\delta}}(u)\big) - F\big(N_{f_0}(u)\big)\big\|_0 &\longrightarrow 0, \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\ \big\|K_1\big(N_{f_{\lambda\delta}}(u)\big) - K_1\big(N_{f_0}(u)\big)\big\|_1 &\longrightarrow 0, \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0, \\ \big|P_1\big(N_{f_{\lambda\delta}}(u)\big) - P_1\big(N_{f_0}(u)\big)\big| &\longrightarrow 0, \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \end{aligned} \tag{4.54}$$

Thus,

$$\|\Psi_\delta(u,\lambda) - \Psi_0(u,\lambda)\|_1 \longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \tag{4.55}$$

Obviously, $\Psi_0(u,\lambda) = \Psi_\delta(u,0) = \Psi_0(u,0)$. Therefore,

$$\|\Psi_\delta(u,\lambda) - \Psi_\delta(u,0)\|_1 \longrightarrow 0 \quad \text{for } (u,\lambda) \in \overline{B(R_0)} \times [0,1] \text{ uniformly, as } \delta \longrightarrow 0. \tag{4.56}$$

Thus, when δ is small enough, from (4.53), we can conclude

$$\begin{aligned}&\inf_{(u,\lambda)\in\partial B(R_0)\times[0,1]} \|u - \Psi_\delta(u,\lambda)\|_1 \\ &\quad \geq \inf_{u\in\partial B(R_0)} \|u - \Psi_\delta(u,0)\|_1 - \sup_{(u,\lambda)\in\overline{B(R_0)}\times[0,1]} \|\Psi_\delta(u,0) - \Psi_\delta(u,\lambda)\|_1 > 0.\end{aligned} \tag{4.57}$$

Thus $u = \Psi_\delta(u,\lambda)$ has no solution on $\partial B(R_0)$ for any $\lambda \in [0,1]$, when δ is small enough. It means that the Leray-Schauder degree $d_{\mathrm{LS}}[I - \Psi_\delta(\cdot,\lambda), B(R_0), 0]$ is well defined for any $\lambda \in [0,1]$ and

$$d_{\mathrm{LS}}[I - \Psi_\delta(u,\lambda), B(R_0), 0] = d_{\mathrm{LS}}[I - \Psi_\delta(u,0), B(R_0), 0]. \tag{4.58}$$

From the proof of Theorem 4.2, we can see that the right hand side is nonzero, then (4.48) with (1.2) has at least one solution, when δ is small enough. This completes the proof. □

5. Existence of Solutions in Case (iii)

In this section, we will apply Leray-Schauder's degree to deal with the existence of solutions for (1.1)-(1.2) when $\sum_{i=1}^{m-2}\alpha_i = 1,\ \sigma = 1$.

Theorem 5.1. *Assume that Ω is an open bounded set in C^1 such that the following conditions hold.*

(1^0) *For each $\lambda \in (0,1)$, the problem*

$$\begin{aligned}&\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' = \lambda\delta f\left(t, u, (w(t))^{1/(p(t)-1)}u', S(u), T(u)\right), \quad t \in (0,+\infty), \\ &u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + \lambda e_1, \\ &\lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + \lambda e_2,\end{aligned} \tag{5.1}$$

has no solution on $\partial\Omega$.

(2^0) *The equation*

$$\omega^*(a) := \sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{+\infty} \delta f(t, a, 0, S(a), T(a))dt - e_2 = 0 \tag{5.2}$$

has no solution on $\partial\Omega \cap \mathbb{R}^N$.

(3^0) *The Brouwer degree $d_B[\omega^*, \Omega \cap \mathbb{R}^N, 0] \neq 0$.*

Then, problems (1.1)-(1.2) *have a solution on $\overline{\Omega}$.*

Proof. Let us consider the following problem:

$$
\begin{gathered}
\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' = \lambda\delta N_f(u) + (1-\lambda)Q^*\left(\delta N_f(u)\right), \quad \forall t \in (0,+\infty), \\
u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + \lambda e_1, \\
\lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + \lambda e_2,
\end{gathered}
\tag{5.3}
$$

where Q and Q^* are defined in (2.79) and (2.80), respectively.

For any $\lambda \in (0,1]$, observe that, if u is a solution to (5.1) or u is a solution to (5.3), we have necessarily

$$
Q\left(N_{\delta f}(u)\right) = 0, \qquad Q^*\left(N_{\delta f}(u)\right) = 0. \tag{5.4}
$$

It means that (5.1) and (5.3) have the same solutions for $\lambda \in (0,1]$.

We denote $N(\cdot,\cdot) : C^1 \times J \to L^1$ defined by

$$
N(u,\lambda) = \lambda N_{\delta f}(u) + (1-\lambda)Q^*\left(N_{\delta f}(u)\right), \tag{5.5}
$$

where $N_{\delta f}(u)$ is defined by (2.32). Let

$$
\begin{aligned}
\Phi_f^*(u,\lambda) &= P_2 u + Q_\lambda(N(u,\lambda)) + K_2^\lambda(N(u,\lambda)) \\
&= P_2 u + Q\left(N_{\delta f}(u)\right) + K_2^\lambda(N(u,\lambda)),
\end{aligned}
\tag{5.6}
$$

where

$$
\begin{gathered}
P_2 : C^1 \longrightarrow C^1, \qquad u \longmapsto u(0), \\
Q_\lambda : L^1 \longrightarrow \mathbb{R}^N, \qquad h \longmapsto \sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{+\infty} h(t)dt - \lambda e_2, \\
Q_\lambda^* : L^1 \longrightarrow L^1, \qquad h \longmapsto \tau(t)\left(\sum_{i=1}^{m-2}\alpha_i \int_{\xi_i}^{+\infty} h(t)dt - \lambda e_2\right), \\
K_2^\lambda(h)(t) := F\left\{\varphi^{-1}\left[t,(w(t))^{-1}\left(\rho_2^\lambda\left((I-Q_\lambda^*)h\right) + F\left((I-Q_\lambda^*)h\right)\right)\right]\right\}(t), \quad \forall t \in [0,+\infty),
\end{gathered}
\tag{5.7}
$$

where ρ_2^λ satisfies

$$
\int_0^{+\infty}\left\{e(t)\int_t^{+\infty}\varphi^{-1}\left[r,(w(r))^{-1}\left(\rho_2^\lambda + F(h)(r)\right)\right]dr\right\}dt - \lambda e_1 = 0, \tag{5.8}
$$

and the fixed point of $\Phi_f^*(u,1)$ is a solution for (5.3). Also problem (5.3) can be written in the equivalent form

$$u = \Phi_f^*(u,\lambda). \tag{5.9}$$

Since f is Caratheodory, it is easy to see that $N(\cdot,\cdot)$ is continuous and sends bounded sets into equi-integrable sets. It is easy to see that P_2 is compact continuous. According to Lemmas 2.8 and 2.9, we can conclude that $\Phi_f^*(\cdot,\cdot)$ is continuous and compact from $C^1 \times [0,1]$ to C^1. We assume that for $\lambda = 1$, (5.9) does not have a solution on $\partial\Omega$, otherwise we complete the proof. Now, from hypothesis (1^0), it follows that (5.9) has no solutions for $(u,\lambda) \in \partial\Omega \times (0,1]$. For $\lambda = 0$, (5.3) is equivalent to the problem

$$\begin{gathered}\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' = Q^*(\delta N_f(u)),\\ u(+\infty) = \int_0^{+\infty} e(t)u(t)dt,\\ \lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i).\end{gathered} \tag{5.10}$$

If u is a solution to this problem, we must have

$$\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty} Q^*(\delta N_f(u))dt = \sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty} \delta f\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right)dt - e_2 = 0. \tag{5.11}$$

Hence,

$$w(t)\left|u'\right|^{p(t)-2}u' \equiv c, \tag{5.12}$$

where $c \in \mathbb{R}^N$ is a constant.

It is easy to see that $(u^i)'$ keeps the same sign of c^i. From $u(+\infty) = \int_0^{+\infty} e(t)u(t)dt$, we have $\int_0^{+\infty} e(t)(u(+\infty) - u(t))dt = 0$. From the continuity of u, there exist $t_i \in (0,+\infty)$, such that $(u^i)'(t_i) = 0$, $i = 1,\dots,N$. Hence, $u' \equiv 0$, it holds $u \equiv d$, a constant. Thus, by (5.11), we have

$$\sum_{i=1}^{m-2}\alpha_i\int_{\xi_i}^{+\infty} \delta f(t,d,0,S(d),T(d))dt - e_2 = 0, \tag{5.13}$$

which together with hypothesis (2^0) implies that $u = d \notin \partial\Omega$. Thus, we have proved that (5.9) has no solution (u,λ) on $\partial\Omega \times [0,1]$, then we get that the Leray-Schauder degree $d_{\mathrm{LS}}[I - \Phi_f^*(\cdot,\lambda),\Omega,0]$ is well defined for $\lambda \in [0,1]$, and from the properties of that degree, we have

$$d_{\mathrm{LS}}\left[I - \Phi_f^*(\cdot,1),\Omega,0\right] = d_{\mathrm{LS}}\left[I - \Phi_f^*(\cdot,0),\Omega,0\right]. \tag{5.14}$$

Now it is clear that problem

$$u = \Phi_f^*(u,1) \tag{5.15}$$

is equivalent to the problem (1.1)-(1.2), and (5.14) tells us that problem (5.15) will have a solution if we can show that

$$d_{\mathrm{LS}}\left[I - \Phi_f^*(\cdot,0),\Omega,0\right] \neq 0. \tag{5.16}$$

Since

$$\Phi_f^*(u,0) = P_2 u + Q\left(N_{\delta f}(u)\right) + K_2^0\left(Q^*\left(N_{\delta f}(u)\right)\right), \tag{5.17}$$

then

$$u - \Phi_f^*(u,0) = u - P_2 u - Q\left(N_{\delta f}(u)\right) - K_2^0(0). \tag{5.18}$$

Similar to Lemma 2.8, we have

$$\left|\rho_2^{\lambda}\left(Q^*\left(N_{\delta f}(u)\right)\right)\right| \leq 3N\left(\frac{E_{\#}+1}{E_{\#}}\right)^{p^+-1}\left[\left\|Q^*\left(N_{\delta f}(u)\right)\right\|_0 + |\lambda e_1|^{p^{\#}-1}\right]. \tag{5.19}$$

Thus, $\rho_2^0(0) = 0$, then $K_2^0(0) \equiv 0$. From (5.18), we have

$$u - \Phi_f^*(u,0) = u - P_2 u - Q\left(N_{\delta f}(u)\right). \tag{5.20}$$

By the properties of the Leray-Schauder degree, we have

$$d_{\mathrm{LS}}\left[I - \Phi_f^*(\cdot,0),\Omega,0\right] = (-1)^N d_B\left[\omega^*,\Omega\cap\mathbb{R}^N,0\right], \tag{5.21}$$

where the function ω^* is defined in (5.2) and d_B denotes the Brouwer degree. By hypothesis (3^0), this last degree is different from zero. This completes the proof. □

Our next theorem is a consequence of Theorem 5.1. As an application of Theorem 5.1. Let us consider the following equation with (1.2)

$$\begin{aligned}\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' &= g\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right)\\ &\quad + b\left(t,u,(w(t))^{1/(p(t)-1)}u',S(u),T(u)\right),\end{aligned} \tag{5.22}$$

where $b : J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, $g = (g^1,\dots,g^N) : J\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}^N$ is continuous and Caratheodory, and, for any fixed $y_0 \in \mathbb{R}^N\setminus\{0\}$, if $y_0^i \neq 0$, then $g^i(t,y_0,0,S(y_0),T(y_0)) \neq 0$, for all $t\in J$, for all $i = 1,\dots,N$.

Theorem 5.2. *Assume that the following conditions hold.*

(1^0) $g(t, kx, ky, kz, kw) = k^{q(t)-1} g(t, x, y, z, w)$ *for all* $k > 0$ *and all* $(t, x, y, z, w) \in J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, *where* $q(t) \in C(J, \mathbb{R})$ *satisfies* $1 < q^- \leq q^+ < p^-$,

(2^0) $\lim_{|x|+|y|+|z|+|w| \to +\infty} (b(t, x, y, z, w)/(|x| + |y| + |z| + |w|)^{q(t)-1}) = 0$, *for* $t \in J$ *uniformly,*

(3^0) *for large enough* $R_0 > 0$, *the equation*

$$\omega_g^*(a) := \sum_{i=1}^{m-2} \alpha_i \int_{\xi_i}^{+\infty} g(t, a, 0, S(a), T(a)) dt - e_2 = 0, \tag{5.23}$$

has no solution on $\partial B(R_0) \cap \mathbb{R}^N$, *where* $B(R_0) = \{u \in C^1 \|u\|_1 < R_0\}$,

(4^0) *the Brouwer degree* $d_B[\omega_g^*, b(R_0), 0] \neq 0$ *for large enough* $R_0 > 0$, *where* $b(R_0) = \{x \in \mathbb{R}^N \mid |x| < R_0\}$.

Then, problem (5.22) *with* (1.2) *has at least one solution.*

Proof. Denote

$$\begin{aligned} N_f(u, \lambda) &= f\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u), \lambda\right) \\ &= g\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u)\right) + \lambda b\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u)\right). \end{aligned} \tag{5.24}$$

At first, we consider the following problem

$$\left(w(t) |u'|^{p(t)-2} u'\right)' = f\left(t, u, (w(t))^{1/(p(t)-1)} u', S(u), T(u), \lambda\right), \quad t \in (0, +\infty). \tag{5.25}$$

According to Lemma 2.10, we know problem (5.25) with (1.2) has the same solution of

$$u = \widetilde{\Phi}_f(u, \lambda) = P_2 u + Q(N_f(u, \lambda)) + K_2(N_f(u, \lambda)). \tag{5.26}$$

Similar to the proof of Theorem 4.2, we obtain that all the solutions of (5.26) are uniformly bounded for $\lambda \in [0, 1]$. Then, there exists a big enough $R_0 > 0$ such that all the solutions of (5.26) belong to $B(R_0)$, and then we have

$$d_{\mathrm{LS}}\left[I - \widetilde{\Phi}_f(\cdot, 1), B(R_0), 0\right] = d_{\mathrm{LS}}\left[I - \widetilde{\Phi}_f(\cdot, 0), B(R_0), 0\right]. \tag{5.27}$$

If we prove that $d_{\mathrm{LS}}[I - \widetilde{\Phi}_f(\cdot, 0), B(R_0), 0] \neq 0$, then we obtain the existence of solutions for (5.22) with (1.2).

Now, we consider the following equation

$$\begin{gathered}\left(w(t)\left|u'\right|^{p(t)-2}u'\right)' = \lambda N_g(u) + (1-\lambda)Q^* N_g(u), \quad t \in (0,+\infty), \\ u(+\infty) = \int_0^{+\infty} e(t)u(t)dt + \lambda e_1, \\ \lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i) + \lambda e_2,\end{gathered} \tag{5.28}$$

where $N_g(u) = g(t,u,(w(t))^{1/(p(t)-1)}u', S(u), T(u))$.

We denote $G(\cdot,\cdot) : C^1 \times J \to L^1$ defined by

$$G(u,\lambda) = \lambda N_g(u) + (1-\lambda)Q^*\left(N_g(u)\right). \tag{5.29}$$

Similar to the proof of Theorem 5.1, we know that (5.28) has the same solution of

$$u = \Phi_g^*(u,\lambda) = P_2 u + Q\left(N_g(u)\right) + K_2^\lambda(G(u,\lambda)). \tag{5.30}$$

Similar to the discussions of Theorem 4.2, we can obtain that all the solutions of (5.28) are uniformly bounded for each $\lambda \in (0,1]$. When $\lambda = 0$, similar to the proof of Theorem 5.1, we can prove that (5.28) has no solution on $\partial B(R_0)$. Then, we get that the Leray-Schauder degree $d_{LS}[I - \Phi_g^*(\cdot,\lambda), B(R_0), 0]$ is well defined for $\lambda \in [0,1]$, which implies that

$$d_{LS}\left[I - \Phi_g^*(\cdot,1), B(R_0), 0\right] = d_{LS}\left[I - \Phi_g^*(\cdot,0), B(R_0), 0\right]. \tag{5.31}$$

Now it is clear that $\Phi_g^*(u,1) = \widetilde{\Phi}_f(u,0)$. So $d_{LS}[I - \Phi_g^*(\cdot,1), B(R_0), 0] = d_{LS}[I - \widetilde{\Phi}_f(\cdot,0), B(R_0), 0]$. If we prove that $d_{LS}[I - \Phi_g^*(\cdot,0), B(R_0), 0] \neq 0$, then we obtain the existence of solutions for (5.22) with (1.2). Similar to the proof of Theorem 5.1, we have

$$d_{LS}\left[I - \Phi_g^*(\cdot,0), B(R_0), 0\right] = (-1)^N d_B\left[\omega_g^*, b(R_0), 0\right]. \tag{5.32}$$

According to hypothesis (4^0), this last degree is different from zero. We obtain that (5.22) with (1.2) has at least one solution. This completes the proof. □

Similar to Corollary 4.3 and Theorem 4.4, we have the following.

Corollary 5.3. *If $b : J \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Caratheodory, which satisfies the conditions of Theorem 5.2, $g(t,u,v,S(u),T(u)) = \beta(t)(|u|^{q(t)-2}u + |v|^{q(t)-2}v + |S(u)|^{q(t)-2}S(u) + |T(u)|^{q(t)-2}T(u))$, where $\beta(t) \in L^1(J,\mathbb{R})$, $\beta(t), q(t) \in C(J,\mathbb{R})$ are positive functions, and satisfies $1 < q^- \leq q^+ < p^-$, $\varphi(s,t)$ and $\chi(s,t)$ are nonnegative, then* (5.22) *with* (1.2) *has at least one solution.*

Theorem 5.4. *We assume that conditions of (1^0), (3^0), and (4^0) of Theorem 5.2 are satisfied, then problem* (4.48) *with* (1.2) *has at least one solution when the parameter δ is small enough.*

6. Examples

Example 6.1. Consider the following problem

$$-\Delta_{p(t)}u - e^{-t}|u|^{q(t)-2}u - S(u)(t) - (t+1)^{-2} = 0, \quad t \in (0,+\infty),$$
$$\lim_{t\to+\infty} u(t) = \int_0^{+\infty} e^{-t}u(t)dt, \qquad \lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i), \tag{S_1}$$

where $p(t) = 6 + e^{-t}\sin t$, $q(t) = 3 + 2^{-t}\cos t$, $S(u)(t) = \int_0^\infty e^{-2s-t}(\sin st + 1)u(s)ds$.

Obviously, $e^{-t}|u|^{q(t)-2}u+S(u)(t)+(t+1)^{-2}$ is Caratheodory, $q(t) \le 4 < 5 \le p(t)$, $\sum_{i=1}^{m-2}\alpha_i < 1$, the conditions of Corollary 4.3 are satisfied, then (S_1) has a solution.

Example 6.2. Consider the following problem:

$$-\Delta_{p(t)}u + f\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u)\right) + \delta h\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u)\right) + e^{-t}$$
$$= 0, \quad t \in (0,+\infty), \tag{S_2}$$
$$\lim_{t\to+\infty} u(t) = \int_0^{+\infty} e^{-2t}u(t)dt, \qquad \lim_{t\to+\infty} w(t)\left|u'\right|^{p(t)-2}u'(t) = \sum_{i=1}^{m-2}\alpha_i w(\xi_i)\left|u'\right|^{p(\xi_i)-2}u'(\xi_i),$$

where h is Caratheodory and

$$f\left(t,u,(w(t))^{1/(p(t)-1)}u', S(u)\right) = e^{-t}|u|^{q(t)-2}u + e^{-t}w(t)^{(q(t)-1)/(p(t)-1)}\left|u'\right|^{q(t)-2}u' + S(u)(t),$$
$$p(t) = 7 + 3^{-t}\cos 3t, \qquad q(t) = 4 + e^{-2t}\sin 2t, \qquad S(u)(t) = \int_0^\infty e^{-s-2t}(\cos st + 1)u(s)ds. \tag{6.1}$$

Obviously, $e^{-t}|u|^{q(t)-2}u + e^{-t}w(t)^{(q(t)-1)/(p(t)-1)}|u'|^{q(t)-2}u' + S(u)(t)$ is Caratheodory, $q(t) \le 5 < 6 \le p(t)$, $\sum_{i=1}^{m-2}\alpha_i < 1$, the conditions of Theorem 3.2 are satisfied, then (S_2) has a solution when δ is small enough.

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