Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 792689, 13 pages
doi:10.1155/2011/792689

Research Article

Existence Conditions for Bounded Solutions of Weakly Perturbed Linear Impulsive Systems

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Received 31 December 2010; Revised 1 July 2011; Accepted 1 July 2011

Academic Editor: Josef Diblík

The weakly perturbed linear nonhomogeneous impulsive systems in the form $\dot{x} = A(t)x + \varepsilon A_1(t)x + f(t)$, $t \in \mathbb{R}$, $t \notin \mathcal{T} := \{\tau_i\}_{\mathbb{Z}}$, $\Delta x|_{t=\tau_i} = \gamma_i + \varepsilon A_{1i}x(\tau_i-)$, $\tau_i \in \mathcal{T} \subset \mathbb{R}$, $\gamma_i \in \mathbb{R}^n$, and $i \in \mathbb{Z}$ are considered. Under the assumption that the generating system (for $\varepsilon = 0$) does not have solutions bounded on the entire real axis for some nonhomogeneities and using the Vishik-Lyusternik method, we establish conditions for the existence of solutions of these systems bounded on the entire real axis in the form of a Laurent series in powers of small parameter ε with finitely many terms with negative powers of ε, and we suggest an algorithm of construction of these solutions.

1. Introduction

In this contribution we study the problem of existence and construction of solutions of weakly perturbed linear differential systems with impulsive action bounded on the entire real axis. The application of the theory of differential systems with impulsive action (developed in [1–3]), the well-known results on the splitting index by Sacker [4] and by Palmer [5] on the Fredholm property of bounded solutions of linear systems of ordinary differential equations [6–9], the theory of pseudoinverse matrices [10] and results obtained in analyzing boundary-value problems for ordinary differential equations (see [10–12]), enables us to obtain existence conditions and to propose an algorithm for the construction of solutions bounded on the entire real axis of weakly perturbed linear impulsive differential systems.

2. Initial Problem

We consider the problem of existence and construction of solutions bounded on the entire real axis of linear systems of ordinary differential equations with impulsive action at fixed points of time

$$\begin{aligned} \dot{x} &= A(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathfrak{T}, \\ \Delta x|_{t=\tau_i} &= \gamma_i, \quad \tau_i \in \mathfrak{T},\ i \in \mathbb{Z}, \end{aligned} \tag{2.1}$$

where $A \in BC_{\mathfrak{T}}(\mathbb{R})$ is an $n \times n$ matrix of functions, $f \in BC_{\mathfrak{T}}(\mathbb{R})$ is an $n \times 1$ vector function, $BC_{\mathfrak{T}}(\mathbb{R})$ is the Banach space of real vector functions bounded on $\mathbb{R}$ and left-continuous for $t \in \mathbb{R}$ with discontinuities of the first kind at $t \in \mathfrak{T} := \{\tau_i\}_{\mathbb{Z}}$ with the norm: $\|x\|_{BC_{\mathfrak{T}}(\mathbb{R})} := \sup_{t\in\mathbb{R}}\|x(t)\|$, γ_i are n-dimensional column constant vectors: $\gamma_i \in \mathbb{R}^n$; $\cdots < \tau_{-2} < \tau_{-1} < \tau_0 = 0 < \tau_1 < \tau_2 < \cdots$, and $\Delta x|_{t=\tau_i} := x(\tau_i+) - x(\tau_i-)$.

The solution $x(t)$ of the system (2.1) is sought in the Banach space of n-dimensional bounded on $\mathbb{R}$ and piecewise continuously differentiable vector functions with discontinuities of the first kind at $t \in \mathfrak{T}:\ x \in BC^1_{\mathfrak{T}}(\mathbb{R})$.

Parallel with the nonhomogeneous impulsive system (2.1), we consider the corresponding homogeneous system

$$\dot{x} = A(t)x, \qquad \Delta x|_{t=\tau_i} = 0, \tag{2.2}$$

which is the homogeneous system without impulses, and let $X(t)$ be the fundamental matrix of (2.2) such that $X(0) = I$.

Assume that the homogeneous system (2.2) is exponentially dichotomous (e-dichotomous) [5, 10] on semiaxes $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, \infty)$, that is, there exist projectors P and Q $(P^2 = P,\ Q^2 = Q)$ and constants $K_i \geq 1$, $\alpha_i > 0$ $(i = 1, 2)$ such that the following inequalities are satisfied:

$$\begin{aligned} \left\|X(t)PX^{-1}(s)\right\| &\leq K_1 e^{-\alpha_1(t-s)}, \quad t \geq s, \\ \left\|X(t)(I-P)X^{-1}(s)\right\| &\leq K_1 e^{-\alpha_1(s-t)}, \quad s \geq t,\ t, s \in \mathbb{R}_+, \\ \left\|X(t)QX^{-1}(s)\right\| &\leq K_2 e^{-\alpha_2(t-s)}, \quad t \geq s, \\ \left\|X(t)(I-Q)X^{-1}(s)\right\| &\leq K_2 e^{-\alpha_2(s-t)}, \quad s \geq t,\ t, s \in \mathbb{R}_-. \end{aligned} \tag{2.3}$$

For getting the solution $x \in BC^1_{\mathcal{T}}(\mathbb{R})$ bounded on the entire axis, we assume that $t = 0 \notin \mathcal{T}$, that is, $x(0+) - x(0-) = \gamma_0 = 0$.

We use the following notation: $D = P - (I - Q)$; D^+ is a Moore-Penrose pseudoinverse matrix to D; P_D and P_{D^*} are $n \times n$ matrices (orthoprojectors) projecting $\mathbb{R}^n$ onto $N(D) = \ker D$ and onto $N(D^*) = \ker D^*$, respectively, that is, $P_D : \mathbb{R}^n \to N(D)$, $P_D^2 = P_D = P_D^*$, and $P_{D^*} : \mathbb{R}^n \to N(D^*)$, $P_{D^*}^2 = P_{D^*} = P_{D^*}^*$; $H(t) = [P_{D^*}Q]X^{-1}(t)$; $d = \operatorname{rank}[P_{D^*}Q] = \operatorname{rank}[P_{D^*}(I - P)]$ and $r = \operatorname{rank}[PP_D] = \operatorname{rank}[(I - Q)P_D]$.

The existence conditions and the structure of solutions of system (2.1) bounded on the entire real axis was analyzed in [13]. Here the following theorem was formulated and proved.

Theorem 2.1. *Assume that the linear nonhomogeneous impulsive differential system* (2.1) *has the corresponding homogeneous system* (2.2) *e-dichotomous on the semiaxes* $\mathbb{R}_- = (-\infty, 0]$ *and* $\mathbb{R}_+ = [0, \infty)$ *with projectors* P *and* Q*, respectively. Then the homogeneous system* (2.2) *has exactly* r *linearly independent solutions bounded on the entire real axis. If nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *satisfy* d *linearly independent conditions*

$$\int_{-\infty}^{\infty} H_d(t) f(t) dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)\gamma_i = 0, \tag{2.4}$$

then the nonhomogeneous system (2.1) *possesses an* r*-parameter family of linearly independent solutions bounded on* $\mathbb{R}$ *in the form*

$$x(t, c_r) = X_r(t)c_r + \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t), \quad \forall c_r \in \mathbb{R}^r. \tag{2.5}$$

Here, $H_d(t) = [P_{D^*}Q]_d X^{-1}(t)$ is a $d \times n$ matrix formed by a complete system of d linearly independent rows of matrix $H(t)$,

$$X_r(t) := X(t)[PP_D]_r = X(t)[(I - Q)P_D]_r \tag{2.6}$$

is an $n \times r$ matrix formed by a complete system of r linearly independent solutions bounded on $\mathbb{R}$ of homogeneous system (2.2), and $\left(G\left[\begin{smallmatrix} f \\ \gamma_i \end{smallmatrix}\right]\right)(t)$ is the generalized Green operator of the problem of finding bounded solutions of the nonhomogeneous impulsive system (2.1), acting upon $f \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$, defined by the formula

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t) = X(t)\begin{cases} \displaystyle\int_0^t PX^{-1}(s)f(s)ds - \int_t^{\infty}(I-P)X^{-1}(s)f(s)ds \\ \displaystyle\quad + \sum_{i=1}^{j} PX^{-1}(\tau_i)\gamma i - \sum_{i=j+1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \\ \displaystyle\quad + PD^{+}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \right. \\ \displaystyle\qquad\quad \left. + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \right\}, & t \geq 0; \\ \displaystyle\int_{-\infty}^{t} QX^{-1}(s)f(s)ds - \int_t^{0}(I-Q)X^{-1}(s)f(s)ds \\ \displaystyle\quad + \sum_{i=-\infty}^{-(j+1)} QX^{-1}(\tau_i)\gamma_i - \sum_{i=-j}^{-1}(I-Q)X^{-1}(\tau_i)\gamma_i \\ \displaystyle\quad + (I-Q)D^{+}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \right. \\ \displaystyle\qquad\quad \left. + \sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \right\}, & t \leq 0, \end{cases} \tag{2.7}$$

with the following property

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0-) - \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0+) = \int_{-\infty}^{\infty} H(t)f(t)dt + \sum_{i=-\infty}^{\infty} H(\tau_i)\gamma_i. \tag{2.8}$$

These results are required to establish new conditions for the existence of solutions of weakly perturbed linear impulsive systems bounded on the entire real axis.

3. Perturbed Problems

Consider a weakly perturbed nonhomogeneous linear impulsive system in the form

$$\begin{aligned} \dot{x} &= A(t)x + \varepsilon A_1(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathcal{T}, \\ \Delta x|_{t=\tau_i} &= \gamma_i + \varepsilon A_{1i}x(\tau_i -), \quad \tau_i \in \mathcal{T},\ \gamma_i \in \mathbb{R}^n,\ i \in \mathbb{Z}, \end{aligned} \tag{3.1}$$

where $A_1 \in BC_{\mathcal{T}}(\mathbb{R})$ is an $n \times n$ matrix of functions, A_{1i} are $n \times n$ constant matrices.

Assume that the condition of solvability (2.4) of the generating system (2.1) (obtained from system (3.1) for $\varepsilon = 0$) is not satisfied for all nonhomogeneities $f \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$, that is, system (2.1) does not have solutions bounded on the entire real axis. Therefore, we analyze whether the system (2.1) can be made solvable by introducing linear perturbations to the differential system and to the pulsed conditions. Also it is important to determine perturbations $A_1(t)$ and A_{1i} required to make the problem (3.1) solvable in the space of functions bounded on the entire real axis, that is, it is necessary to specify pertubations for which the corresponding homogeneous system

$$\begin{gathered}\dot{x} = A(t)x + \varepsilon A_1(t)x, \quad t \in \mathbb{R} \setminus \mathcal{T}, \\ \Delta x|_{t=\tau_i} = \varepsilon A_{1i}x(\tau_i-), \quad \tau_i \in \mathcal{T}, \; i \in \mathbb{Z},\end{gathered} \tag{3.2}$$

turns into a system e-trichotomous or e-dichotomous on the entire real axis [10].

We show that this problem can be solved using the $d \times r$ matrix

$$B_0 = \int_{-\infty}^{\infty} H_d(t)A_1(t)X_r(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}X_r(\tau_i-), \tag{3.3}$$

constructed with the coefficients of the system (3.1). The Vishik-Lyusternik method developed in [14] enables us to establish conditions under which a solution of impulsive system (3.1) can be represented by a function bounded on the entire real axis in the form of a Laurent series in powers of the small fixed parameter ε with finitely many terms with negative powers of ε.

We use the following notation: B_0^+ is the unique matrix pseudoinverse to B_0 in the Moore-Penrose sense, P_{B_0} is the $r \times r$ matrix (orthoprojector) projecting the space R^r to the null space $N(B_0)$ of the $d \times r$ matrix B_0, that is, $P_{B_0}: R^r \to N(B_0)$, and $P_{B_0^*}$ is the $d \times d$ matrix (orthoprojector) projecting the space $\mathbb{R}^d$ to the null space $N(B_0^*)$ of the $r \times d$ matrix B_0^* ($B_0^* = B^T$), that is, $P_{B_0^*} : \mathbb{R}^d \to N(B_0^*)$.

Now we formulate and prove a theorem that enables us to solve indicated problem.

Theorem 3.1. *Suppose that the system* (3.1) *satisfies the conditions imposed above, and the homogeneous system* (2.2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively. Let nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *be given such that the condition* (2.4) *is not satisfied and the generating system* (2.1) *does not have solutions bounded on the entire real axis. If*

$$P_{B_0^*} = 0, \tag{3.4}$$

then the system (3.2) *is e-trichotomous on* $\mathbb{R}$ *and, for all nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$*, the system* (3.1) *possesses at least one solution bounded on* $\mathbb{R}$ *in the form of a series*

$$x(t,\varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k x_k(t), \tag{3.5}$$

uniformly convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon_*]$.

Here, ε_* is a proper constant characterizing the range of convergence of the series (3.5) and the coefficients $x_k(t)$ of the series (3.5) are determined from the corresponding impulsive systems as

$$\begin{aligned}
x_k(t) = x_k(t,c_k) &= X_r(t)c_k + \left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot,c_{k-1}) \\ A_{1i}x(\tau_i-,c_{k-1}) \end{bmatrix}\right)(t) \quad \text{for } k = 1,2,\ldots, \\
c_k &= -B_0^+\Bigg[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot,c_{k-1}) \\ A_{1i}x_{k-1}(\tau_i-,c_{k-1}) \end{bmatrix}\right)(t)dt \\
&\quad + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot,c_{k-1}) \\ A_{1i}x_{k-1}(\cdot,c_{k-1}) \end{bmatrix}\right)(\tau_i-)\Bigg], \\
x_{-1}(t) = x_{-1}(t,c_{-1}) &= X_r(t)c_{-1}, \quad c_{-1} = B_0^+\left\{\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i\right\}, \\
x_0(t) = x_0(t,c_0) &= X_r(t)c_0 + \left(G\begin{bmatrix} A_1(\cdot)X_r(t)c_{-1} + f(\cdot) \\ \gamma_i + A_{1i}X_r(\tau_i-)c_{-1} \end{bmatrix}\right)(t), \\
c_0 &= -B_0^+\Bigg[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\tau_i-,c_{-1}) + \gamma_i \end{bmatrix}\right)(t)dt \\
&\quad + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\cdot,c_{-1}) + \gamma_i \end{bmatrix}\right)(\tau_i-)\Bigg].
\end{aligned} \tag{3.6}$$

Proof. We suppose that the problem (3.1) has a solution in the form of a Laurent series (3.5). We substitute this solution into the system (3.1) and equate the coefficients at the same powers of ε. The problem of determination of the coefficient $x_{-1}(t)$ of the term with ε^{-1} in series (3.5) is reduced to the problem of finding solutions of homogeneous system without impulses

$$\begin{aligned}
\dot{x}_{-1} &= A(t)x_{-1}, \quad t \notin \mathcal{T}, \\
\Delta x_{-1}|_{t=\tau_i} &= 0, \quad i \in \mathbb{Z},
\end{aligned} \tag{3.7}$$

bounded on the entire real axis. According to the Theorem 2.1, the homogeneous system (3.7) possesses r-parameter family of solutions

$$x_{-1}(t,c_{-1}) = X_r(t)c_{-1} \tag{3.8}$$

bounded on the entire real axis, where c_{-1} is an r-dimensional vector column $c_{-1} \in \mathbb{R}^r$ and is determined from the condition of solvability of the problem used for determining the coefficient x_0 of the series (3.5).

For ε^0, the problem of determination of the coefficient $x_0(t)$ of series (3.5) reduces to the problem of finding solutions of the following nonhomogeneous system:

$$\begin{aligned} \dot{x}_0 &= A(t)x_0 + A_1(t)x_{-1} + f(t), \quad t \notin \mathcal{T}, \\ \Delta x_0|_{t=\tau_i} &= A_{1i}x_{-1}(\tau_i-) + \gamma_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{3.9}$$

bounded on the entire real axis. According to the Theorem 2.1, the condition of solvability of this problem takes the form

$$\int_{-\infty}^{\infty} H_d(t)\left[A_1(t)X_r(t)c_{-1} + f(t)\right]dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)\left[A_{1i}X_r(\tau_i-)c_{-1} + \gamma_i\right] = 0. \tag{3.10}$$

Using the matrix B_0, we get the following algebraic system for $c_{-1} \in \mathbb{R}^r$:

$$B_0 c_{-1} = -\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i, \tag{3.11}$$

which is solvable if and only if the condition

$$P_{B_0^*}\left\{\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i\right\} = 0 \tag{3.12}$$

is satisfied, that is, if

$$P_{B_0^*} = 0. \tag{3.13}$$

In this case, this algebraic system is solvable with respect to $c_{-1} \in \mathbb{R}^r$ within an arbitrary vector constant $P_{B_0}c(\forall c \in \mathbb{R}^r)$ from the null space of the matrix B_0, and one of its solutions has the form

$$c_{-1} = B_0^+\left\{\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i\right\}. \tag{3.14}$$

Therefore, under condition (3.4), the nonhomogeneous system (3.9) possesses an r-parameter set of solution bounded on $\mathbb{R}$ in the form

$$x_0(t, c_0) = X_r(t)c_0 + \left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot, c_{-1}) + f(\cdot) \\ \gamma_i + A_{1i}x_{-1}(\tau_i-, c_{-1}) \end{bmatrix}\right)(t), \tag{3.15}$$

where $(G[{*\atop *}])(t)$ is the generalized Green operator (2.7) of the problem of finding bounded solutions of system (3.9), and c_0 is an r-dimensional constant vector determined in the next step of the process from the condition of solvability of the impulsive problem for coefficient $x_1(t)$.

We continue this process by problem of determination of the coefficient $x_1(t)$ of the term with ε^1 in the series (3.5). It reduces to the problem of finding solutions of the system

$$\begin{aligned} \dot{x}_1 &= A(t)x_1 + A_1(t)x_0, \quad t \notin \mathcal{T}, \\ \Delta x_1|_{t=\tau_i} &= A_{1i}x_0(\tau_i-), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.16}$$

bounded on the entire real axis. If the condition (3.4) is satisfied and by using the condition of solvability of this problem, that is,

$$\begin{aligned} &\int_{-\infty}^{\infty} H_d(t)A_1(t)\left[X_r(t)c_0 + \left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\tau_i-,c_{-1}) + \gamma_i \end{bmatrix}\right)(t)\right]dt \\ &\quad + \sum_{i=-\infty}^{\infty} H_d(\tau_{i-})A_{1i}\left[X_r(\tau_i-)c_0 + \left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\cdot,c_{-1}) + \gamma_i \end{bmatrix}\right)(\tau_i-)\right] = 0, \end{aligned} \tag{3.17}$$

we determine the vector $c_0 \in \mathbb{R}^r$ (within an arbitrary vector constant $P_{B_0}c$, $\forall c \in \mathbb{R}^r$) as

$$\begin{aligned} c_0 = -B_0^+ &\left[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\tau_i-,c_{-1}) + \gamma_i \end{bmatrix}\right)(t)dt \right. \\ &\quad \left. + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\cdot,c_{-1}) + \gamma_i \end{bmatrix}\right)(\tau_i-)\right]. \end{aligned} \tag{3.18}$$

Thus, under the condition (3.4), system (3.16) possesses an r-parameter set of solutions bounded on $\mathbb{R}$ in the form

$$x_1(t,c_1) = X_r(t)c_1 + \left(G\begin{bmatrix} A_1(\cdot)x_0(\cdot,c_0) \\ A_{1i}x(\tau_i-,c_0) \end{bmatrix}\right)(t), \tag{3.19}$$

where $(G[{*\atop *}])(t)$ is the generalized Green operator (2.7) of the problem of finding bounded solutions of system (3.16), and c_1 is an r-dimensional constant vector determined in the next stage of the process from the condition of solvability of the problem for $x_2(t)$.

If we continue this process, we prove (by induction) that the problem of determination of the coefficient $x_k(t)$ in the series (3.5) is reduced to the problem of finding solutions of the system

$$\begin{aligned} \dot{x}_k &= A(t)x_k + A_1(t)x_{k-1}, \quad t \notin \mathcal{T}, \\ \Delta x_k|_{t=\tau_i} &= A_{1i}x_{k-1}(\tau_i-), \quad i \in \mathbb{Z}, \; k = 1,2,\ldots, \end{aligned} \tag{3.20}$$

bounded on the entire real axis. If the condition (3.4) is satisfied, then a solution of this problem bounded on $\mathbb{R}$ has the form

$$x_k(t) = x_k(t, c_k) = X_r(t)c_k + \left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot, c_{k-1}) \\ A_{1k}x_{k-1}(\tau_i -, c_{k-1}) \end{bmatrix} \right)(t), \tag{3.21}$$

where $(G[{}^*_*])(t)$ is the generalized Green operator of the problem of finding bounded solutions of impulsive system (3.20) and the constant vector $c_k \in R^r$ is given by the formula

$$\begin{aligned} c_k = -B_0^+ \Bigg[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot, c_{k-1}) \\ A_{1i}x_{k-1}(\tau_i -, c_{k-1}) \end{bmatrix} \right)(t)dt \\ + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot, c_{k-1}) \\ A_{1i}x_{k-1}(\cdot, c_{k-1}) \end{bmatrix} \right)(\tau_i -) \Bigg] \end{aligned} \tag{3.22}$$

(within an arbitrary vector constant $P_{B_0}c$, $c \in R^r$).

The fact that the series (3.5) is convergent can be proved by using the procedure of majorization. □

In the case where the number $r = \operatorname{rank} PP_D = \operatorname{rank}(I - Q)P_D$ of linear independent solutions of system (2.2) bounded on $\mathbb{R}$ is equal to the number $d = \operatorname{rank}[P_{D^*}Q] = \operatorname{rank}[P_{D^*}(I - P)]$, Theorem 3.1 yields the following assertion.

Corollary 3.2. *Suppose that the system* (3.1) *satisfies the conditions imposed above, and the homogeneous system* (2.2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively. Let nonhomogeneities* $f \in BC_{\tau}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *be given such that the condition* (2.4) *is not satisfied, and the generating system* (2.1) *does not have solutions bounded on the entire real axis. If condition*

$$\det B_0 \neq 0 \quad (r = d), \tag{3.23}$$

is satisfied, then the system (3.1) *possesses a unique solution bounded on* $\mathbb{R}$ *in the form of series* (3.5) *uniformly convergent for sufficiently small fixed* $\varepsilon \in (0, \varepsilon_*]$.

Proof. If $r = d$, then B_0 is a square matrix. Therefore, it follows from condition (3.4) that $P_{B_0} = P_{B_0^*} = 0$, which is equivalent to the condition (3.23). In this case, the constant vectors $c_k \in \mathbb{R}^r$ are uniquely determined from (3.22). The coefficients of the series (3.5) are also uniquely determined by (3.21), and, for all $f \in BC_{\tau}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$, the system (3.1) possesses a unique solution bounded on $\mathbb{R}$, which means that system (3.2) is e-dichotomous. □

We now illustrate the assertions proved above.

Example 3.3. Consider the impulsive system

$$\dot{x} = A(t)x + \varepsilon A_1(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathcal{T},$$
$$\Delta x|_{t=\tau_i} = \gamma_i + \varepsilon A_{1i}x(\tau_i -), \quad \gamma_i = \begin{Bmatrix} \gamma_i^{(1)} \\ \gamma_i^{(2)} \\ \gamma_i^{(3)} \end{Bmatrix} \in \mathbb{R}^3, \; i \in \mathbb{Z}, \tag{3.24}$$

where

$$A(t) = \operatorname{diag}\{-\tanh t, \; -\tanh t, \tanh t\},$$
$$f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC_{\mathcal{T}}(\mathbb{R}), \tag{3.25}$$
$$A_1(t) = \{a_{ij}(t)\}_{i,j=1}^3 \in BC_{\mathcal{T}}(\mathbb{R}), \qquad A_{1i} = \{\tilde{a}_{ij}\}_{i,j=1}^3.$$

The generating homogenous system (for $\varepsilon = 0$) has the form

$$\dot{x} = A(t)x, \qquad \Delta x|_{t=\tau_i} = 0 \tag{3.26}$$

and is e-dichotomous (as shown in [6]) on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors $P = \operatorname{diag}\{1,1,0\}$ and $Q = \operatorname{diag}\{0,0,1\}$. The normal fundamental matrix of this system is

$$X(t) = \operatorname{diag}\left\{\frac{2}{e^t + e^{-t}}, \frac{2}{e^t + e^{-t}}, \frac{e^t + e^{-t}}{2}\right\}. \tag{3.27}$$

Thus, we have

$$D = 0, \qquad D^+ = 0, \qquad P_D = P_{D^*} = I_3,$$
$$r = \operatorname{rank} PP_D = 2, \qquad d = \operatorname{rank} P_{D^*}Q = 1,$$
$$X_r(t) = \begin{pmatrix} \dfrac{2}{e^t + e^{-t}} & 0 \\ 0 & \dfrac{2}{e^t + e^{-t}} \\ 0 & 0 \end{pmatrix}, \tag{3.28}$$
$$H_d(t) = \left(0, 0, \frac{2}{e^t + e^{-t}}\right). \tag{3.29}$$

In order that the generating impulsive system (2.1) with the matrix $A(t)$ specified above has solutions bounded on the entire real axis, the nonhomogeneities $f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i = \operatorname{col}\{\gamma_i^{(1)}, \gamma_i^{(2)}, \gamma_i^{(3)}\} \in \mathbb{R}^3$ must satisfy condition (2.4). In this analyzed impulsive problem, this condition takes the form

$$\int_{-\infty}^{\infty} \frac{2 f_3(t)}{e^t + e^{-t}} dt + \sum_{i=-\infty}^{\infty} \frac{2}{e^{\tau_i} + e^{-\tau_i}} \gamma_i^{(3)} = 0, \quad \forall f_1(t), f_2(t) \in BC_{\mathcal{T}}(\mathbb{R}), \; \forall \gamma_i^{(1)}, \gamma_i^{(2)} \in \mathbb{R}. \tag{3.30}$$

Let f_3 and $\gamma_i^{(3)}$ be given such that the condition (3.30) is not satisfied and the corresponding generating system (2.1) does not have solutions bounded on the entire real axis. The system (3.24) will be an e-trichotomous on $\mathbb{R}$ if the coefficients $a_{31}(t), a_{32}(t) \in BC_{\mathcal{T}}(\mathbb{R})$ of the perturbing matrix $A_1(t)$ and the coefficients $\tilde{a}_{31}, \tilde{a}_{32} \in \mathbb{R}$ of the perturbing matrix A_{1i} satisfy condition (3.4), that is, $P_{B_0^*} = 0$, where the matrix B_0 has the form

$$B_0 = \int_{-\infty}^{\infty} \left[\frac{a_{31}(t)}{(e^t + e^{-t})^2}, \frac{a_{32}(t)}{(e^t + e^{-t})^2} \right] dt + \sum_{i=-\infty}^{\infty} \left[\frac{\tilde{a}_{31}}{(e^{\tau_i-} + e^{-\tau_i-})^2}, \frac{\tilde{a}_{32}}{(e^{\tau_i-} + e^{-\tau_i-})^2} \right]. \tag{3.31}$$

Therefore, if $a_{31}(t), a_{32}(t) \in BC_{\mathcal{T}}(\mathbb{R})$ and $\tilde{a}_{31}, \tilde{a}_{32} \in \mathbb{R}$ are such that at least one of the following inequalities

$$\begin{gathered} \int_{-\infty}^{\infty} \frac{a_{31}(t)}{(e^t + e^{-t})^2} dt + \sum_{i=-\infty}^{\infty} \frac{\tilde{a}_{31}}{(e^{\tau_i-} + e^{-\tau_i-})^2} \neq 0, \\ \int_{-\infty}^{\infty} \frac{a_{32}(t)}{(e^t + e^{-t})^2} dt + \sum_{i=-\infty}^{\infty} \frac{\tilde{a}_{32}}{(e^{\tau_i-} + e^{-\tau_i-})^2} \neq 0 \end{gathered} \tag{3.32}$$

is satisfied, then either the condition (3.4) or the equivalent condition rank $B_0 = d = 1$ from Theorem 3.1 is satisfied and the system (3.2) is e-trichotomous on $\mathbb{R}$. In this case, the coefficients $a_{11}(t), a_{12}(t), a_{13}(t), a_{21}(t), a_{22}(t), a_{23}(t), a_{33}(t)$ are arbitrary functions from the space $BC_{\mathcal{T}}(\mathbb{R})$, and $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{13}, \tilde{a}_{21}, \tilde{a}_{22}, \tilde{a}_{23}, \tilde{a}_{33}$ are arbitrary constants from $\mathbb{R}$. Moreover, for any

$$f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC_{\mathcal{T}}(\mathbb{R}) \tag{3.33}$$

a solution of the system (3.24) bounded on $\mathbb{R}$ is given by the series (3.5) (within a constant from the null space $N(B_0)$, $\dim N(B_0) = r - \operatorname{rank} B_0 = 1$).

Another Perturbed Problem

In this part, we show that the problem of finding bounded solutions of nonhomogeneous system (2.1), in the case if the condition (2.4) is not satisfied, can be made solvable by introducing linear perturbations only to the pulsed conditions.

Therefore, we consider the weakly perturbed nonhomogeneous linear impulsive system in the form

$$\begin{gathered} \dot{x} = A(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathcal{T},\ A, f \in BC_{\mathcal{T}}(\mathbb{R}), \\ \Delta x|_{t=\tau_i} = \gamma_i + \varepsilon A_{1i} x(\tau_i-), \quad \gamma_i \in \mathbb{R}^n,\ i \in \mathbb{Z}, \end{gathered} \tag{3.34}$$

where A_{1i} are $n \times n$ constant matrices. For $\varepsilon = 0$, we obtain the generating system (2.1). We assume that this generating system does not have solutions bounded on the entire real axis, which means that the condition of solvability (2.4) is not satisfied (for some nonhomogeneities $f \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$). Let us show that it is possible to make this problem solvable by adding linear perturbation only to the pulsed conditions. In the case, if this is possible, it is necessary to determine perturbations A_{1i} for which the corresponding homogeneous system

$$\dot{x} = A(t)x, \quad t \in \mathbb{R} \setminus \mathcal{T}, \qquad \Delta x|_{t=\tau_i} = \varepsilon A_{1i} x(\tau_i -), \quad i \in \mathbb{Z}, \tag{3.35}$$

turns into the system e-trichotomous or e-dichotomous on the entire real axis.

This problem can be solved with help of the $d \times r$ matrix

$$B_0 = \sum_{i=-\infty}^{\infty} H_d(\tau_i) A_{1i} X_r(\tau_i -) \tag{3.36}$$

constructed with the coefficients from the impulsive system (3.34).

By using Theorem 3.1, we seek a solution in the form of the series (3.5). Thus, we have the following corollary.

Corollary 3.4. *Suppose that the system* (3.34) *satisfies the conditions imposed above and the generating homogeneous system* (2.2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q, *respectively. Let nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *be given such that the condition* (2.4) *is not satisfied, and the generating system* (2.1) *does not have solutions bounded on the entire real axis. If the condition* (3.4) *is satisfied, then the system* (3.35) *is e-trichotomous on* $\mathbb{R}$, *and the system* (3.34) *possesses at least one solution bounded on* $\mathbb{R}$ *in the form of series* (3.5) *uniformly convergent for sufficiently small fixed* $\varepsilon \in (0, \varepsilon_*]$.

Acknowledgments

This research was supported by Grants 1/0771/08 of the Grant Agency of Slovak Republic (VEGA) and project APVV-0700-07 of Slovak Research and Development Agency.

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