Research Article

Normal Criteria of Function Families Concerning Shared Values

Wenjun Yuan,[1] Bing Zhu,[2] and Jianming Lin[3]

[1] School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
[2] College of Computer Engineering Technology, Guangdong Institute of Science and Technology, Zhuhai 519090, China
[3] School of Economic and Management, Guangzhou University of Chinese Medicine, Guangzhou 510006, China

Correspondence should be addressed to Jianming Lin, ljmguanli@21cn.com

We study the normality of families of meromorphic functions concerning shared values. We consider whether a family of meromorphic functions $\mathcal{F}$ is normal in D, if, for every pair of functions f and g in $\mathcal{F}$, $f' - af^{-n}$ and $g' - ag^{-n}$ share the value b, where a and b are two finite complex numbers such that $a \neq 0$, n is a positive integer. Some examples show that the conditions in our results are best possible.

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subseteq \mathbb{C}$, and let a be a finite complex value. We say that f and g share a CM (or IM) in D provided that $f - a$ and $g - a$ have the same zeros counting (or ignoring) multiplicity in D. When $a = \infty$, the zeros of $f - a$ mean the poles of f (see [1]). It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory ([2–4] or [1]).

Bloch's principle [5] states that every condition which reduces a meromorphic function in the plane $\mathbb{C}$ to be a constant forces a family of meromorphic functions in a domain D to be normal. Although the principle is false in general (see [6]), many authors proved normality criterion for families of meromorphic functions corresponding to Liouville-Picard type theorem (see [7] or [4]).

It is also more interesting to find normality criteria from the point of view of shared values. In this area, Schwick [8] first proved an interesting result that a family of

meromorphic functions in a domain is normal if every function shares three distinct finite complex numbers with its first derivative. And later, more results about normality criteria concerning shared values can be found, for instance, in [9–11] and so on. In recent years, this subject has attracted the attention of many researchers worldwide.

We now first introduce a normality criterion related to a Hayman normal conjecture [12].

Theorem 1.1. *Let $\mathcal{F}$ be a family of holomorphic (meromorphic) functions defined in a domain D, $n \in \mathbb{N}, a \neq 0, b \in \mathbb{C}$. If $f'(z) + af^n(z) - b \neq 0$ for each function $f(z) \in \mathcal{F}$ and $n \geq 2$ ($n \geq 3$), then $\mathcal{F}$ is normal in D.*

The results for the holomorphic case are due to Drasin [7] for $n \geq 3$, Pang [13] for $n = 3$, Chen and Fang [14] for $n = 2$, Ye [15] for $n = 2$, and Chen and Gu [16] for the generalized result with a and b replaced by meromorphic functions. The results for the meromorphic case are due to Li [17], Li [18] and Langley [19] for $n \geq 5$, Pang [13] for $n = 4$, Chen and Fang [14] for $n = 3$, and Zalcman [20] for $n = 3$, obtained independently.

When $n = 2$ and $\mathcal{F}$ is meromorphic, Theorem 1.1 is not valid in general. Fang and Yuan [21] gave an example to this, and moreover a result added other conditions below.

Example 1.2. The family of meromorphic functions $\mathcal{F} = \{f_j(z) = jz/(\sqrt{j}z - 1)^2 : j = 1, 2, \ldots, \}$ is not normal in $D = \{z : |z| < 1\}$. This is deduced by $f_j^\#(0) = j \to \infty$, as $j \to \infty$ and Marty's criterion [2], although, for any $f_j(z) \in \mathcal{F}, f_j' + f_j^2 = j(\sqrt{j}z - 1)^{-4} \neq 0$.

Here $f^\#(\xi)$ denotes the spherical derivative

$$f^\#(\xi) = \frac{|f'(\xi)|}{1 + |f(\xi)|^2}. \tag{1.1}$$

Theorem 1.3. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and $a \neq 0, b \in \mathbb{C}$. If $f'(z) + a(f(z))^2 - b \neq 0$ and the poles of $f(z)$ are of multiplicity ≥ 3 for each $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 2008, by the ideas of shared values, Zhang [11] proved the following.

Theorem 1.4. *Let $\mathcal{F}$ be a family of meromorphic (holomorphic) functions in D, n a positive integer, and a, b two finite complex numbers such that $a \neq 0$. If $n \geq 4$ ($n \geq 2$) and, for every pair of functions f and g in $\mathcal{F}$, $f' - af^n$ and $g' - ag^n$ share the value b, then $\mathcal{F}$ is normal in D.*

Example 1.5 (see [11]). The family of meromorphic functions $\mathcal{F} = \{f_j(z) = 1/(\sqrt{j}(z - (1/j))) : j = 1, 2, \ldots, \}$ is not normal in $D = \{z : |z| < 1\}$. Obviously $f_j' - f_j^3 = -z/(\sqrt{j}(z - (1/j))^3)$. So for each pair $m, j, f_j' - f_j^3$ and $f_m' - f_m^3$ share the value 0 in D, but $\mathcal{F}$ is not normal at the point $z = 0$, since $f_j^\#(0) = 2(\sqrt{j})^3/(1 + j) \to \infty$, as $j \to \infty$.

Remark 1.6. Example 1.5 shows that Theorem 1.4 is not valid when $n = 3$, and the condition $n = 4$ is best possible for meromorphic case.

In this paper, we will consider the similar relations and prove the following results.

Theorem 1.7. *Let $\mathcal{F}$ be a family of meromorphic functions in D, n a positive integer, and a, b two finite complex numbers such that $a \neq 0$. If $n \geq 2$ and, for every pair of functions f and g in $\mathcal{F}$, $f' - a f^{-n}$ and $g' - a g^{-n}$ share the value b, then $\mathcal{F}$ is normal in D.*

Example 1.8. The family of holomorphic functions $\mathcal{F} = \{ f_j(z) = \sqrt{j}(z - (1/j)) : j = 1, 2, \ldots, \}$ is not normal in $D = \{ z : |z| < 1 \}$. This is deduced by $f_j^{\#}(0) = j\sqrt{j}/(j+1) \rightarrow \infty$, as $j \rightarrow \infty$ and Marty's criterion [2], although, for any $f_j(z) \in \mathcal{F}$, $f_j' + f_j^{-1} = j\sqrt{j}z/(jz - 1)$.

Remark 1.9. Example 1.8 shows that the condition that added $n \geq 2$ in Theorem 1.7 is best possible. In Theorem 1.7 taking $b = 0$ we get Corollary 1.10 obtained by Zhang [22].

Corollary 1.10. *Let $\mathcal{F}$ be a family of meromorphic functions in D, $n \geq 2$, and let a be a nonzero finite complex number. If, for every pair of functions f and g in $\mathcal{F}$, $f^n f'$ and $g^n g'$ share the value a, then $\mathcal{F}$ is normal in D.*

A natural problem is what conditions are added such that Theorem 1.7 holds when $n = 1$. Next we give an answer.

Theorem 1.11. *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let a and b be two finite complex numbers such that $a \neq 0$. Suppose that all of zeros are multiple for each $f(z) \in \mathcal{F}$. If, for every pair of functions f and g in $\mathcal{F}$, $f' - a f^{-1}$ and $g' - a g^{-1}$ share the value b, then $\mathcal{F}$ is normal in D.*

Remark 1.12. Example 1.8 shows that the condition that all of zeros are multiple for each $f(z) \in \mathcal{F}$ added in Theorem 1.7 is best possible. In Theorem 1.11 taking $b = 0$ we get Corollary 1.13.

Corollary 1.13. *Let $\mathcal{F}$ be a family of meromorphic functions in D, and let a be a nonzero finite complex number. Suppose that all of zeros are multiple for each $f(z) \in \mathcal{F}$. If, for every pair of functions f and g in $\mathcal{F}$, $f f'$ and $g g'$ share the value a, then $\mathcal{F}$ is normal in D.*

From the proof of Theorem 1.7 we know that the following corollary holds.

Corollary 1.14. *Let $\mathcal{F}$ be a family of meromorphic functions in D, n be a positive integer and a, b be two finite complex numbers such that $a \neq 0$. If for each function f in $\mathcal{F}$, $f' - a f^{-n} \neq b$, then $\mathcal{F}$ is normal in D.*

2. Preliminary Lemmas

In order to prove our result, we need the following lemmas. The first one extends a famous result by Zalcman [23] concerning normal families.

Lemma 2.1 (see [24]). *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc satisfying all zeros of functions in $\mathcal{F}$ that have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ that have multiplicity $\geq q$. Let α be a real number satisfying $-q < \alpha < p$. Then $\mathcal{F}$ is not normal at 0 if and only if there exist*

    (a) *a number $0 < r < 1$;*

    (b) *points z_n with $|z_n| < r$;*

(c) *functions $f_n \in \mathscr{F}$;*

(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := \rho^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges spherically uniformly on each compact subset of $\mathbb{C}$ to a nonconstant meromorphic function $g(\zeta)$, whose all zeros have multiplicity $\geq p$ and all poles have multiplicity $\geq q$ and order is at most 2.

Remark 2.2. If $\mathscr{F}$ is a family of holomorphic functions on the unit disc in Lemma 2.1, then $g(\zeta)$ is a nonconstant entire function whose order is at most 1.

The order of g is defined by using Nevanlinna's characteristic function $T(r, g)$:

$$\rho(g) = \lim_{r \to \infty} \sup \frac{\log T(r, g)}{\log r}. \tag{2.1}$$

Lemma 2.3 (see [25] or [26]). *Let $f(z)$ be a meromorphic function and $c \in \mathbb{C} \setminus \{0\}$. If $f(z)$ has neither simple zero nor simple pole, and $f'(z) \neq c$, then $f(z)$ is constant.*

Lemma 2.4 (see [27]). *Let $f(z)$ be a transcendental meromorphic function of finite order in $\mathbb{C}$ and have no simple zero, then $f'(z)$ assumes every nonzero finite value infinitely often.*

3. Proof of the Results

Proof of Theorem 1.7. Suppose that $\mathscr{F}$ is not normal in D. Then there exists at least one point z_0 such that $\mathscr{F}$ is not normal at the point z_0. Without loss of generality we assume that $z_0 = 0$. By Lemma 2.1, there exist points $z_j \to 0$, positive numbers $\rho_j \to 0$, and functions $f_j \in \mathscr{F}$ such that

$$g_j(\xi) = \rho_j^{-1/(n+1)} f_j(z_j + \rho_j \xi) \Longrightarrow g(\xi) \tag{3.1}$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function in $\mathbb{C}$. Moreover, the order of g is ≤ 2.

From (3.1) we know

$$g_j'(\xi) = \rho_j^{n/(n+1)} f_j'(z_j + \rho_j \xi) \Longrightarrow g'(\xi),$$

$$\rho_j^{n/(n+1)} \left(f_j'(z_j + \rho_j \xi) - a f_j^{-n}(z_j + \rho_j \xi) - b \right) = g_j'(\xi) - a g_j^{-n}(\xi) - \rho_j^{n/(n+1)} b \Longrightarrow g'(\xi) - a g^{-n}(\xi) \tag{3.2}$$

in $\mathbb{C} \setminus \mathbf{S}$ locally uniformly with respect to the spherical metric, where $\mathbf{S}$ is the set of all poles of $g(\xi)$.

If $g'g^n - a \equiv 0$, then $-1/(n+1)g^{n+1} \equiv a\xi + c$, where c is a constant. This contradicts with g being a meromorphic function. So $g'g^n - a \not\equiv 0$.

If $g'g^n - a \neq 0$, by Lemma 2.3, then g is also a constant which is a contradiction with g being a nonconstant. Hence, $g'g^n - a$ is a nonconstant meromorphic function and has at least one zero.

Next we prove that $g'g^n - a$ has just a unique zero. On the contrary, let ξ_0 and ξ_0^* be two distinct zeros of $g'g^n - a$, and choose $\delta(> 0)$ small enough such that $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$, where $D(\xi_0, \delta) = \{\xi : |\xi - \xi_0| < \delta\}$ and $D(\xi_0^*, \delta) = \{\xi : |\xi - \xi_0^*| < \delta\}$. From (3.2), by *Hurwitz's* theorem, there exist points $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$ such that for sufficiently large j

$$
f_j'\left(z_j + \rho_j \xi_j\right) - a f_j^{-n}\left(z_j + \rho_j \xi_j\right) - b = 0,
$$
$$
f_j'\left(z_j + \rho_j \xi_j^*\right) - a f_j^{-n}\left(z_j + \rho_j \xi_j^*\right) - b = 0.
$$
(3.3)

By the hypothesis that, for each pair of functions f and g in $\mathcal{F}$, $f' - af^{-n}$ and $g' - ag^{-n}$ share b in D, we know that for any positive integer m

$$
f_m'\left(z_j + \rho_j \xi_j\right) - a f_m^{-n}\left(z_j + \rho_j \xi_j\right) - b = 0,
$$
$$
f_m'\left(z_j + \rho_j \xi_j^*\right) - a f_m^{-n}\left(z_j + \rho_j \xi_j^*\right) - b = 0.
$$
(3.4)

Fix m, take $j \to \infty$, and note $z_j + \rho_j \xi_j \to 0$, $z_j + \rho_j \xi_j^* \to 0$, then $f_m'(0) - a f_m^{-n}(0) - b = 0$. Since the zeros of $f_m' - af_m^{-n} - b$ have no accumulation point, so

$$
z_j + \rho_j \xi_j = 0, \qquad z_j + \rho_j \xi_j^* = 0.
$$
(3.5)

Hence, $\xi_j = -z_j/\rho_j$, $\xi_j^* = -z_j/\rho_j$. This contradicts with $\xi_j \in D(\xi_0, \delta)$, $\xi_j^* \in D(\xi_0^*, \delta)$, and $D(\xi_0, \delta) \cap D(\xi_0^*, \delta) = \phi$. So $g'g^n - a$ has just a unique zero, which can be denoted by ξ_0. By Lemma 2.4, g is not any transcendental function.

If g is a nonconstant polynomial, then $g'g^n - a = A(\xi - \xi_0)^l$, where A is a nonzero constant, l is a positive integer, because $l \geq n \geq 3$. Set $\phi = (1/(n+1))g^{n+1}$, then $\phi' = A(\xi - \xi_0)^l + a$ and $\phi'' = Al(\xi - \xi_0)^{l-1}$. Note that the zeros of ϕ are of multiplicity ≥ 4. But ϕ'' has only one zero ξ_0, so ϕ has only the same zero ξ_0 too. Hence, $\phi'(\xi_0) = 0$ which contradicts with $\phi'(\xi_0) = a \neq 0$. Therefore, g and ϕ are rational functions which are not polynomials, and $\phi' - a$ has just a unique zero ξ_0.

Next we prove that there exists no rational function such as ϕ. Noting that $\phi = (1/(n+1))g^{n+1}$, we can set

$$
\phi(\xi) = A \frac{(\xi - \xi_1)^{m_1}(\xi - \xi_2)^{m_2} \cdots (\xi - \xi_s)^{m_s}}{(\xi - \eta_1)^{n_1}(\xi - \eta_2)^{n_2} \cdots (\xi - \eta_t)^{n_t}},
$$
(3.6)

where A is a nonzero constant, $s \geq 1$, $t \geq 1$, $m_i \geq n+1 \geq 3$ $(i = 1, 2, \ldots, s)$, $n_j \geq n+1 \geq 3$ $(j = 1, 2, \ldots, t)$. For stating briefly, denote

$$
m = m_1 + m_2 + \cdots + m_s \geq 3s, \qquad N = n_1 + n_2 + \cdots + n_t \geq 3t.
$$
(3.7)

From (3.6),

$$\phi'(\xi) = \frac{A(\xi - \xi_1)^{m_1-1}(\xi - \xi_2)^{m_2-1}\cdots(\xi - \xi_s)^{m_s-1}h(\xi)}{(\xi - \eta_1)^{n_1+1}(\xi - \eta_2)^{n_2+1}\cdots(\xi - \eta_t)^{n_t+1}} = \frac{p_1(\xi)}{q_1(\xi)}, \qquad (3.8)$$

where

$$h(\xi) = (m - N - t)\xi^{s+t-1} + a_{s+t-2}\xi^{s+t-2} + \cdots + a_0,$$
$$p_1(\xi) = A(\xi - \xi_1)^{m_1-1}(\xi - \xi_2)^{m_2-1}\cdots(\xi - \xi_s)^{m_s-1}h(\xi), \qquad (3.9)$$
$$q_1(\xi) = (\xi - \eta_1)^{n_1+1}(\xi - \eta_2)^{n_2+1}\cdots(\xi - \eta_t)^{n_t+1}$$

are polynomials. Since $\phi'(\xi) + a$ has only a unique zero ξ_0, set

$$\phi'(\xi) + a = \frac{B(\xi - \xi_0)^l}{(\xi - \eta_1)^{n_1+1}(\xi - \eta_2)^{n_2+1}\cdots(\xi - \eta_t)^{n_t+1}}, \qquad (3.10)$$

where B is a nonzero constant, so

$$\phi''(\xi) = \frac{(\xi - \xi_0)^{l-1}p_2(\xi)}{(\xi - \eta_1)^{n_1+2}(\xi - \eta_2)^{n_2+2}\cdots(\xi - \eta_t)^{n_t+2}}, \qquad (3.11)$$

where $p_2(\xi) = B(l - N - 2t)\xi^t + b_{t-1}\xi^{t-1} + \cdots + b_0$ is a polynomial. From (3.8) we also have

$$\phi''(\xi) = \frac{(\xi - \xi_1)^{m_1-2}(\xi - \xi_2)^{m_2-2}\cdots(\xi - \xi_s)^{m_s-2}p_3(\xi)}{(\xi - \eta_1)^{n_1+2}(\xi - \eta_2)^{n_2+2}\cdots(\xi - \eta_t)^{n_t+2}}, \qquad (3.12)$$

where $p_3(\xi)$ is also a polynomial.

Let $\deg(p)$ denote the degree of a polynomial $p(\xi)$.

From (3.8) and (3.9),

$$\deg(h) \le s + t - 1, \qquad \deg(p_1) \le m + t - 1, \qquad \deg(q_1) = N + t. \qquad (3.13)$$

Similarly from (3.11), (3.12) and noting (3.13),

$$\deg(p_2) \le t, \qquad (3.14)$$

$$\deg(p_3) \le \deg(p_1) + t - 1 - (m - 2s) \le 2t + 2s - 2. \qquad (3.15)$$

Note that $m_i \ge 3$ $(i = 1, 2, \ldots, s)$, it follows from (3.8) and (3.10) that $\phi'(\xi_i) = 0$ $(i = 1, 2, \ldots, s)$ and $\phi'(\xi_0) = a \ne 0$. Thus, $\xi_0 \ne \xi_i$ $(i = 1, 2, \ldots, s)$, and then $(\xi - \xi_0)^{l-1}$ is a factor of

$p_3(\xi)$. Hence, we get that $l - 1 \le \deg(p_3)$. Combining (3.11) and (3.12) we also have $m - 2s = \deg(p_2) + l - 1 - \deg(p_3) \le \deg(p_2)$. By (3.14) we obtain

$$m - 2s \le \deg(p_2) \le t. \qquad (3.16)$$

Since $m \ge 3s$, we know by (3.16) that

$$s \le t. \qquad (3.17)$$

If $l \ge N + t$, by (3.15), then

$$4t - 1 \le N + t - 1 \le l - 1 \le \deg(p_3) \le 2t + 2s - 2. \qquad (3.18)$$

Noting (3.17), we obtain $1 \le 0$; a contradiction.

If $l < N + t$, from (3.8) and (3.10), then $\deg(p_1) = \deg(q_1)$. Noting that $\deg(h) \le s + t - 1$, $\deg(p_1) \le m + t - 1$, and $\deg(q_1) = N + t$, hence $m \ge N + 1 \ge 3t + 1$. By (3.16), $2t + 1 \le 2s$. From (3.17), we obtain $1 \le 0$; a contradiction.

The proof of **Theorem 1.7** is complete. □

Proof of Theorem 1.11. The proof of this theorem is the same as the proof of **Theorem 1.7**, some different places are stated as follows.

The zeros of g are multiple;

$$l \ge 2n + 1 = 3. \qquad (3.19)$$

The zeros of ϕ are of multiplicity ≥ 4:

$$m_i \ge 2(n + 1) = 4 \quad (i = 1, 2, \ldots, s), \qquad n_j \ge n + 1 = 2 \quad (j = 1, 2, \ldots, t); \qquad (3.20)$$

$$m = m_1 + m_2 + \cdots + m_s \ge 4s, \qquad N = n_1 + n_2 + \cdots + n_t \ge 2t. \qquad (3.7)'$$

Noting $m \ge 4s$, by (3.16) we have

$$2s \le t. \qquad (3.17)'$$

If $l \ge N + t$, by (3.15), then

$$3t - 1 \le N + t - 1 \le l - 1 \le \deg(p_3) \le 2t + 2s - 2. \qquad (3.21)$$

Noting (3.17), we obtain $1 \le 0$; a contradiction.

If $l < N + t$, from (3.8) and (3.10), then $\deg(p_1) = \deg(q_1)$. Noting that $\deg(h) \le s + t - 1$, $\deg(p_1) \le m + t - 1$, and $\deg(q_1) = N + t$, hence $m \ge N + 1 \ge 2t + 1$. By (3.16), $2t + 1 \le 2s + t$. From (3.17)$'$, we obtain $1 \le 0$; a contradiction.

The proof of **Theorem 1.11** is complete. □

Acknowledgment

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