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Review Article

On Random Topological Structures

Ravi P. Agarwal,[1] Yeol Je Cho,[2] and Reza Saadati[3]

[1] *Department of Mathematics, Florida Institute of Technology, FL 32901, USA*
[2] *Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*
[3] *Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

Correspondence should be addressed to
Yeol Je Cho, yjcho@gnu.ac.kr and Reza Saadati, rsaadati@eml.cc

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We present some topics about random spaces. The main purpose of this paper is to study topological structure of random normed spaces and random functional analysis. These subjects are important to the study of nonlinear analysis in random normed spaces.

1. Introduction

The purpose of this paper is to give a comprehensive text to the study of nonlinear random analysis such as the study of fixed point theory, approximation theory and stability of functional equations in random normed spaces. The notion of random normed space goes back to Šherstnev [1] and Hadžić [2–4] who were dulled from Menger [5], and Schweizer and Sklar [4] works. Some authors [6–11] considered some properties of probabilistic normed and metric spaces also fuzzy metric and normed spaces [12–21]. Fixed point theory [3, 22–26], approximation theory [27–31], and stability of functional equations [32–38] are studied at random normed space and its depended space that is, fuzzy normed space.

This paper is introduced as a survey of the latest and new results on the following topics.

(i) Basic theory of triangular norms.

(ii) Topological structure of random normed spaces.

(iii) Random functional analysis.

(iv) Relationship between random normed spaces and fuzzy normed spaces.

2. Triangular Norms

Triangular norms first appeared in the framework of probabilistic metric spaces in the work Menger [5]. It also turns out that this is a crucial operation in several fields. Triangular norms are an indispensable tool for the interpretation of the conjunction in fuzzy logics [39] and, subsequently, for the intersection of fuzzy sets [40]. They are, however, interesting mathematical objects themselves. We refer to some papers and books for further details (see [2, 4, 41–44]).

Definition 2.1. A *triangular norm* (shorter *t-norm*) is a binary operation on the unit interval $[0,1]$, that is, a function $T : [0,1] \times [0,1] \to [0,1]$ such that for all $a,b,c \in [0,1]$ the following four axioms are satisfied:

(T1) $T(a,b) = T(b,a)$ (commutativity);
(T2) $T(a,(T(b,c))) = T(T(a,b),c)$ (associativity);
(T3) $T(a,1) = a$ (boundary condition);
(T4) $T(a,b) \leq T(a,c)$ whenever $b \leq c$ (monotonicity).

The commutativity of (T1), boundary condition (T3), and the the monotonicity (T4) imply that for each t-norm T and for each $x \in [0,1]$ the following boundary conditions are also satisfied:

$$\begin{aligned} T(x,1) &= T(1,x) = x, \\ T(x,0) &= T(0,x) = 0, \end{aligned} \tag{2.1}$$

and therefore all t-norms coincide on the boundary of the unit square $[0,1]^2$.

The monotonicity of a t-norm T in its second component (T3) is, together with the commutativity (T1), equivalent to the (joint) monotonicity in both components, that is, to

$$T(x_1,y_1) \leq T(x_2,y_2) \quad \text{whenever } x_1 \leq x_2,\ y_1 \leq y_2. \tag{2.2}$$

Basic examples are the Łukasiewicz t-norm T_L:

$$T_L(a,b) = \max(a+b-1,0), \quad \forall a,b \in [0,1] \tag{2.3}$$

and the t-norms T_P, T_M, T_D, where

$$\begin{aligned} T_P(a,b) &:= ab, \\ T_M(a,b) &:= \min\{a,b\}, \\ T_D(a,b) &:= \begin{cases} \min(a,b), & \text{if } \max(\text{a},\text{b}) = 1, \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{2.4}$$

If, for two t-norms T_1 and T_2, the inequality $T_1(x,y) \leq T_2(x,y)$ holds for all $(x,y) \in [0,1]^2$, then we say that T_1 is weaker than T_2 or, equivalently, that T_2 is stronger than T_2.

As a result of (2.2) we obtain, for each $(x,y) \in [0,1]^2$,

$$T(x,y) \leq T(x,1) = x,$$
$$T(x,y) \leq T(1,y) = y. \tag{2.5}$$

Since, for all $(x,y) \in (0,1)^2$, trivially $T(x,y) \geq 0 = T_D(x,y)$, we get for an arbitrary t-norm T,

$$T_D \leq T \leq T_M. \tag{2.6}$$

That is, T_D is weaker and T_M is stronger than any other t-norms. Also since $T_L < T_P$, we obtain the following ordering for four basic t-norms

$$T_D < T_L < T_P < T_M. \tag{2.7}$$

Proposition 2.2 (see [2]). (i) *The minimum T_M is the only t-norm satisfying $T(x,x) = x$ for all $x \in (0,1)$.*

(ii) *The weakest t-norm T_D is the only t-norm satisfying $T(x,x) = 0$ for all $x \in (0,1)$.*

Proposition 2.3 (see [2]). *A t-norm T is continuous if and only if it is continuous in its first component, that is, if for each $y \in [0,1]$ the one-place function*

$$T(\cdot,y) : [0,1] \longrightarrow [0,1], \qquad x \longmapsto T(x,y), \tag{2.8}$$

is continuous.

For example, the minimum T_M and Łukasiewicz t-norm T_L are continuous but the t-norm T^{Δ} defined by

$$T^{\Delta}(x,y) := \begin{cases} \dfrac{xy}{2}, & \text{if } \max(x,y) < 1, \\ xy, & \text{otherwise}, \end{cases} \tag{2.9}$$

for $x,y \in [0,1]$, is not continuous.

Definition 2.4. (i) A t-norm T is said to be *strictly monotone* if

$$T(x,y) < T(x,z) \quad \text{whenever } x \in (0,1),\ y < z. \tag{2.10}$$

(ii) A t-norm T is said to be *strict* if it is continuous and strictly monotone.

For example, the t-norm T^{Δ} is strictly monotone but the minimum T_M and Łukasiewicz t-norm T_L are not.

Proposition 2.5 (see [2]). *A t-norm T is strictly monotone if and only if*

$$T(x,y) = T(x,z), \quad x > 0 \Longrightarrow y = z. \tag{2.11}$$

If T is a t-norm, then $x_T^{(n)}$ is defined for every $x \in [0,1]$ and $n \in N \cup \{0\}$ by 1, if $n = 0$ and $T(x_T^{(n-1)}, x)$, if $n \geq 1$.

Definition 2.6. A t-norm T is said to be *Archimedean* if for all $(x,y) \in (0,1)^2$ there is an integer $n \in \mathbb{N}$ such that

$$x_T^{(n)} < y. \tag{2.12}$$

Proposition 2.7 (see [2]). *A t-norm T is Archimedean if and only if for each $x \in (0,1)$ one has*

$$\lim_{n\to\infty} x_T^{(n)} = 0. \tag{2.13}$$

Proposition 2.8 (see [2]). *If t-norm T is Archimedean, then for each $x \in (0,1)$ one has*

$$T(x,x) < x. \tag{2.14}$$

For example, the product T_P, Łukasiewicz t-norm T_L, and weakest t-norm T_D are all Archimedean but the minimum T_M is not an Archimedean t-norm.

A t-norm T is said to be *of Hadžić-type* (we denote by $T \in \mathcal{H}$) if the family $(x_T^{(n)})_{n\in N}$ is equicontinuous at $x = 1$, that is,

$$\forall \varepsilon \in (0,1) \exists\ \delta \in (0,1), \qquad x > 1-\delta \Longrightarrow x_T^{(n)} > 1-\varepsilon, \quad \forall n \geq 1. \tag{2.15}$$

T_M is a trivial example of a t-norm of Hadžić type, but T_P is not of Hadžić type.

Proposition 2.9 (see [2]). *If a continuous t-norm T is Archimedean, then it cannot be a t-norm of Hadžić type.*

Other important triangular norms are [see [45]]:

(i) the *Sugeno-Weber family* $\{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty]}$ is defined by $T_{-1}^{\mathrm{SW}} = T_D$, $T_\infty^{\mathrm{SW}} = T_P$ and

$$T_\lambda^{\mathrm{SW}}(x,y) = \max\left(0, \frac{x+y-1+\lambda xy}{1+\lambda}\right) \tag{2.16}$$

if $\lambda \in (-1,\infty)$;

(ii) the *Domby family* $\{T_\lambda^D\}_{\lambda\in[0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^D(x,y) = \frac{1}{1+\left(((1-x)/x)^\lambda + ((1-y)/y)^\lambda\right)^{1/\lambda}} \tag{2.17}$$

if $\lambda \in (0,\infty)$;

(iii) the *Aczel-Alsina family* $\{T_\lambda^{\mathrm{AA}}\}_{\lambda\in[0,\infty]}$, defined by T_D, if $\lambda = 0$, T_M, if $\lambda = \infty$ and

$$T_\lambda^{\mathrm{AA}}(x,y) = e^{-(|\log x|^\lambda + |\log y|^\lambda)^{1/\lambda}} \tag{2.18}$$

if $\lambda \in (0,\infty)$.

A t-norm T can be extended (by associativity) in a unique way to an n-array operation taking for any $(x_1,\ldots,x_n) \in [0,1]^n$ the value $T(x_1,\ldots,x_n)$ defined by

$$\mathrm{T}_{i=1}^{0} x_i = 1, \qquad \mathrm{T}_{i=1}^{n} x_i = T\left(\mathrm{T}_{i=1}^{n-1} x_i, x_n\right) = T(x_1,\ldots,x_n). \tag{2.19}$$

T can also be extended to a countable operation taking for any sequence $(x_n)_{n\in N}$ in $[0,1]$ the value

$$\mathrm{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathrm{T}_{i=1}^{n} x_i. \tag{2.20}$$

The limit on the right side of (2.20) exists since the sequence $(\mathrm{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

Proposition 2.10 (see [45]). (i) *For $T \geq T_L$, the following implication holds:*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty. \tag{2.21}$$

(ii) *If T is of Hadžić-type, then*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \tag{2.22}$$

for every sequence $(x_n)_{n\in N}$ in $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$.

(iii) *If $T \in \{T_\lambda^{\mathrm{AA}}\}_{\lambda\in(0,\infty)} \cup \{T_\lambda^{D}\}_{\lambda\in(0,\infty)}$, then*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n)^\lambda < \infty. \tag{2.23}$$

(iv) *If $T \in \{T_\lambda^{\mathrm{SW}}\}_{\lambda\in[-1,\infty)}$, then*

$$\lim_{n\to\infty} T_{i=1}^{\infty} x_{n+i} = 1 \Longleftrightarrow \sum_{n=1}^{\infty}(1-x_n) < \infty. \tag{2.24}$$

Definition 2.11. Let T and T' be two continuous t-norms. Then T' *dominates* T, denoted by $T' \gg T$, if for all $x_1, x_2, y_1, y_2 \in [0,1]$,

$$T\left[T'(x_1,x_2), T'(y_1,y_2)\right] \leq T'\left[T(x_1,y_1), T(x_2,y_2)\right]. \tag{2.25}$$

Now, we extended definitions and results on the triangular norm on lattices. Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice, that is, a partially ordered set in which every nonempty subset admits supremum and infimum, and $0_{\mathcal{L}} = \inf L$, $1_{\mathcal{L}} = \sup L$.

Definition 2.12 (see [46]). A *t-norm* on L is a mapping $\mathcal{T} : (L)^2 \to L$ satisfying the following conditions:

(a) $(\forall x \in L)(\mathcal{T}(x, 1_{\mathcal{L}}) = x)$ (boundary condition);
(b) $(\forall (x, y) \in (L)^2)(\mathcal{T}(x, y) = \mathcal{T}(y, x))$ (commutativity);
(c) $(\forall (x, y, z) \in (L)^3)(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$ (associativity);
(d) $(\forall (x, x', y, y') \in (L)^4)(x \leq_L x'$ and $y \leq_L y' \Rightarrow \mathcal{T}(x, y) \leq_L \mathcal{T}(x', y'))$ (monotonicity).

Let $\{x_n\}$ be a sequence in L converges to $x \in L$ (equipped order topology). The t-norm $\mathcal{T}$ is said to be a *continuous t-norm* if

$$\lim_{n\to\infty} \mathcal{T}(x_n, y) = \mathcal{T}(x, y), \tag{2.26}$$

for each $y \in L$.

Note that we put $\mathcal{T} = T$ whenever $L = [0, 1]$.

Definition 2.13 (see [46]). A continuous t-norm $\mathcal{T}$ on $L = [0, 1]^2$ is said to be *continuous t-representable* if there exist a continuous t-norm $*$ and a continuous t-conorm $\diamond$ on $[0, 1]$ such that, for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in L$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2). \tag{2.27}$$

For example,

$$\begin{aligned} \mathcal{T}(a, b) &= (a_1 b_1, \min\{a_2 + b_2, 1\}), \\ \mathbf{M}(a, b) &= (\min\{a_1, b_1\}, \max\{a_2, b_2\}) \end{aligned} \tag{2.28}$$

for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in [0, 1]^2$ are continuous t–representable.

Define the mapping $\mathcal{T}_\wedge$ from L^2 to L by:

$$\mathcal{T}_\wedge(x, y) = \begin{cases} x, & \text{if } y \geq_L x, \\ y, & \text{if } x \geq_L y. \end{cases} \tag{2.29}$$

A *negation* on $\mathcal{L}$ is any decreasing mapping $\mathcal{N} : L \to L$ satisfying $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L$, then $\mathcal{N}$ is called an *involutive negation*. In the following $\mathcal{L}$ is endowed with a (fixed) negation $\mathcal{N}$.

3. Distribution Functions and Fuzzy Sets

We denote, Δ^+, the space of all distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$, such that F is left-continuous and nondecreasing on $\mathbb{R}$,

$F(0) = 0$ and $F(+\infty) = 1$. D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0, \end{cases} \tag{3.1}$$

Example 3.1. The function $G(t)$, defined by

$$G(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1 - e^{-t}, & \text{if } t > 0, \end{cases} \tag{3.2}$$

is a distribution function. Since $\lim_{t \to \infty} G(t) = 1$, then $G \in D^+$. Note that $G(t+s) \geq T_P(G(t), G(s))$ for each $t, s > 0$.

Example 3.2. The function $F(t)$, defined by

$$F(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ t & \text{if } 0 \leq t \leq 1, \\ 1 & \text{if } 1 \leq t, \end{cases} \tag{3.3}$$

is a distribution function. Since $\lim_{t \to \infty} F(t) = 1$, then $F \in D^+$. Note that $F(t+s) \geq T_M(F(t), F(s))$ for each $t, s > 0$.

Example 3.3 (see [6]). The function $G_p(t)$, $p > 0$, defined by

$$G_p(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \exp\left(-|p|^{1/2}\right) & \text{if } 0 < t < +\infty, \\ 1 & \text{if } t = +\infty, \end{cases} \tag{3.4}$$

is a distribution function. Since $\lim_{t \to \infty} G_p(t) \neq 1$, then $G \in \Delta^+ \setminus D^+$. Note that $G_p(t+s) \geq T_M(G_p(t), G_p(s))$ for each $t, s > 0$.

Definition 3.4. If W is a collection of objects denoted generically by w, then a *fuzzy set* A in W is a set of ordered pairs:

$$A = \{(w, \lambda_A(w)) : w \in W\}. \tag{3.5}$$

$\lambda_A(w)$ is called the *membership function* or grade of membership of w in A which maps W to the membership space M. Note that, when M contains only the two points 0 and 1, A is nonfuzzy and $\lambda_A(w)$ is identical to the characteristic function of a non-fuzzy set. The range of the membership function is $[0,1]$ or a complete lattice.

Example 3.5. Consider the fuzzy set A which is real numbers considerably larger than 10,

$$A = \{(w, \lambda_A(w)) : w \in W\}, \tag{3.6}$$

where

$$\lambda_A(w) = \begin{cases} 0, & \text{if } w < 10, \\ \dfrac{1}{1+(w-10)^{-2}}, & \text{if } w \geq 10. \end{cases} \tag{3.7}$$

Example 3.6. Consider the fuzzy set A which is real numbers close to 10, like (3.6) where

$$\lambda_A(w) = \frac{1}{1+(w-10)^2}. \tag{3.8}$$

Note that, in this paper, in short, we apply membership functions instead of fuzzy sets.

Definition 3.7 (see [47]). Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice and let U be a nonempty set called the universe. An $\mathcal{L}$*- fuzzy set* in U is defined as a mapping $\mathcal{A} : U \to L$. For each u in U, $\mathcal{A}(u)$ represents the *degree* (in L) to which u is an element of $\mathcal{A}$.

Lemma 3.8 (see [46]). *Consider the set L^* and operation $\leq_{L^*}$ defined by*

$$\begin{gathered} L^* = \left\{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \textit{ and } \ x_1 + x_2 \leq 1\right\}, \\ (x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1,\ x_2 \geq y_2 \end{gathered} \tag{3.9}$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 3.9 (see [48]). An *intuitionistic fuzzy set* $\mathcal{A}_{\zeta,\eta}$ in the universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(u, \zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) : u \in U\}$, where $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ for all $u \in U$ are called the *membership degree* and the *nonmembership degree*, respectively, of u in $\mathcal{A}_{\zeta,\eta}$ and, furthermore, satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

Example 3.10. Consider the intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ which is real numbers considerably larger than 10 for first place and real numbers close to 10 in second place,

$$\mathcal{A}_{\zeta,\eta} = \{(w, \zeta_A(w), \eta_A(w)) : w \in W\}, \tag{3.10}$$

where

$$(\zeta_A(w), \eta_A(w)) = \begin{cases} \left(0, \dfrac{1}{1+(w-10)^2}\right), & \text{if } w < 10, \\ \left(\dfrac{1}{1+(w-10)^{-2}}, \dfrac{1}{1+(w-10)^2}\right), & \text{if } w \geq 10. \end{cases} \tag{3.11}$$

As we said above, we apply $\mathcal{A}_{\zeta,\eta}(w) = (\zeta_A(w), \eta_A(w))$.

4. Random Normed Spaces

Random (probabilistic) normed spaces were introduced by Šerstnev in 1962 [1] by means of a definition that was closely modelled on the theory of (classical) normed spaces and used to study the problem of best approximation in statistics. In the sequel, we will adopt usual terminology, notation, and conventions of the theory of random normed spaces, as in [4, 6–8].

Definition 4.1. A *Menger probabilistic metric space* (or *random metric spaces*) is a triple $(X, \mathcal{F}, T)$, where X is a nonempty set, T is a continuous t-norm, and $\mathcal{F}$ is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of $\mathcal{F}$ at the pair (x, y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;

(PM2) $F_{x,y}(t) = F_{y,x}(t)$;

(PM3) $F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in \mathrm{X}$ and $t, s \geq 0$.

Definition 4.2 (see [1]). A *random normed space* (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in \mathrm{X}$, $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(\mathrm{s}))$ for all $x, y \in \mathrm{X}$ and $t, s \geq 0$.

Example 4.3. Let $(\mathrm{X}, \|\cdot\|)$ be a linear normed space. Define

$$\mu_x(t) = \begin{cases} 0, & \text{if } \ t \leq 0, \\ \dfrac{t}{t+\|x\|}, & \text{if } \ t > 0. \end{cases} \tag{4.1}$$

Then (X, μ, T_P) is a random normed space. (RN1) and (RN2) are obvious and we show (RN3).

$$\begin{aligned} T_P(\mu_x(t), \mu_y(s)) &= \frac{t}{t+\|x\|} \cdot \frac{s}{s+\|y\|} \\ &= \frac{1}{1+(\|x\|/t)} \cdot \frac{1}{1+(\|y\|/s)} \\ &\leq \frac{1}{1+(\|x\|/(t+s))} \cdot \frac{1}{1+(\|y\|/(t+s))} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{1+\left(\left(\|x\|+\|y\|\right)/(t+s)\right)}\\
&\leq \frac{1}{1+\left(\left(\|x+y\|\right)/(t+s)\right)}\\
&= \frac{t+s}{t+s+\|x+y\|}\\
&= \mu_{x+y}(t+s)
\end{aligned} \tag{4.2}$$

for all $x, y \in X$ and $t, s \geq 0$. Also, (X, μ, T_M) is a random normed space.

Example 4.4. Let $(X, \|\cdot\|)$ be a linear normed spaces. Define

$$\mu_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ e^{-(\|x\|/t)}, & \text{if } t > 0. \end{cases} \tag{4.3}$$

Then (X, μ, T_P) is a random normed space. (RN1) and (RN2) are obvious and we show (RN3).

$$\begin{aligned}
T_P\left(\mu_x(t), \mu_y(s)\right) &= e^{-(\|x\|/t)} \cdot e^{-(\|y\|/s)}\\
&\leq e^{-(\|x\|/(t+s))} \cdot e^{-(\|y\|/(t+s))}\\
&= e^{-((\|x\|+\|y\|)/(t+s))}\\
&\leq e^{-((\|x+y\|)/(t+s))}\\
&= \mu_{x+y}(t+s)
\end{aligned} \tag{4.4}$$

for all $x, y \in X$ and $t, s \geq 0$. Also, (X, μ, T_M) is a random normed space.

Example 4.5 (see [36]). Let $(X, \|\cdot\|)$ be a linear normed space. Define

$$\mu_x(t) = \begin{cases} \max\left\{1 - \dfrac{\|x\|}{t}, 0\right\}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases} \tag{4.5}$$

Then $(X, \mu,\ T_L)$ is an RN-space (this was essentially proved by Mušthari in [49], see also [50]). Indeed, $\mu_x(t) = 1$ for all $t > 0 \Rightarrow \|x\|/t = 0$ for all $t > 0 \Rightarrow x = 0$ and obviously$\mu_{\lambda x}(t) =$

$\mu_x(t/\lambda)$ for all $x \in X$ and $t > 0$. Next, for every $x, y \in X$ and $t, s > 0$ we have

$$\begin{aligned}
\mu_{x+y}(t+s) &= \max\left\{1 - \frac{\|x+y\|}{t+s}, 0\right\} = \max\left\{1 - \left\|\frac{x+y}{t+s}\right\|, 0\right\} \\
&= \max\left\{1 - \left\|\frac{x}{t+s} + \frac{y}{t+s}\right\|, 0\right\} \geq \max\left\{1 - \left\|\frac{x}{t}\right\| - \left\|\frac{y}{s}\right\|, 0\right\} \\
&= T_L\left(\mu_x(t), \mu_y(s)\right).
\end{aligned} \tag{4.6}$$

Let φ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

(a) $\varphi(-t) = \varphi(t)$ for every $t \in \mathbb{R}$;

(b) $\varphi(1) = 1$;

(c) φ is strictly increasing and continuous on $[0, \infty)$, $\varphi(0) = 0$ and $\lim_{\alpha \to \infty} \varphi(\alpha) = \infty$.

Examples of such functions are $\varphi(t) = |t|$; $\varphi(t) = |t|^p$, $p \in (0, \infty)$; $\varphi(t) = 2t^{2n}/(|t| + 1)$, $n \in \mathbb{N}$.

Definition 4.6 (see [51]). A *random φ-normed space* is a triple (X, ν, T), where X is a real vector space, T is a continuous t–norm, and ν is a mapping from X into D^+ such that the following conditions hold:

(φ-RN1) $\nu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(φ-RN2) $\nu_{\alpha x}(t) = \nu_x(t/\varphi(\alpha))$ for all x in X, $\alpha \neq 0$ and $t > 0$;

(φ-RN3) $\nu_{x+y}(t+s) \geq T(\nu_x(t), \nu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

Example 4.7 (see [37]). An important example is the space (X, ν, T_M), where $(X, \|\cdot\|^p)$ is a p-normed space and

$$\nu_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \dfrac{t}{t + \|x\|^p}, \quad p \in (0, 1], & \text{if } t > 0. \end{cases} \tag{4.7}$$

(φ-RN1) and (φ-RN2) are obvious and we show (φ-RN3). Let $\nu_x(t) \leq \nu_y(s)$, then for each $x, y \in X$ we have

$$\frac{\|y\|^p}{s} \leq \frac{\|x\|^p}{t}. \tag{4.8}$$

Now, if $x = y$, then we have $t \leq s$. Now, since we have

$$\begin{aligned}
\frac{\|x\|^p}{t} + \frac{\|x\|^p}{t} &\geq \frac{\|x\|^p}{t} + \frac{\|y\|^p}{s} \\
&\geq 2\frac{\|x\|^p}{t+s} + 2\frac{\|y\|^p}{t+s} \\
&\geq 2\frac{\|x+y\|^p}{t+s},
\end{aligned} \tag{4.9}$$

then it follows that

$$1 + \frac{\|x\|^p}{t} \geq 1 + \frac{\|x+y\|^p}{t+s}, \tag{4.10}$$

which implies that $\nu_x(t) \leq \nu_{x+y}(t+s)$. Hence $\nu_{x+y}(t+s) \geq T_M(\nu_x(t), \nu_y(s))$ for all $x, y \in X$ and $t, s \geq 0$.

By a *non-Archimedean field* we mean a field $\mathcal{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathcal{K}$ into $[0,\infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r|\,|s|$, and $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. By the trivial valuation we mean the mapping $|\cdot|$ taking everything but 0 into 1 and $|0| = 0$. Let X be a vector space over a field $\mathcal{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$, that is, that there is an $a_0 \in \mathcal{K}$ such that $|a_0|$ is not in $\{0, 1\}$.

The most important examples of non-Archimedean spaces are p-adic numbers. In 1897, Hensel [52] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in \mathbb{Z}$ such that $x = (a/b)p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x,y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, which is called the p-adic number field.

A function $\|\cdot\| : X \to [0,\infty)$ is called a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) for any $r \in \mathcal{K}, x \in X$, $\|rx\| = |r|\|x\|$;

(iii) the strong triangle inequality (ultrametric); namely,

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X. \tag{4.11}$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\}, \quad \forall n, m \in \mathbb{N}\ (n > m), \tag{4.12}$$

a sequence $\{x_n\}$ is a Cauchy sequence if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space. By a *complete non-Archimedean normed space* we mean one in which every Cauchy sequence is convergent.

Definition 4.8. A *non-Archimedean random normed space* (briefly, non-Archimedean RN-space) is a triple (X, μ, T), where X is a linear space over a non-Archimedean field $\mathcal{K}$, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(NA-RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(NA-RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $t > 0$, $\alpha \neq 0$;

(NA-RN3) $\mu_{x+y}(\max\{t,s\}) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y, z \in X$ and $t, s \geq 0$.

It is easy to see that, if (NA-RN3) holds then so will (RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$.

Example 4.9. As a classical example, if $(X, \|\cdot\|)$ is a non-Archimedean normed linear space, then the triple (X, μ, T_M), where

$$\mu_x(t) = \begin{cases} 0 & t \leq \|x\|, \\ 1 & t > \|x\|, \end{cases} \tag{4.13}$$

is a non-Archimedean RN-space.

Example 4.10. Let $(X, \|\cdot\|)$ be a non-Archimedean normed linear space. Define

$$\mu_x(t) = \frac{t}{t + \|x\|}, \quad \forall x \in X, \ t > 0. \tag{4.14}$$

Then (X, μ, T_M) is a non-Archimedean RN-space.

Definition 4.11. Let (X, μ, T) be a non-Archimedean RN-space. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in X$ such that

$$\lim_{n \to \infty} \mu_{x_n - x}(t) = 1 \tag{4.15}$$

for all $t > 0$. In that case, x is called the *limit* of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in X is called a *Cauchy* sequence if for each $\varepsilon > 0$ and $t > 0$ there exists n_0 such that for all $n \geq n_0$ and $p > 0$ we have $\mu_{x_{n+p} - x_n}(t) > 1 - \varepsilon$.

If each Cauchy sequence is convergent, then the random normed space is said to be *complete* and the non-Archimedean RN-space is called a *non-Archimedean random Banach space*.

Remark 4.12 (see [53]). Let (X, μ, T_M) be a non-Archimedean RN-space, then

$$\mu_{x_{n+p} - x_n}(t) \geq \min\left\{\mu_{x_{n+j+1} - x_{n+j}}(t) : j = 0, 1, 2, \ldots, p-1\right\}. \tag{4.16}$$

So, the sequence $\{x_n\}$ is a Cauchy sequence if for each $\varepsilon > 0$ and $t > 0$ there exists n_0 such that for all $n \geq n_0$ we have

$$\mu_{x_{n+1} - x_n}(t) > 1 - \varepsilon. \tag{4.17}$$

5. Topological Structure of Random Normed Spaces

Definition 5.1. Let (X, μ, T) be an RN-space. We define the *open ball* $B_x(r, t)$ and the *closed ball* $B_x[r, t]$ with center $x \in X$ and radius $0 < r < 1$, $t > 0$ as follows:

$$\begin{aligned} B_x(r, t) &= \{y \in X : \mu_{x-y}(t) > 1 - r\}, \\ B_x[r, t] &= \{y \in X : \mu_{x-y}(t) \geq 1 - r\}. \end{aligned} \tag{5.1}$$

Theorem 5.2. *Let (X, μ, T) be an RN-space. Every open ball $B_x(r, t)$ is an open set.*

Proof. Let $B_x(r,t)$be an open ball with center x and radius r with respect t. Let $y \in B_x(r,t)$. Then $\mu_{x-y}(t) > 1-r$. Since $\mu_{x-y}(t) > 1-r$, there exists $t_0 \in (0,t)$ such that $\mu_{x-y}(t_0) > 1-r$. Put $r_0 = \mu_{x,y}(t_0)$. Since $r_0 > 1-r$, there exists $s \in (0,1)$ such that $r_0 > 1-s > 1-r$. Now for given r_0 and s such that $r_0 > 1-s$, there exists $r_1 \in (0,1)$ such that $T(r_0,r_1) > 1-s$. Consider the open ball $B_y(1-r_1,t-t_0)$. We claim $B_y(1-r_1,t-t_0) \subset B_x(r,t)$. Now, let $z \in B_y(1-r_1,t-t_0)$. Then $\mu_{y-z}(t-t_0) > r_1$. Therefore, we have

$$\begin{aligned}\mu_{x-z}(t) &\geq T\left(\mu_{x-y}(t_0),\mu_{y-z}(t-t_0)\right)\\ &\geq T(r_0,r_1)\\ &\geq 1-s\\ &> 1-r.\end{aligned} \tag{5.2}$$

Thus $z \in B_x(r,t)$ and hence $B_y(1-r_1,t-t_0) \subset B_x(r,t)$. □

Different kinds of topologies can be introduced in a random normed space [4]. The (r,t)-*topology* is introduced by a family of neighborhoods

$$\{B_x(r,t)\}_{x\in X,\ t>0,\ r\in(0,1)}. \tag{5.3}$$

In fact, every random norm μ on X generates a topology ((r,t)-topology) on X which has as a base the family of open sets of the form

$$\{B_x(r,t)\}_{x\in X,\ t>0,\ r\in(0,1)}. \tag{5.4}$$

Remark 5.3. Since $\{B_x(1/n,1/n) : n = 1,2,3,\ldots\}$ is a local base at x, the (r,t)-topology is first countable.

Theorem 5.4. *Every RN-space (X,μ,T) is a Hausdorff space.*

Proof. Let (X,μ,T) be an RN-space. Let x and y be two distinct points in X and $t > 0$. Then $0 < \mu_{x-y}(t) < 1$. Put $r = \mu_{x-y}(t)$. For each $r_0 \in (r,1)$, there exists r_1 such that $T(r_1,r_1) \geq r_0$. Consider the open balls $B_x(1-r_1,t/2)$ and $B_y(1-r_1,t/2)$. Then clearly $B_x(1-r_1,t/2) \cap B_y(1-r_1,t/2) = \emptyset$. For, if there exists

$$z \in B_x\left(1-r_1,\frac{t}{2}\right) \cap B_y\left(1-r_1,\frac{t}{2}\right), \tag{5.5}$$

then we have

$$\begin{aligned} r &= \mu_{x-y}(t) \\ &\geq T\left(\mu_{x-z}\left(\frac{t}{2}\right), \mu_{y-z}\left(\frac{t}{2}\right)\right) \\ &\geq T(r_1, r_1) \\ &\geq r_0 \\ &> r, \end{aligned} \tag{5.6}$$

which is a contradiction. Hence (X, μ, T) is a Hausdorff space. □

Definition 5.5. Let (X, μ, T) be an RN-space. A subset A of X is said to be *R-bounded* if there exists $t > 0$ and $r \in (0, 1)$ such that $\mu_{x-y}(t) > 1 - r$ for all $x, y \in A$.

Theorem 5.6. *Every compact subset A of an RN-space (X, μ, T) is R-bounded.*

Proof. Let A be a compact subset of an RN-space (X, μ, T). Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B_x(r, t) : x \in A\}$. Since A is compact, there exist $x_1, x_2, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_{x_i}(r, t). \tag{5.7}$$

Let $x, y \in A$. Then $x \in B_{x_i}(r, t)$ and $y \in B_{x_j}(r, t)$ for some i, j. Thus we have $\mu_{x-x_i}(t) > 1 - r$ and $\mu_{y-x_j}(t) > 1 - r$. Now, let

$$\alpha = \min\left\{\mu_{x_i, x_j}(t) : 1 \leq i, j \leq n\right\}. \tag{5.8}$$

Then $\alpha > 0$. Now, we have

$$\begin{aligned} \mu_{x-y}(3t) &\geq T^2\left(\mu_{x-x_i}(t), \mu_{x_i, x_j}(t), \mu_{y-x_j}(t)\right) \\ &\geq T^2(1 - r, 1 - r, \alpha) \\ &> 1 - s. \end{aligned} \tag{5.9}$$

Taking $t' = 3t$, we have $\mu_{x-y}(t') > 1 - s$ for all $x, y \in A$. Hence A is R-bounded. □

Remark 5.7. In an RN-space (X, μ, T) every compact set is closed and R-bounded.

Theorem 5.8 (see [4]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then* $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ *almost everywhere.*

Theorem 5.9. *Let (X, μ, T) be an RN-space such that every Cauchy sequence in X has a convergent subsequence. Then (X, μ, T) is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence and let $\{x_{i_n}\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $x_n \to x$. Let $t > 0$ and $\epsilon \in (0, 1)$ such that

$$T(1-r, 1-r) \geq 1-\epsilon. \tag{5.10}$$

Since $\{x_n\}$ is a Cauchy sequence, there is $n_0 \in \mathbb{N}$ such that

$$\mu_{x_m - x_n}(t) > 1-r \tag{5.11}$$

for all $m, n \geq n_0$. Since $x_{i_n} \to x$, there is positive integer i_p such that $i_p > n_0$ and

$$\mu_{x_{i_p} - x}\left(\frac{t}{2}\right) > 1-r. \tag{5.12}$$

Then, if $n \geq n_0$, then we have

$$\begin{aligned}
\mu_{x_n - x}(t) &\geq T\left(\mu_{x_n - x_{i_p}}\left(\frac{t}{2}\right), \mu_{x_{i_p} - x}\left(\frac{t}{2}\right)\right) \\
&> T(1-r, 1-r) \\
&\geq 1-\epsilon.
\end{aligned} \tag{5.13}$$

Therefore, $x_n \to x$ and hence (X, μ, T) is complete. □

Lemma 5.10. *Let (X, μ, T) be an RN-space. If one defines*

$$F_{x,y}(t) = \mu_{x-y}(t), \tag{5.14}$$

then F is a random (probabilistic) metric on X, which is called the random (probabilistic) metric induced by the random norm μ.

Lemma 5.11. *A random (probabilistic) metric F which is induced by a random norm on a RN-space (X, μ, T) has the following properties for all $x, y, z \in X$ and every scalar $\alpha \neq 0$:*

(i) $F_{x+z, y+z}(t) = F_{x,y}(t)$,

(ii) $F_{\alpha x, \alpha y}(t) = F_{x,y}(t/|\alpha|)$.

Proof. We have

$$\begin{aligned}
F_{x+z, y+z}(t) &= \mu_{(x+z)-(y+z)}(t) \\
&= \mu_{x-y}(t) = F_{x,y}(t).
\end{aligned} \tag{5.15}$$

Also, we have

$$\begin{aligned} F_{\alpha x,\alpha y}(t) &= \mu_{\alpha x-\alpha y}(t) \\ &= \mu_{x-y}\left(\frac{t}{|\alpha|}\right) \\ &= F_{x,y}\left(\frac{t}{|\alpha|}\right). \end{aligned} \tag{5.16}$$

□

Lemma 5.12. *If* (X, μ, T) *is an RN-space, then*

(i) *the function* $(x, y) \to x + y$ *is continuous,*

(ii) *the function* $(\alpha, x) \to \alpha x$ *is continuous.*

Proof. If $x_n \to x$ and $y_n \to y$, then as $n \to \infty$

$$\mu_{(x_n+y_n)-(x+y)}(t) \geq T\left(\mu_{x_n-x}\left(\frac{t}{2}\right), \mu_{y_n-y}\left(\frac{t}{2}\right)\right) \longrightarrow 1. \tag{5.17}$$

This proves (i).

Now, if $x_n \to x$, $\alpha_n \to \alpha$ and $\alpha_n \neq 0$, then

$$\begin{aligned} \mu_{\alpha_n x_n-\alpha x}(t) &= \mu_{\alpha_n(x_n-x)+x(\alpha_n-\alpha)}(t) \\ &\geq T\left(\mu_{\alpha_n(x_n-x)}\left(\frac{t}{2}\right)\mu_{x(\alpha_n-\alpha)}\left(\frac{t}{2}\right)\right) \\ &= T\left(\mu_{x_n-x}\left(\frac{t}{2\alpha_n}\right), \mu_x\left(\frac{t}{2(\alpha_n-\alpha)}\right)\right) \longrightarrow 1 \end{aligned} \tag{5.18}$$

as $n \to \infty$ and this proves (ii). □

Definition 5.13. The RN-space (X, μ, T) is said to be a *random Banach space* whenever X is complete with respect to the random metric induced by random norm.

Lemma 5.14. *Let* (X, μ, T) *be an RN-space and define*

$$E_{\lambda,\mu} : X \to \mathbb{R}^+ \cup \{0\} \tag{5.19}$$

by

$$E_{\lambda,\mu}(x) = \inf\{t > 0 : \mu_x(t) > 1 - \lambda\} \tag{5.20}$$

for each $\lambda \in (0, 1)$ *and* $x \in X$. *Then we have*

(i) $E_{\lambda,\mu}(\alpha x) = |\alpha| E_{\lambda,\mu}(x)$ *for every* $x \in X$ *and* $\alpha \in \mathbb{R}$;

(ii) *if T satisfies* (2.15)*, then for any $\gamma \in (0,1)$, there is $\lambda \in (0,1)$ such that*

$$E_{\gamma,\mu}(x_1 + \cdots + x_n) \leq E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n) \tag{5.21}$$

for any $x, y \in X$;

(iii) *the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent with respect to a random norm μ if and only if $E_{\lambda,\mu}(x_n - x) \to 0$. Also the sequence $\{x_n\}$ is a Cauchy sequence with respect to a random norm μ if and only if it is a Cauchy sequence with $E_{\lambda,\mu}$.*

Proof. For (i), we find

$$\begin{aligned} E_{\lambda,\mu}(\alpha x) &= \inf\{t > 0 : \mu_{\alpha x}(t) > 1 - \lambda\} \\ &= \inf\left\{t > 0 : \mu_x\left(\frac{t}{|\alpha|}\right) > 1 - \lambda\right\} \\ &= |\alpha| \inf\{t > 0 : \mu_x(t) > 1 - \lambda\} \\ &= |\alpha| E_{\lambda,\mu}(x). \end{aligned} \tag{5.22}$$

For (ii), by (2.15), for every $\alpha \in (0,1)$ we can find $\lambda \in (0,1)$ such that

$$T^{n-1}(1 - \lambda, \ldots, 1 - \lambda) \geq 1 - \alpha. \tag{5.23}$$

Thus we have

$$\begin{aligned} \mu_{x_1+\cdots+x_n}\left(E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n) + n\delta\right) &\geq_L T^{n-1}\left(\mu_{x_1}(E_{\lambda,\mathcal{M}}(x_1) + \delta), \ldots, \mu_{x_n}(E_{\lambda,\rho}(x_n) + \delta)\right) \\ &\geq T(1 - \lambda, \ldots, 1 - \lambda) \\ &\geq 1 - \alpha \end{aligned} \tag{5.24}$$

for every $\delta > 0$, which implies that

$$E_{\alpha,\mu}(x_1 + \cdots + x_n) \leq E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n) + n\delta. \tag{5.25}$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\alpha,\mu}(x_1 + \cdots + x_n) \leq E_{\lambda,\mu}(x_1) + \cdots + E_{\lambda,\mu}(x_n). \tag{5.26}$$

For (iii), note that since μ is continuous, $E_{\lambda,\mu}(x)$ is not an element of the set $\{t > 0 : \mu_x(t) > 1 - \lambda\}$ as soon as $x \neq 0$. Hence we have

$$\mu_{x_n - x}(\eta) > 1 - \lambda \Longleftrightarrow E_{\lambda,\mu}(x_n - x) < \eta \tag{5.27}$$

for every $\eta > 0$. This completes the proof. □

Definition 5.15. A function f from an RN-space (X, μ, T) to an RN-space (Y, ν, T') is said to be *uniformly continuous* if for given $r \in (0,1)$ and $t > 0$, there exist $r_0 \in (0,1)$ and $t_0 > 0$ such that

$$\mu_{x-y}(t_0) > 1 - r_0 \tag{5.28}$$

implies

$$\nu_{f(x),f(y)}(t) > 1 - r. \tag{5.29}$$

Theorem 5.16 (uniform continuity theorem). *If f is continuous function from a compact RN-space (X, μ, T) to an RN-space (Y, ν, T'), then f is uniformly continuous.*

Proof. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that

$$T'(1-r, 1-r) > 1 - s. \tag{5.30}$$

Since $f : X \to Y$ is continuous, for each $x \in X$, we can find $r_x \in (0,1)$ and $t_x > 0$ such that

$$\mu_{x-y}(t_x) > 1 - r_x \tag{5.31}$$

implies

$$\nu_{f(x)-f(y)}\left(\frac{t}{2}\right) > 1 - r. \tag{5.32}$$

But $r_x \in (0,1)$ and then we can find $s_x < r_x$ such that

$$T(1 - s_x, 1 - s_x) > 1 - r_x. \tag{5.33}$$

Since X is compact and

$$\left\{ B_x\left(s_x, \frac{t_x}{2}\right) : x \in X \right\} \tag{5.34}$$

is an open covering of X, there exist $x_1, x_2, \ldots, x_k$ in X such that

$$X = \bigcup_{i=1}^{k} B_{x_i}\left(s_{x_i}, \frac{t_{x_i}}{2}\right). \tag{5.35}$$

Put $s_0 = \min s_{x_i}$ and $t_0 = \min(t_{x_i}/2)$, $i = 1, 2, \ldots, k$. For any $x, y \in X$, if

$$\mu_{x-y}(t_0) > 1 - s_0, \tag{5.36}$$

then

$$\mu_{x-y}\left(\frac{t_{x_i}}{2}\right) > 1 - s_{x_i}. \tag{5.37}$$

Since $x \in X$, there exists x_i such that

$$\mu_{x-x_i}\left(\frac{t_{x_i}}{2}\right) > 1 - s_{x_i}. \tag{5.38}$$

Hence we have

$$\nu_{f(x),f(x_i)}\left(\frac{t}{2}\right) > 1 - r. \tag{5.39}$$

Now, since we have

$$\begin{aligned}\mu_{y-x_i}(t_{x_i}) &\geq T\left(\mu_{x-y}\left(\frac{t_{x_i}}{2}\right), \mu_{x-x_i}\left(\frac{t_{x_i}}{2}\right)\right)\\ &\geq T(1 - s_{x_i}, 1 - s_{x_i})\\ &> 1 - r_{x_i},\end{aligned} \tag{5.40}$$

it follows that

$$\nu_{f(y)-f(x_i)}\left(\frac{t}{2}\right) > 1 - r. \tag{5.41}$$

Now, we have

$$\begin{aligned}\nu_{f(x)-f(y)}(t) &\geq T\left(\nu_{f(x)-f(x_i)}\left(\frac{t}{2}\right), \nu_{f(y)-f(x_i)}\left(\frac{t}{2}\right)\right)\\ &\geq T(1-r, 1-r)\\ &> 1 - s.\end{aligned} \tag{5.42}$$

Hence f is uniformly continuous. □

Remark 5.17. Let f be a uniformly continuous function from RN-space (X, μ, T) to RN-space (Y, ν, T'). If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is also a Cauchy sequence in Y.

Theorem 5.18. *Every compact RN-space is separable.*

Proof. Let (X, μ, T) be the given compact RN-space. Let $r \in (0,1)$ and $t > 0$. Since X is compact, there exist $x_1, x_2, \ldots, x_n$ in X such that

$$X = \bigcup_{i=1}^{n} B_{x_i}(r,t). \tag{5.43}$$

In particular, for each $n \in \mathbb{N}$, we can choose a finite subset A_n such that

$$X = \bigcup_{a\in A_n} B_a\left(r_n, \frac{1}{n}\right), \tag{5.44}$$

in which $r_n \in (0,1)$. Let

$$A = \bigcup_{n\in\mathbb{N}} A_n. \tag{5.45}$$

Then A is countable. We claim that $X \subset \overline{A}$. Let $x \in X$. Then for each $n \in \mathbb{N}$, there exists $a_n \in A_n$ such that $x \in B_{a_n}(r_{\mathrm{n}}, 1/n)$. Thus a_n converges to x. But since $a_n \in A$ for all n, $x \in \overline{A}$. Hence A is dense in X and thus X is separable. □

Definition 5.19. Let X be any nonempty set and (Y, ν, T') be an RN-space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be *uniformly convergent* to a function f from X to Y if for given $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\nu_{f_n(x)-f(x)}(t) > 1 - r \tag{5.46}$$

for all $n \geq n_0$ and $x \in X$.

Definition 5.20. A family $\mathcal{F}$ of functions from an RN-space (X, μ, T) to a complete RN-space (Y, ν, T') is said to be *equicontinuous* if for any $r \in (0,1)$ and $t > 0$, there exist $r_0 \in (0,1)$ and $t_0 > 0$ such that

$$\mu_{x-y}(t_0) > 1 - r_0 \Longrightarrow \nu_{f(x)-f(y)}(\mathrm{t}) > 1 - r \tag{5.47}$$

for all $f \in \mathcal{F}$.

Lemma 5.21. *Let $\{f_n\}$ be an equicontinuous sequence of functions from an RN-space (X, μ, T) to a complete RN-space (Y, ν, T'). If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges for each point of X and the limit function is continuous.*

Proof. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that

$$T'^{2}(1-r, 1-r, 1-r) > 1 - s. \tag{5.48}$$

Since $\mathcal{F} = \{f_n\}$ is equicontinuous family, for given $r \in (0,1)$ and $t > 0$, there exist $r_1 \in (0,1)$ and $t_1 > 1$ such that, for each $x, y \in X$,

$$\mu_{x-y}(t_1) > 1 - r_1 \Longrightarrow \nu_{f_n(x)-f_n(y)}\left(\frac{t}{3}\right) > 1 - r \tag{5.49}$$

for all $f_n \in \mathcal{F}$. Since D is dense in X, there exists

$$y \in B_x(r_1, t_1) \cap D, \tag{5.50}$$

and $\{f_n(y)\}$ converges for that y. Since $\{f_n(y)\}$ is a Cauchy sequence, for any $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\nu_{f_n(y)-f_m(y)}\left(\frac{t}{3}\right) > 1 - r \tag{5.51}$$

for all $m, n \geq n_0$. Now, for any $x \in X$, we have

$$\begin{aligned}
\nu_{f_n(x)-f_m(x)}(t) &\geq T'^2\left(\nu_{f_n(x)-f_n(y)}\left(\frac{t}{3}\right), \nu_{f_n(y)-f_m(y)}\left(\frac{t}{3}\right), \nu_{f_m(x)-f_m(y)}\left(\frac{t}{3}\right)\right) \\
&\geq T'^2(1-r, 1-r, 1-r) \\
&> 1 - s.
\end{aligned} \tag{5.52}$$

Hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $f_n(x)$ converges. Let $f(x) = \lim f_n(x)$. We claim that f is continuous. Let $s_o \in 1 - r$ and $t_0 > 0$ be given. Then we can find $r_0 \in 1 - r$ such that

$$T^{'2}(1 - r_0, 1 - r_0, 1 - r_0) > 1 - s_0. \tag{5.53}$$

Since $\mathcal{F}$ is equicontinuous, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that

$$\mu_{x-y}(t_2) > 1 - r_2 \Longrightarrow \nu_{f_n(x)-f_n(y)}\left(\frac{t_0}{3}\right) > 1 - r_0 \tag{5.54}$$

for all $f_n \in \mathcal{F}$. Since $f_n(x)$ converges to $f(x)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$\nu_{f_n(x)-f(x)}\left(\frac{t_0}{3}\right) > 1 - r_0. \tag{5.55}$$

Also since $f_n(y)$ converges to $f(y)$, for given $r_0 \in (0,1)$ and $t_0 > 0$, there exists $n_2 \in \mathbb{N}$ such that

$$\nu_{f_n(y)-f(y)}\left(\frac{t_0}{3}\right) > 1 - r_0 \tag{5.56}$$

for all $n \geq n_2$. Now, for all $n \geq \max\{n_1, n_2\}$, we have

$$\begin{aligned}
\nu_{f(x)-f(y)}(t_0) &\geq T'^2\left(\nu_{f(x)-f_n(x)}\left(\frac{t_0}{3}\right), \nu_{f_n(x)-f_n(y)}\left(\frac{t_0}{3}\right), \nu_{f_n(y)-f(y)}\left(\frac{t_0}{3}\right)\right) \\
&\geq T'^2(1-r_0, 1-r_0, 1-r_0) \\
&> 1-s_0.
\end{aligned} \tag{5.57}$$

Hence f is continuous. □

Theorem 5.22 (Ascoli-Arzela Theorem). *Let (X, μ, T) be a compact RN-space and (Y, ν, T') be a complete RN-space. Let $\mathcal{F}$ be an equicontinuous family of functions from X to Y. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{F}$ such that*

$$\overline{\{f_n(x) : n \in \mathbb{N}\}} \tag{5.58}$$

is a compact subset of Y for each $x \in X$, then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.

Proof. Since (X, μ, T) be a compact RN-space, by Theorem 5.18, X is separable. Let

$$D = \{x_i : i = 1, 2, \ldots\} \tag{5.59}$$

be a countable dense subset of X. By hypothesis, for each i,

$$\overline{\{f_n(x_i) : n \in \mathbb{N}\}} \tag{5.60}$$

is compact subset of Y. Since every RN-space is first countable space, every compact subset of Y is sequentially compact. Thus by standard argument, we have a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each $i = 1, 2, \ldots$. By Lemma 5.21, there exists a continuous function f from X to Y such that $\{g_n(x)\}$ converges to $f(x)$ for all $x \in X$.

Now we claim that $\{g_n\}$ converges uniformly to f on X. Let $s \in (0,1)$ and $t > 0$ be given. Then we can find $r \in (0,1)$ such that

$$T'^2(1-r, 1-r, 1-r) > 1-s. \tag{5.61}$$

Since $\mathcal{F}$ is equicontinuous, there exist $r_1 \in (0,1)$ and $t_1 > 0$ such that

$$\mu_{x-y}(t_1) > 1-r_1 \Longrightarrow \nu_{g_n(x),g_n(y)}\left(\frac{t}{3}\right) > 1-r \tag{5.62}$$

for all $n \in \mathbb{N}$. Since X is compact, by Theorem 5.16, f is uniformly continuous. Hence for given $r \in (0,1)$ and $t > 0$, there exist $r_2 \in (0,1)$ and $t_2 > 0$ such that

$$\mu_{x-y}(t_2) > 1-r_2 \Longrightarrow \nu_{f(x)-f(y)}\left(\frac{t}{3}\right) > 1-r \tag{5.63}$$

for all $x, y \in X$. Let $r_0 = \min\{r_1, r_2\}$ and $t_0 = \min\{t_1, t_2\}$. Since X is compact and D is dense in X,

$$X = \bigcup_{i=1}^{k} B_{x_i}(r_0, t_0) \tag{5.64}$$

for some k. Thus for each $x \in X$, there exists i, $i \leq i \leq k$, such that

$$\mu_{x-x_i}(t_0) > 1 - r_0. \tag{5.65}$$

But since $r_0 = \min\{r_1, r_2\}$ and $t_0 = \min\{t_1, t_2\}$, we have, by the equicontinuity of $\mathcal{F}$,

$$\nu_{g_n(x)-g_n(x_i)}\left(\frac{t}{3}\right) > 1 - r \tag{5.66}$$

and we also have, by the uniform continuity of f,

$$\nu_{f(x)-f(x_i)}\left(\frac{t}{3}\right) > 1 - r. \tag{5.67}$$

Since $\{g_n(x_j)\}$ converges to $f(x_j)$, for any $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\nu_{g_n(x_j)-f(x_j)}\left(\frac{t}{3}\right) > 1 - r \tag{5.68}$$

for all $j = 1, 2, \ldots, n$.
Now, for each $x \in X$, we have

$$\begin{aligned} \nu_{g_n(x)-f(x)}(t) &\geq T'^2\left(\nu_{g_n(x)-g_n(x_i)}\left(\frac{t}{3}\right), \nu_{g_n(x_i)-f(x_i)}\left(\frac{t}{3}\right), \nu_{f(x_i)-f(x)}\left(\frac{t}{3}\right)\right) \\ &\geq T'^2(1-r, 1-r, 1-r) \\ &> 1 - s. \end{aligned} \tag{5.69}$$

Hence $\{g_n\}$ converges uniformly to f on X. □

Lemma 5.23. *A subset A of $\mathbb{R}$ is R-bounded in $(\mathbb{R}, \mu, T)$ if and only if it is bounded in $\mathbb{R}$.*

Proof. Let A be a subset in $\mathbb{R}$ which is R-bounded in $(\mathbb{R}, \mu, T)$, then there exist $t_0 > 0$ and $r_0 \in (0,1)$ such that $\mu_x(t_0) > 1 - r_0$ for each $x \in A$ and therefore we have

$$t_0 \geq E_{r_0,\mu}(x) = |x| E_{r_0,\mu}(1). \tag{5.70}$$

Now, $E_{r_0,\mu}(1) \neq 0$. If we put $k = t_0/(E_{r_0,\mu}(1))$, then we have $|x| \leq k$ for each $x \in A$, that is, A is bounded in $\mathbb{R}$. The converse is easy to see. □

Lemma 5.24. *A sequence* $\{\beta_n\}_{n\in\mathbb{N}}$ *is convergent in the RN-space* $(\mathbb{R},\mu,T)$ *if and only if it is convergent in* $(\mathbb{R},|\cdot|)$.

Proof. Let $\beta_n \to \beta$ in $\mathbb{R}$, by Lemma 5.14 (i), we have

$$E_{\lambda,\mu}(\beta_n-\beta) = |\beta_n-\beta|E_{\lambda,\mu}(1) \longrightarrow 0. \tag{5.71}$$

Then, by Lemma 5.14 (iii), $\beta_n \xrightarrow{\mu} \beta$.

Conversely, let $\beta_n \xrightarrow{\mu} \beta$, then, by Lemma 5.14, we have

$$\lim_{n\to+\infty}|\beta_n-\beta|E_{\lambda,\mu}(1) = \lim_{n\to+\infty}E_{\lambda,\mu}(\beta_n-\beta) = 0. \tag{5.72}$$

Now, $E_{\lambda,\mu}(1)\neq 0$ and so $\beta_n \to \beta$ in $\mathbb{R}$. □

Corollary 5.25. *If the real sequence* $\{\beta_n\}_{n\in\mathbb{N}}$ *is R-bounded, then it has at least one limit point.*

Definition 5.26. The 3-tuple $(\mathbb{R}^n,\Phi,T)$ is called a random Euclidean normed space if T is a continuous t-norm and $\Phi_x(t)$ is a random Euclidean norm defined by

$$\Phi_x(t) = \prod_{j=1}^{n}\mu_{x_j}(t), \tag{5.73}$$

where $\prod_{j=1}^{n}a_j = T'^{n-1}(a_1,\ldots,a_n)$, $T'\gg T$, $x=(x_1,\ldots,x_n)$, $t>0$, and μ is a random norm.

For example, let $\Phi_x(t) = \exp(\|x\|/t)^{-1}$, $\mu_{x_j}(t) = \exp(|x_j|/t)^{-1}$, and $T=\min$. Then we have $\Phi_x(t)=\min_j\mu_{x_j}(t)$ or equivalently $\|x\|=\max_j|x_j|$.

Lemma 5.27. *Suppose that the hypotheses of Definition 5.26 are satisfied. Then* $(\mathbb{R}^n,\Phi,T)$ *is an RN-space.*

Proof . The properties of (RN1) and (RN2) follow immediately from the definition. For the triangle inequality (RN3), suppose that $x,y\in$ X and $t,s>0$. Then

$$\begin{aligned}
T(\Phi_x(t),\Phi_y(s)) &= T\left(\prod_{j=1}^{n}\rho_{x_j}(t),\prod_{j=1}^{n}\rho_{y_j}(s)\right)\\
&= T\left(T'^{n-1}(\rho_{x_1}(t),\ldots,\rho_{x_n}(t)),T'^{n-1}(\rho_{y_1}(t),\ldots,\rho_{y_n}(t))\right)\\
&\leq T'^{n-1}(T(\rho_{x_1}(t),\rho_{y_1}(t)),\ldots,T(\rho_{x_n}(t),\rho_{y_n}(t)))\\
&\leq T'^{n-1}(\rho_{x_1+y_1}(t+s),\ldots,\rho_{x_n+y_n}(t+s))\\
&= \prod_{j=1}^{n}\rho_{x_j+y_j}(t+s)\\
&= \Phi_{x+y}(t+s).
\end{aligned} \tag{5.74}$$

□

Lemma 5.28. *Suppose that $(\mathbb{R}^n, \Phi, T)$ is a random Euclidean normed space and A is an infinite and R-bounded subset of $\mathbb{R}^n$. Then A has at least one limit point.*

Proof. Let $\{x^{(m)}\}_{m\in\mathbb{N}} \subseteq A$ be an infinite sequence. Since A is R-bounded and so is $\{x^{(m)}\}_{m\in\mathbb{N}}$. Therefore there exist $t_0 > 0$ and $r_0 \in (0,1)$ such that $1 - r_0 < \Phi_x(t_0)$ for each $x \in A$, which implies that $E_{r_0,\Phi}(x) \le t_0$. However, we have

$$\begin{aligned}
E_{r_0,\Phi}(x) &= \inf\{t > 0 : 1 - r_0 < \Phi_x(t)\} \\
&= \inf\left\{ t > 0 : 1 - r_0 < \prod_{j=1}^{n} \mu_{x_j}(t) \right\} \\
&\ge \inf\left\{ t > 0 : 1 - r_0 < \mu_{x_j}(t) \right\} \\
&= E_{r_0,\mu}(x_j)
\end{aligned} \tag{5.75}$$

for each $1 \le j \le n$. Therefore $|x_j| \le k$ in which $k = t_0/(E_{r_0,\mu}(1))$, that is, the real sequences $\{x_j^{(m)}\}$ for all $j \in \{1, \ldots, n\}$ are bounded. Hence there exists a subsequence $\{x_1^{(m_{k_1})}\}$ which converges to x_1 in A w.r.t. the random norm μ. The corresponding sequence $\{x_2^{(m_{k_1})}\}$ is bounded and so there exists a subsequence $\{x_2^{(m_{k_2})}\}$ of $\{x_2^{(m_{k_1})}\}$ which converges to x_2 with respect to the random norm μ. Continuing like this, we find a subsequence $\{x^{(m_k)}\}$ converging to $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. □

Lemma 5.29. *Let $(\mathbb{R}^n, \Phi, T)$ be a random Euclidean normed space. Let $\{Q_1, Q_2, \ldots\}$ be a countable collection of nonempty subsets in $\mathbb{R}^n$ such that $Q_{k+1} \subseteq Q_k$, each Q_k is closed and Q_1 is R-bounded. Then $\cap_{k=1}^{\infty} Q_k$ is nonempty and closed.*

Proof. Using the above lemma, the proof proceeds as in the classical case [see Theorem 3.25 in [54]]. □

We call an n-dimensional ball $B_x(r,t)$ a rational ball if $x \in \mathbb{Q}^n$, $r_0 \in (0,1)$, and $t \in \mathbb{Q}^+$.

Theorem 5.30. *Let $(\mathbb{R}^n, \Phi, T)$ be a random Euclidean normed space in which T satisfies* (2.15). *Let $G = \{A_1, A_2, \ldots\}$ be a countable collection of n-dimensional rational open balls. If $x \in \mathbb{R}^n$ and S is an open subset of $\mathbb{R}^n$ containing x, then there exists a $A_k \in G$ such that $x \in A_k \subseteq S$, for some $k \ge 1$.*

Proof. Since $x \in S$ and S is an open, there exist $r \in (0,1)$ and $t > 0$ such that $B_x(r,t) \subseteq S$. By (2.15), we can find $\eta \in (0,1)$ such that $1 - r < T(1 - \eta, 1 - \eta)$. Let $\{\xi_k\}_{k=1}^{n}$ be a finite sequence such that $1 - \eta < \prod_{k=1}^{n}(1 - \xi_k)$ and $x = (x_1, \ldots, x_n)$ and then we can find $y = (y_1, \ldots, y_n) \in \mathbb{Q}^n$ such that $1 - \xi_k) < \mu_{x_k - y_k}(t/2)$. Therefore, we have

$$1 - \eta < \prod_{k=1}^{n}(1 - \xi_k) \le \Phi_{x-y}\left(\frac{t}{2}\right) = \prod_{k=1}^{n}\mu_{x_k - y_k}\left(\frac{t}{2}\right) \tag{5.76}$$

and so $x \in B_y(\eta, t/2)$. Now we prove that $B_y(\eta, t/2) \subseteq B_x(r, t)$. Let $z \in B_y(\eta, t/2)$. Then $\Phi_{y-z}(t/2) > 1-\eta$ and hence

$$1 - r < T(1-\eta, 1-\eta) \leq T\left(\Phi_{x-y}\left(\frac{t}{2}\right), \Phi_{y-z}\left(\frac{t}{2}\right)\right) \leq \Phi_{x-z}(t). \tag{5.77}$$

On the other hand, there exists $t_0 \in \mathbb{Q}$ such that $t_0 < t/2$ and $x \in B_y(\eta, t_0) \subseteq B_y(\eta, t/2) \subseteq B_x(r, t) \subseteq S$. Now $B_y(\eta, t_0) \in G$ and the proof is complete. □

Corollary 5.31. *In a random Euclidean normed space $(\mathbb{R}^n, \Phi, T)$ in which T satisfies* (2.15)*, every closed and R-bounded set is compact.*

Proof. The proof is similar to the proof of Theorem 3.29 in [54]. □

Corollary 5.32. *Let $(\mathbb{R}^n, \Phi, T)$ be a random Euclidean normed space in which T satisfies* (2.15) *and $S \subseteq \mathbb{R}^n$. Then S is compact set if and only if it is R-bounded and closed.*

Corollary 5.33. *The random Euclidean normed space $(\mathbb{R}^n, \Phi, T)$ is complete.*

Proof. Let $\{x_m\}$ be a Cauchy sequence in the random Euclidean normed space $(\mathbb{R}^n, \Phi, T)$. Since

$$\begin{aligned}
E_{\lambda,\Phi}(x_n - x_m) &= \inf\{t > 0 : \Phi_{x_n - x_m}(t) > 1 - \lambda\} \\
&= \inf\left\{t > 0 : \prod_{j=1}^{n} \rho_{x_{m,j} - x_{n,j}}(t) > 1 - \lambda\right\} \\
&\geq \inf\left\{t > 0 : \rho_{x_{m,j} - x_{n,j}}(t) > 1 - \lambda\right\} \\
&= E_{\lambda,\rho}(x_{m,j} - x_{n,j}) = |x_{m,j} - x_{n,j}| E_{\lambda,\rho}(1),
\end{aligned} \tag{5.78}$$

the sequence $\{x_{m,j}\}$ in which $j = 1, \ldots, n$ is a Cauchy sequence in $\mathbb{R}$ and convergent to $x_j \in \mathbb{R}$ then, and by Lemma 5.14, the sequence $\{x_{m,j}\}$ is convergent in RN-space $(\mathbb{R}, \mu, T)$. We prove that $\{x_m\}$ is convergent to $x = (x_1, \ldots, x_n)$ and

$$\lim_{m\to\infty} \Phi_{x_m - x}(t) = \lim_{m\to\infty} \prod_{j=1}^{n} \rho_{x_{m,j} - x_j}(t) = T'^{\,n-1}(1, \ldots, 1) = 1. \tag{5.79}$$

□

6. Random Functional Analysis

Theorem 6.1. *Let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in vector space X and (X, μ, T) be an RN-space. Then there is $c \neq 0$ and an RN-space $(\mathbb{R}, \mu', T)$ such that for every choice of the n real scalars $\alpha_1, \ldots, \alpha_n$ we have*

$$\mu_{\alpha_1 x_1 + \cdots + \alpha_n x_n}(t) \leq \mu'_c \sum_{j=1}^{n} |\alpha_j| (t). \tag{6.1}$$

Proof. Put $s = |\alpha_1| + \cdots + |\alpha_n|$. If $s = 0$, all α_js must be zero and so (6.1) holds for any c. Let $s > 0$. Then (6.1) is equivalent to the inequality which we obtain from (6.1) by dividing by s and putting $\beta_j = (\alpha_j)/s$, that is,

$$\mu_{\beta_1 x_1 + \cdots + \beta_n x_n}(t') \le \mu'_c(t'), \tag{6.2}$$

where $t' = \dfrac{t}{s}$ and $\sum_{j=1}^{n} |\beta_j| = 1$. Hence it suffices to prove the existence of a $c \neq 0$ and random norm μ' such that (6.2) holds. Suppose that this is not true. Then there exists a sequence $\{y_m\}$ of vectors,

$$y_m = \beta_{1,m} x_1 + \cdots + \beta_{n,m} x_n, \tag{6.3}$$

where $\sum_{j=1}^{n} |\beta_{j,m}| = 1$, such that

$$\mu_{y_m}(t) \longrightarrow 1 \tag{6.4}$$

as $m \to \infty$ for every $t > 0$. Since $\sum_{j=1}^{n} |\beta_{j,m}| = 1$, we have $|\beta_{j,m}| \le 1$ and then, by Lemma 5.23, the sequence of $\{\beta_{j,m}\}$ is R-bounded. According the Corollary 5.25, $\{\beta_{1,m}\}$ has a convergent subsequence. Let β_1 denote the limit of that subsequence, and let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument, $\{y_{1,m}\}$ has a subsequence $\{y_{2,m}\}$ for which the corresponding of real scalars $\beta_2^{(m)}$ convergence; let β_2 denote the limit. Continuing this process, after n steps, we obtain a subsequence $\{y_{n,m}\}_m$ of $\{y_m\}$ such that

$$y_{n,m} = \sum_{j=1}^{n} \gamma_{j,m} x_j, \tag{6.5}$$

where $\sum_{j=1}^{n} |\gamma_{j,m}| = 1$, and $\gamma_{j,m} \to \beta_j$ as $m \to \infty$. By the Lemma 5.14 (ii) for any $\alpha \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned} E_{\alpha,\mu}\left(y_{n,m} - \sum_{j=1}^{n} \beta_j x_j \right) &= E_{\alpha,\mu}\left(\sum_{j=1}^{n} (\gamma_{j,m} - \beta_j) x_j \right) \\ &\le \sum_{j=1}^{n} \left| \gamma_{j,m} - \beta_j \right| E_{\lambda,\mu}(x_j) \longrightarrow 0 \end{aligned} \tag{6.6}$$

as $m \to \infty$. By the Lemma 5.14 (iii), we conclude

$$\lim_{m \to \infty} y_{n,m} = \sum_{j=1}^{n} \beta_j x_j, \tag{6.7}$$

where $\sum_{j=1}^{n}|\beta_j| = 1$, so that not all β_j can be zero. Put $y = \sum_{j=1}^{n} \beta_j x_j$. Since $\{x_1, \ldots, x_n\}$ is a linearly independent set, we thus have $y \neq 0$. Since

$$\mu_{y_m}(t) \longrightarrow 1, \tag{6.8}$$

by assumption, we have

$$\mu_{y_{n,m}}(t) \longrightarrow 1. \tag{6.9}$$

Hence it follows that

$$\begin{aligned} \mu_y(t) &= \mu_{(y-y_{n,m})+y_{n,m}}(t) \\ &\geq T\left(\left(\mu_{y-y_{n,m}}\frac{t}{2}\right), \mu_{y_{n,m}}\left(\frac{t}{2}\right)\right) \longrightarrow 1 \end{aligned} \tag{6.10}$$

and so $y = 0$, which is a contradiction. □

Definition 6.2. Let (X, μ, T) and (X, ν, T') be two RN-spaces. Then two random norms μ and ν are said to be *equivalent* whenever $x_n \xrightarrow{\mu} x$ in (X, μ, T) if and only if $x_n \xrightarrow{\nu} x$ in (X, ν, T').

Theorem 6.3. *On a finite dimensional vector space X, every two random norms μ and ν are equivalent.*

Proof. Let $\dim X = n$ and $\{v_1, \ldots, v_n\}$ be a basis for X. Then every $x \in X$ has a unique representation $x = \sum_{j=1}^{n} \alpha_j v_j$. Let $x_m \xrightarrow{\mu} x$ in (X, μ, T), but, for each $m \in \mathbb{N}$, x_m has a unique representation, that is,

$$x_m = \alpha_{1,m}v_1 + \cdots + \alpha_{n,m}v_n. \tag{6.11}$$

By Theorem 6.1, there is $c \neq 0$ and a random norm μ' such that (6.1) holds. So

$$\mu_{x_m - x}(t) \leq \mu'_{c\sum_{j=1}^{n}|\alpha_{j,m}-\alpha_j|}(t) \leq \mu'_{c|\alpha_{j,m}-\alpha_j|}(t). \tag{6.12}$$

Now, if $m \to \infty$, then

$$\mu_{x_m - x}(t) \longrightarrow 1 \tag{6.13}$$

for every $t > 0$ and hence $|\alpha_{j,m} - \alpha_j| \to 0$ in $\mathbb{R}$. On the other hand, by Lemma 5.14 (ii), for any $\alpha \in (0,1)$, there exists $\lambda \in (0,1)$ such that

$$E_{\alpha,\nu}(x_m - x) \leq \sum_{j=1}^{n}\left|\alpha_{j,m} - \alpha_j\right| E_{\lambda,\nu}(v_j). \tag{6.14}$$

Since $|\alpha_{j,m} - \alpha_j| \to 0$, then we have $x_m \xrightarrow{\nu} x$ in (X, ν, T'). With the same argument $x_m \to x$ in (X, ν, T') imply $x_m \to x$ in (X, μ, T). □

Definition 6.4. A linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is said to be *random bounded* if there exists a constant $h \in \mathbb{R} - \{0\}$ such that for every $x \in X$ and $t > 0$

$$\nu_{\Lambda x}(t) \geq \mu_{hx}(t). \tag{6.15}$$

Note that, by Lemma 5.14 and last definition, we have

$$\begin{aligned} E_{\lambda,\nu}(\Lambda x) &= \inf\{t > 0 : \nu_{\Lambda x}(t) > 1 - \lambda\} \\ &\leq \inf\left\{t > 0 : \mu_x\left(\frac{t}{|h|}\right) > 1 - \lambda\right\} = |h| \inf\{t > 0 : \mu_x(t) > 1 - \lambda\} \\ &= |h| E_{\lambda,\mu}(x). \end{aligned} \tag{6.16}$$

Theorem 6.5. *Every linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is random bounded if and only if it is continuous.*

Proof. By (6.15) every random bounded linear operator is continuous. Now, we prove that the converse it. Let the linear operator Λ be continuous but not random bounded. Then, for each $n \in \mathbb{N}$ there is x_n in X such that $E_{\lambda,\nu}(\Lambda x_n) \geq n E_{\lambda,\mu}(p_n)$. If we let

$$y_n = \frac{x_n}{n E_{\lambda,\mu}(x_n)}, \tag{6.17}$$

then it is easy to see $y_n \to 0$ but Λy_n do not tend to 0. □

Definition 6.6. A linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is called a *random topological isomorphism* if Λ is one-to-one and onto and both Λ and Λ^{-1} are continuous. RN-spaces (X, μ, T) and (Y, ν, T') for which such a Λ exists are said to be *random topologically isomorphic*.

Lemma 6.7. *A linear operator $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ is random topological isomorphism if Λ is onto and there exists constants $a, b \neq 0$ such that $\mu_{ax}(t) \leq \nu_{\Lambda x}(t) \leq \mu_{bx}(t)$.*

Proof. By hypothesis Λ is random bounded and, by last theorem, is continuous and, since $\Lambda x = 0$, we have

$$1 = \nu_{\Lambda x}(t) \leq \mu_x\left(\frac{t}{|b|}\right) \tag{6.18}$$

and consequently $x = 0$. So Λ is one-to-one. Thus Λ^{-1} exists and, since

$$\nu_{\Lambda x}(t) \leq \mu_{bx}(t) \tag{6.19}$$

is equivalent to

$$\nu_y(t) \le \mu_{b\Lambda^{-1}y}(t) = \mu_{\Lambda^{-1}y}\left(\frac{t}{|b|}\right) \tag{6.20}$$

or

$$\nu_{(\frac{1}{b})y}(t) \le \mu_{\Lambda^{-1}y}(t), \tag{6.21}$$

where $y = \Lambda x$, we see that Λ^{-1} is random bounded and, by last theorem, is continuous. Hence Λ is an random topological isomorphism. □

Corollary 6.8. *Random topologically isomorphism preserves completeness.*

Theorem 6.9. *Every linear operator* $\Lambda : (X, \mu, T) \to (Y, \nu, T')$ *where* $\dim X < \infty$ *but other, not necessarily finite dimensional, is continuous.*

Proof. If we define

$$\eta_x(t) = T'(\mu_x(t), \nu_{\Lambda x}(t)), \tag{6.22}$$

where $T' \gg T$, then (X, η, T) is an RN-space because (RN1) and (RN2) are immediate from definition, for the triangle inequality (RN3),

$$\begin{aligned} T(\eta_x(t), \eta_z(s)) &= T[T'(\mu_x(t), \nu_{\Lambda x}(t)), T'(\mu_z(s), \nu_{\Lambda z}(s))] \\ &\le T'[T(\mu_x(t), \mu_z(s)) T(\nu_{\Lambda x}(t), \nu_{\Lambda z}(s))] \\ &\le T'(\mu_{x+z}(t+s), \nu_{\Lambda(x+z)}(t+s)) \\ &= \eta_{x+z}(t+s). \end{aligned} \tag{6.23}$$

Now, let $x_n \xrightarrow{\mu} x$, then, by Theorem 6.3, $x_n \xrightarrow{\eta} x$ but since, by (6.15),

$$\nu_{\Lambda x}(t) \ge \eta_x(t), \tag{6.24}$$

then $\Lambda x_n \xrightarrow{\nu} \Lambda x$. Hence Λ is continuous. □

Corollary 6.10. *Every linear isomorphism between finite dimensional RN-spaces is topological isomorphism.*

Corollary 6.11. *Every finite dimensional RN-space* (X, μ, T) *is complete.*

Proof. By Corollary 6.10, (X, μ, T) and $(\mathbb{R}^n, \Phi, T)$ are random topologically isomorphic. Since $(\mathbb{R}^n, \Phi, T)$ is complete and random topological isomorphism preserves completeness, (X, μ, T) is complete. □

Definition 6.12. Let (V, μ, T) be an RN-space, W be a linear manifold in V and $Q : V \to V/W$ be the natural mapping, $Qx = x + W$. For any $t > 0$, we define

$$\overline{\mu}(x + W, t) = \sup\{\mu_{x+y}(t) : y \in W\}. \tag{6.25}$$

Theorem 6.13. *Let W be a closed subspace of an RN-space (V, μ, T). If $x \in V$ and $\epsilon > 0$, then there is x' in V such that $x' + W = x + W$, $E_{\lambda,\mu}(x') < E_{\overline{\lambda,\mu}}(x + W) + \epsilon$.*

Proof. By the properties of sup, there always exists $y \in W$ such that $E_{\lambda,p}(x + y) < E_{\overline{\lambda,\mu}}(x + W) + \epsilon$. Now, it is enough to put $x' = x + y$. □

Theorem 6.14. *Let W be a closed subspace of RN-space (V, μ, T) and $\overline{\mu}$ be given in the above definition. Then*

(1) *$\overline{\mu}$ is an RN-space on V/W,*

(2) *$\overline{\mu}_{Qx}(t) \geq \mu_x(t)$,*

(3) *if (V, μ, T) is a random Banach space, then so is $(V/W, \overline{\mu}, T)$.*

Proof. It is clear that $\overline{\mu}_{x+W}(t) > 0$. Let $\overline{\mu}_{x+W}(t) = 1$. By definition, there is a sequence $\{x_n\}$ in W such that $\mu_{x+x_n}(t) \to 1$. Thus $x + x_n \to 0$ or, equivalently, $x_n \to (-x)$ and since W is closed, $x \in W$ and $x + W = W$, the zero element of V/W. Now, we have

$$\begin{aligned}\overline{\mu}_{(x+W)+(y+W)}(t) &= \overline{\mu}_{(x+y)+W}(t)\\ &\geq \mu_{(x+m)+(y+n)}(t)\\ &\geq T\left(\mu_{x+m}(t_1), \mu_{y+n}(t_2)\right)\end{aligned} \tag{6.26}$$

for $m, n \in W$, $x, y \in V$, and $t_1 + t_2 = t$. Now, if we take the sup, then we have

$$\overline{\mu}_{(x+W)+(y+W)}(t) \geq T\Big(\overline{\mu}_{x+W}(t_1), \overline{\mu}_{y+W}(t_2)\Big). \tag{6.27}$$

Therefore, $\overline{\mu}$ is random norm on V/W.

(2) By Definition 6.12, we have

$$\overline{\mu}_{Qx}(t) = \overline{\mu}_{x+W}(t) = \sup\{\mu_{x+y}(t) : y \in W\} \geq \mu_x(t). \tag{6.28}$$

Note that, by Lemma 5.14,

$$\begin{aligned}E_{\lambda,\overline{\mu}}(Qx) &= \inf\Big\{t > 0 : \overline{\mu}_{Qx}(t) > 1 - \lambda\Big\}\\ &\leq \inf\{t > 0 : \mu_x(t) > 1 - \lambda\}\\ &= E_{\lambda,\mu}(x).\end{aligned} \tag{6.29}$$

(3) Let $\{x_n + W\}$ be a Cauchy sequence in V/W. Then there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $E_{\lambda,\overline{\mu}}((x_n + W) - (x_{n+1} + W)) \leq 2^{-n}$. Let $y_1 = 0$. Choose $y_2 \in W$ such that

$$E_{\lambda,\mu}(x_1 - (x_2 - y_2), t) \leq E_{\lambda,\overline{\mu}}((x_1 - x_2) + W) + \frac{1}{2}. \tag{6.30}$$

However, $E_{\overline{\lambda,\mu}}((x_1 - x_2) + W) \leq 1/2$ and so $E_{\lambda,\mu}(x_1 - (x_2 - y_2)) \leq 1/2^2$.

Now, suppose that y_{n-1} has been chosen and so choose $y_n \in W$ such that

$$E_{\lambda,\mu}((x_{n-1} + y_{n-1}) - (x_n + y_n)) \leq E_{\lambda,\overline{\mu}}((x_{n-1} - x_n) + W) + 2^{-n+1}. \tag{6.31}$$

Hence we have

$$E_{\lambda,\mu}((x_{n-1} + y_{n-1}) - (x_n + y_n)) \leq 2^{-n+2}. \tag{6.32}$$

However, for every positive integer $m > n$ and $\lambda \in (0,1)$, by Lemma 5.14, there exists $\gamma \in (0,1)$ such that

$$\begin{aligned} &E_{\lambda,\mu}((x_m + y_m) - (x_n + y_n)) \\ &\quad \leq E_{\gamma,\mu}((x_{n+1} + y_{n+1}) - (x_n + y_n)) + \cdots + E_{\gamma,\mu}((x_m + y_m) - (x_{m-1} + y_{m-1})) \\ &\quad \leq \sum_{i=n}^{m} 2^{-i}. \end{aligned} \tag{6.33}$$

By Lemma 5.14, $\{x_n + y_n\}$ is a Cauchy sequence in V. Since V is complete, there is x_0 in V such that $x_n + y_n \to x_0$ in V. On the other hand,

$$x_n + W = Q(x_n + y_n) \longrightarrow Q(x_0) = x_0 + W. \tag{6.34}$$

Therefore, every Cauchy sequence $\{x_n + W\}$ is convergent in V/W and so V/W is complete. Thus $(V/W, \overline{\mu}, T)$ is a random Banach space. □

Theorem 6.15. *Let W be a closed subspace of an RN-space (V, μ, T). If two of the spaces V, W, and V/W are complete, then so is the third one.*

Proof. If V is a random Banach space, then so are V/W and W. Hence all that needs to be checked is that V is complete whenever both W and V/W are complete. Suppose that W and V/W are random Banach spaces and let $\{x_n\}$ be a Cauchy sequence in V. Since

$$E_{\lambda,\overline{\mu}}((x_n - x_m) + W) \leq E_{\lambda,\mu}(x_n - x_m) \tag{6.35}$$

for each $m, n \in \mathbb{N}$, the sequence $\{x_n + \mathrm{W}\}$ is a Cauchy sequence in V/W and so it converges to $y + W$ for some $y \in W$. Thus there is $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,

$$E_{\lambda,\overline{\mu}}((x_n - y) + W) < 2^{-n}. \tag{6.36}$$

Now, by the last theorem, there exist a sequence $\{y_n\}$ in V such that $y_n + W = (x_n - y) + W$ and

$$E_{\lambda,\mu}(y_n) < E_{\lambda,\overline{\mu}}((x_n - y) + W) + 2^{-n}. \tag{6.37}$$

Thus we have

$$\lim_n E_{\lambda,\mu}(y_n) \le 0 \tag{6.38}$$

and, by Lemma 5.14, $\mu_{y_n}(t) \to 1$ for every $t > 0$, that is, $\lim_n y_n = 0$. Therefore, $\{x_n - y_n - y\}$ is a Cauchy sequence in W and thus is convergent to a point $z \in W$. This implies that $\{x_n\}$ converges to $z + y$ and hence V is complete. □

Theorem 6.16 (open mapping theorem). *If T is a random bounded linear operator from an RN-space (V, μ, T) onto an RN-space (V', ν, T), then T is an open mapping.*

Proof. The theorem will be proved in several steps.

Step 1. Let E be a neighborhood of the 0 in V. We show $0 \in (\overline{T(E)})^o$. Let W be a balanced neighborhood of 0 such that $W + W \subset E$. Since $T(V) = V'$ and W is absorbing, it follows that $V' = \cap_n T(nW)$ and so there exists an $n_0 \in \mathbb{N}$ such that $\overline{T(n_0 W)}$ has nonempty interior. Therefore, $0 \in (\overline{T(W)})^o - (\overline{T(W)})^o$. On the other hand,

$$\begin{aligned}\left(\overline{T(W)}\right)^o - \left(\overline{T(W)}\right)^o &\subset \overline{T(W)} - \overline{T(W)} = \overline{T(W) + T(W)}\\ &\subset \overline{T(E)}.\end{aligned} \tag{6.39}$$

So, the set $\overline{T(E)}$ includes the neighborhood $(\overline{T(W)})^o - (\overline{T(W)})^o$ of 0.

Step 2. We show $0 \in (T(E))^o$. Since $0 \in E$ and E is an open set, there exist $0 < \alpha < 1$ and $t_0 \in (0, \infty)$ such that $B_0(\alpha, t_0) \subset E$. However, $0 < \alpha < 1$ and so a sequence $\{\epsilon_n\}$ can be found such that

$$\begin{aligned}&T^{m-n}(1-\epsilon_{n+1}, \ldots, 1-\epsilon_m) \longrightarrow 1,\\ &1-\alpha < \lim_n T^{n-1}(1-\epsilon_1, 1-\epsilon_n),\end{aligned} \tag{6.40}$$

in which $m > n$. On the other hand, $0 \in \overline{T(B_0(\epsilon_n, t'_n))}$, where $t'_n = (1/2^n)t_0$ and so, by Step 1, there exist $0 < \sigma_n < 1$ and $t_n > 0$ such that

$$B_0(\sigma_n, t_n) \subset \overline{T(B_0(\epsilon_n, t'_n))}. \tag{6.41}$$

Since the set $\{B_0(r, 1/n)\}$ is a countable local base at zero and $t'_n \to 0$ as $n \to \infty$, so t_n and σ_n can be chosen such that $t_n \to 0$ and $\sigma_n \to 0$ as $n \to \infty$.

Now, we show that

$$B_0(\sigma_1, t_1) \subset (T(E))^o. \tag{6.42}$$

Suppose $y_0 \in B_0(, \sigma_1, t_1)$. Then $y_0 \in \overline{T(B_0(\epsilon_1, t_1'))}$ and so for $0 < \sigma_2$ and $t_2 > 0$ the ball $B_{y_0}(\sigma_2, t_2)$ intersects $T(B_0(\epsilon_1, t_1'))$. Therefore, there exists $x_1 \in B_0(\epsilon_1, t_1')$ such that $Tx_1 \in B_{y_0}(\sigma_2, t_2)$, that is,

$$\nu_{y_0 - Tx_1}(t_2) > 1 - \sigma_2 \tag{6.43}$$

or equivalently

$$y_0 - Tx_1 \in B_0(\sigma_2, t_2) \subset \overline{T(B_0(\epsilon_1, t_1'))}. \tag{6.44}$$

By the similar argument, there exist x_2 in $B_0(\epsilon_2, t_2')$ such that

$$\nu_{y_0-(Tx_1+Tx_2)}(t_3) = \nu_{(y_0-Tx_1)-Tx_2}(t_3) > 1 - \sigma_3. \tag{6.45}$$

If this process is continued, then it leads to a sequence $\{x_n\}$ such that $x_n \in B_0(\epsilon_n, t_n')$ and

$$\nu_{y_0 - \sum_{j=1}^{n-1} Tx_j}(t_n) > 1 - \sigma_n. \tag{6.46}$$

Now, if $n, m \in \mathbb{N}$ and $m > n$, then we have

$$\begin{aligned} \mu_{\sum_{j=1}^{n} x_j - \sum_{j=n+1}^{m} x_j}(t) &= \mu_{\sum_{j=n+1}^{m} x_j}(t) \\ &\geq T^{m-n}\left(\mu_{x_{n+1}}(t_{n+1}), \mu_{x_m}(t_m)\right), \end{aligned} \tag{6.47}$$

where $t_{n+1} + t_{n+2} + \cdots + t_m = t$. Put $t_0' = \min\{t_{n+1}, t_{n+2}, \ldots, t_m\}$. Since $t_n' \to 0$, there exists $n_0 \in \mathbb{N}$ such that $0 < t_n' \leq t_0'$ for $n > n_0$. Therefore, for $m > n$ we have

$$\begin{aligned} T^{m-n}\left(\mu_{x_{n+1}}(t_0'), \mu_{x_m}(t_0')\right) &\geq T^{m-n}\left(\mu_{x_{n+1}}(t_{n+1}'), \mu_{x_m}(t_m')\right) \\ &\geq T^{m-n}(1 - \epsilon_{n+1}, 1 - \epsilon_m). \end{aligned} \tag{6.48}$$

Hence it follows that

$$\lim_{n \to \infty} \mu_{\sum_{j=n+1}^{m} x_j}(t) \geq \lim_{n \to \infty} T^{m-n}(1 - \epsilon_{n+1}, 1 - \epsilon_m) = 1. \tag{6.49}$$

That is,

$$\mu_{\sum_{j=n+1}^{m} x_j}(t) \longrightarrow 1 \tag{6.50}$$

for all $t > 0$. Thus the sequence $\{\sum_{j=1}^{n} x_j\}$ is a Cauchy sequence and consequently the series $\{\sum_{j=1}^{\infty} x_j\}$ converges to some point $x_0 \in V$ because V is a complete space.

By fixing $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $t > t_n$ for $n > n_0$ because $t_n \to 0$. Thus

$$\begin{aligned} \nu_{y_0 - T(\sum_{j=1}^{n-1} x_j)}(t) &\geq \nu_{y_0 - T(\sum_{j=1}^{n-1} x_j)}(t_n) \\ &\geq 1 - \sigma_n \end{aligned} \tag{6.51}$$

and thus

$$\nu_{y_0 - T(\sum_{j=1}^{n-1} x_j)}(t) \longrightarrow 1. \tag{6.52}$$

Therefore, we have

$$y_0 = \lim_n T\left(\sum_{j=1}^{n-1} x_j\right) = T\left(\lim_n \sum_{j=1}^{n-1} x_j\right) = Tx_0. \tag{6.53}$$

But it follows that

$$\begin{aligned} \mu_{x_0}(t_0) &= \lim_n \mu_{\sum_{j=1}^{n} x_j}(t_0) \\ &\geq T^n\left(\lim_n (\mu_{x_1}(t_1'), \mu_{x_n}(t_n'))\right) \\ &\geq \lim_n T^{n-1}(1 - \epsilon_1, \ldots, 1 - \epsilon_n) \\ &> 1 - \alpha. \end{aligned} \tag{6.54}$$

Hence $x_0 \in B_0(\alpha, t_0)$.

Step 3. Let G be an open subset of V and $x \in G$. Then we have

$$T(G) = Tx + T(-x + G) \supset Tx + (T(-x + G))^o. \tag{6.55}$$

Hence $T(G)$ is open, because it includes a neighborhood of each of its point.

□

Corollary 6.17. *Every one-to-one random bounded linear operator from a random Banach space onto random Banach space has a random bounded inverse.*

Theorem 6.18 (closed graph theorem). *Let T be a linear operator from the random Banach space (V, μ, T) into the random Banach space (V', ν, T). Suppose that, for every sequence $\{x_n\}$ in V such that $x_n \to x$ and $Tx_n \to y$ for some elements $x \in V$ and $y \in V'$, it follows that $Tx = y$. Then T is random bounded.*

Proof. For any $t > 0$, $x \in X$ and $y \in V'$, define

$$\Phi_{(x,y)}(t) = T'(\mu_x(t), \nu_y(t)), \tag{6.56}$$

where $T' \gg T$.

First we show that $(V \times V', \Phi, T)$ is a complete RN-space. The properties of (RN1) and (RN2) are immediate from the definition. For the triangle inequality (RN3), suppose that $x, z \in V$, $y, u \in V'$, and $t, s > 0$, then

$$\begin{aligned} T\left(\Phi_{(x,y)}(t), \Phi_{(z,u)}(s)\right) &= T\left[T'\left(\mu_x(t), \nu_y(t)\right), T'\left(\mu_z(s), \nu_u(s)\right)\right] \\ &\leq T'\left[T\left(\mu_x(t), \mu_z(s)\right), T\left(\nu_y(t), \nu_u(s)\right)\right] \\ &\leq T'\left(\mu_{x+z}(t+s), \nu_{y+u}(t+s)\right) \\ &= \Phi_{(x+z,y+u)}(t+s). \end{aligned} \tag{6.57}$$

Now, if $\{(x_n, y_n)\}$ is a Cauchy sequence in $V \times V'$, then for every $\epsilon > 0$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\Phi_{(x_n,y_n)-(x_m,y_m)}(t) > 1 - \epsilon \tag{6.58}$$

for $m, n > n_0$. Thus for $m, n > n_0$,

$$\begin{aligned} T'\left(\mu_{x_n - x_m}(t), \nu_{y_n - y_m}(t)\right) &= \Phi_{(x_n - x_m, y_n - y_m)}(t) \\ &= \Phi_{(x_n,y_n)-(x_m,y_m)}(t) \\ &> 1 - \epsilon. \end{aligned} \tag{6.59}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V and V', respectively, and there exist $x \in V$ and $y \in V'$ such that $x_n \to x$ and $y_n \to y$ and consequently $(x_n, y_n) \to (x, y)$. Hence $(V \times V', \Phi, T)$ is a complete RN-space. The remainder of the proof is the same as the classical case. □

7. Fuzzy Normed Spaces

Now, we define the fuzzy normed spaces and give an example of these spaces. Here the t-norms notation is showed by $*$.

Definition 7.1. The triple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

(FM1) $M(x, y, 0) > 0$;

(FM2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ for all $t, s > 0$;

(FM5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Definition 7.2. The triple $(X, N, *)$ is said to be a *fuzzy normed space* if X is a vector space, $*$ is a continuous t-norm, and N is a fuzzy set on $X \times (0, \infty)$ satisfying the following conditions: for every $x, y \in X$ and $t, s > 0$;

(FN1) $N(x,t) > 0$;
(FN2) $N(x,t) = 1$ if and only if $x = 0$;
(FN3) $N(\alpha x,t) = N(x,t/|\alpha|)$, for all $\alpha \neq 0$;
(FN4) $N(x,t) * N(y,s) \leq N(x+y,t+s)$;
(FN5) $N(x,\cdot) : (0,\infty) \to [0,1]$ is continuous;
(FN6) $\lim_{t\to\infty} N(x,t) = 1$.

Lemma 7.3. *Let N be a fuzzy norm, then we have*
(i) *$N(x,t)$ is nondecreasing with respect to t for each $x \in X$,*
(ii) *$N(x-y,t) = N(y-x,t)$.*

Proof. Let $t < s$ then $k = s - t > 0$ and we have

$$\begin{aligned} N(x,t) &= N(x,t) * 1 \\ &= N(x,t) * N(0,k) \\ &\leq N(x,s) \end{aligned} \tag{7.1}$$

and this proves (i).
To prove (ii) we have

$$\begin{aligned} N(x-y,t) &= N((-1)(y-x),t) \\ &= N\left(y-x,\frac{t}{|-1|}\right) \\ &= N(y-x,t). \end{aligned} \tag{7.2}$$

□

Example 7.4. Let $(x,\|\cdot\|)$ be a normed space, define $a * b = ab$ or $a * b = \min(a,b)$ and

$$N(x,t) = \frac{kt^n}{kt^n + m\|x\|} \tag{7.3}$$

for all $k,m,n \in \mathbb{R}^+$. Then $(X,N,*)$ is a fuzzy normed space. In particular, if $k = n = m = 1$, then we have

$$N(x,t) = \frac{t}{t + \|x\|}, \tag{7.4}$$

which is called the *standard fuzzy norm* induced by norm $\|\cdot\|$.

Lemma 7.5. *Let $(X,N,*)$ be a fuzzy normed space. If one defines*

$$M(x,y,t) = N(x-y,t), \tag{7.5}$$

then M is a fuzzy metric on X, which is called the fuzzy metric induced by the fuzzy norm N.

We can see both definition and properties fuzzy normed spaces are very similar to random normed spaces. Then X equipped with $\mu_x(t) = N(x,t)$ and $T = *$ can be regarded as an RN-space. Now, we extend the definition of fuzzy metric space. In fact we extend the range of fuzzy sets to arbitrary lattice.

Definition 7.6. The triple $(X, \mathcal{P}, \mathcal{T})$ is said to be an *$\mathcal{L}$-fuzzy normed space* (briefly, $\mathcal{L}F$-normed space) if X is a vector space, $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}$, and $\mathcal{P}$ is an $\mathcal{L}$-fuzzy set on $X\times\,]0,+\infty[$ satisfying the following conditions: for every x, y in X and t, s in $]0,+\infty[$;

(a) $\mathcal{P}(x,t) >_L 0_{\mathcal{L}}$;
(b) $\mathcal{P}(x,t) = 1_{\mathcal{L}}$ if and only if $x = 0$;
(c) $\mathcal{P}(\alpha x, t) = \mathcal{P}(x, t/|\alpha|)$ for each $\alpha \neq 0$;
(d) $\mathcal{T}(\mathcal{P}(x,t), \mathcal{P}(y,s)) \leq_L \mathcal{P}(x+y, t+s)$;
(e) $\mathcal{P}(x,\cdot) :]0,\infty[\ \to L$ is continuous;
(f) $\lim_{t\to\infty}\mathcal{P}(x,t) = 1_{\mathcal{L}}$.

In this case, $\mathcal{P}$ is called an *$\mathcal{L}$-fuzzy norm* (briefly, $\mathcal{L}F$-norm). If $\mathcal{P} = \mathcal{P}_{\mu,\nu}$ is an intuitionistic fuzzy set and the t-norm $\mathcal{T}$ is t-representable then the triple $(X, \mathcal{P}_{\mu,\nu}, \mathcal{T})$ is said to be an *intuitionistic fuzzy normed space* (briefly, IF-normed space).

Example 7.7. Let $(X, \|\cdot\|)$ be a normed space. Denote $\mathcal{T}(a,b) = (a_1b_1, \min(a_2+b_2, 1))$ for all $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ and let M and N be fuzzy sets on $X \times (0,\infty)$ defined as follows:

$$\mathcal{P}_{M,N}(x,t) = \left(\frac{ht^n}{ht^n + m\|x\|}, \frac{m\|x\|}{ht^n + m\|x\|}\right) \tag{7.6}$$

for all $t, h, m, n \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{M,N}, \mathcal{T})$ is an IF-normed space.

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