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Research Article

On Existence and Uniqueness of Solutions of a Nonlinear Integral Equation

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The purpose of this paper is to study the existence of fixed point for a nonlinear integral operator in the framework of Banach space $X := C([a,b],\mathbb{R}^n)$. Later on, we give some examples of applications of this type of results.

1. Introduction

In this paper, we intend to prove the existence and uniqueness of the solutions of the following nonhomogeneous nonlinear Volterra integral equation:

$$u(x) = f(x) + \varphi\left(\int_a^x F(x,t,u(t))dt\right) \equiv Tu, \quad u \in X, \tag{1.1}$$

where $x, t \in [a,b]$, $-\infty < a < b < \infty$, $f : [a,b] \to \mathbb{R}^n$ is a mapping, and F is a continuous function on the domain $D := \{(x,t,u) : x \in [a,b], t \in [a,x), u \in X\}$.

The solutions of integral equations have a major role in the fields of science and engineering [1, 2]. A physical event can be modeled by the differential equation, an integral equation, an integrodifferential equation, or a system of these [3, 4]. Investigation on existence theorems for diverse nonlinear functional-integral equations has been presented in other references such as [5–10].

In this study, we will use an iterative method to prove that (1.1) has the mentioned cases under some appropriate conditions. Finally, we offer some examples that verify the application of this kind of nonlinear functional-integral equations.

2. Basic Concepts

In this section, we recall basic result which we will need in this paper.

Consider the nonhomogeneous nonlinear Volterra integral equation (1.1). Through this article, we consider the complete metric space (X, d), which $d(f, g) = \max_{x\in[a,b]}|f(x) - g(x)|$, for all $f, g \in X$ and assume that φ is a bounded linear transformation on X.

Note that the linear mapping $\varphi : X \to X$ is called bounded, if there exists $M > 0$ such that $\|\varphi x\| \leq M\|x\|$; for all $x \in X$. In this case, we define $\|\varphi\| = \sup\{\|\varphi x\|/\|x\|; x \neq 0,\ x \in X\}$. Thus, φ is bounded if and only if $\|\varphi\| < \infty$, [11].

Note 1. As φ is a bounded linear mapping on X, then $\varphi(x) = \lambda x$, where λ does not depend on $x \in X$.

Definition 2.1. Let S denote the class of those functions $\alpha : [0, \infty) \to [0, 1)$ satisfying the condition

$$\limsup_{s\to t^+} \alpha(s) < 1, \quad \forall t \in [0, \infty). \tag{2.1}$$

Definition 2.2. Let $\mathfrak{B}$ denoted the class of those functions $\phi : [0, \infty) \to [0, \infty)$ which satisfies the following conditions:

(i) ϕ is increasing,

(ii) for each $x > 0, \phi(x) < x$,

(iii) $\alpha(x) = \phi(x)/x \in S,\ x \neq 0$.

For example, $\phi(t) = \mu t$, where $0 \leq \mu < 1,\ \phi(t) = t/(t + 1)$ and $\phi(t) = \ln(1 + t)$ are in $\mathfrak{B}$.

3. Existence and Uniqueness of the Solution of Nonlinear Integral Equations

In this section, we will study the existence and uniqueness of the nonlinear functional-integral equation (1.1) on X.

Theorem 3.1. *Consider the integral equation* (1.1) *such that*

(i) $\varphi : X \to X$ *is a bounded linear transformation,*

(ii) $F : D \to \mathbb{R}^n$ *and* $f : [a, b] \to \mathbb{R}^n$ *are continuous,*

(iii) *there exists a integrable function* $p : [a,b] \times [a,b] \to \mathbb{R}$ *such that*

$$|F(x,t,u) - F(x,t,v)| \le p(x,t)\phi(|u-v|), \tag{3.1}$$

for each $x, t \in [a,b]$ *and* $u, v \in \mathbb{R}^n$.

(iv) $\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt \le 1/\|\varphi\|^2(b-a)$.

Then, the integral equation (1.1) *has a unique fixed point* u *in* X.

Proof. Consider the iterative scheme

$$u_{n+1}(x) = f(x) + \varphi\left(\int_a^x F(x,t,u_n(t))dt\right) \equiv Tu_n, \quad n = 0, 1, \ldots. \tag{3.2}$$

$$\begin{aligned} |Tu_n(x) - Tu_{n-1}(x)| &= \left|\varphi\left(\int_a^x F(x,t,u_n(t))dt\right) - \varphi\left(\int_a^x F(x,t,u_{n-1}(t))dt\right)\right| \\ &\le \left|\varphi\left(\int_a^x F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))dt\right)\right| \\ &\le \|\varphi\|\left|\int_a^x F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))dt\right| \\ &\le \|\varphi\| \int_a^x |F(x,t,u_n(t)) - F(x,t,u_{n-1}(t))|dt \\ &\le \|\varphi\| \int_a^b p(x,t)\phi(|u_n(t) - u_{n-1}(t)|)dt \\ &\le \|\varphi\|\left(\int_a^b p^2(x,t)dt\right)^{1/2}\left(\int_a^b \phi^2(|u_n(t) - u_{n-1}(t)|)dt\right)^{1/2}. \end{aligned} \tag{3.3}$$

As the function ϕ is increasing then

$$\phi(|u_n(t) - u_{n-1}(t)|) \le \phi(d(u_n, u_{n-1})), \tag{3.4}$$

so, we obtain

$$\begin{aligned} d^2(u_{n+1}, u_n) &\le \|\varphi\|^2\left(\sup_{x\in[a,b]} \int_a^b p^2(x,t)dt\right)\left(\int_a^b \phi^2(d(u_n, u_{n-1}))dt\right) \\ &\le \phi^2(d(u_n, u_{n-1})). \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned} d(u_{n+1}, u_n) \le \phi(d(u_n, u_{n-1})) &= \frac{\phi(d(u_n, u_{n-1}))}{d(u_n, u_{n-1})} d(u_n, u_{n-1}) \\ &= \alpha(d(u_n, u_{n-1}))d(u_n, u_{n-1}), \end{aligned} \tag{3.6}$$

and so the sequence $\{d(u_{n+1}, u_n)\}$ is nonincreasing and bounded below. Thus, there exists $\tau \geq 0$ such that $\lim_{n\to\infty} d(u_{n+1}, u_n) = \tau$. Since $\limsup_{s\to\tau^+} \alpha(s) < 1$ and $\alpha(\tau) < 1$, then there exist $r \in [0, 1)$ and $\epsilon > 0$ such that $\alpha(s) < r$ for all $s \in [\tau, \tau + \epsilon]$. We can take $\nu \in \mathbb{N}$ such that $\tau \leq d(u_{n+1}, u_n) \leq \tau + \epsilon$ for all $n \in \mathbb{N}$ with $n \geq \nu$. On the other hand, we have

$$d(u_{n+2}, u_{n+1}) \leq \alpha(d(u_{n+1}, u_n))d(u_{n+1}, u_n) \leq rd(u_{n+1}, u_n), \tag{3.7}$$

for all $n \in \mathbb{N}$ with $n \geq \nu$. It follows that

$$\sum_{n=1}^{\infty} d(u_{n+1}, u_n) \leq \sum_{n=1}^{\nu} d(u_{n+1}, u_n) + \sum_{n=1}^{\infty} r^n d(u_{\nu+1}, u_\nu) < \infty, \tag{3.8}$$

and hence, $\{u_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there exists a $u \in X$ such that $\lim_{n\to\infty} u_n = u$. Now, by taking the limit of both sides of (3.2), we have

$$\begin{aligned} u = \lim_{n\to\infty} u_{n+1}(x) &= \lim_{n\to\infty} \left(f(x) + \varphi\left(\int_a^x F(x, t, u_n(t))dt \right) \right) \\ &= f(x) + \varphi\left(\int_a^x F\left(x, t, \lim_{n\to\infty} u_n(t)\right) dt \right) \\ &= f(x) + \varphi\left(\int_a^x F(x, t, u(t))dt \right). \end{aligned} \tag{3.9}$$

So, there exists a solution $u \in X$ such that $Tu = u$. It is clear that the fixed point of T is unique. □

Note 2. Theorem 3.1 was proved with the condition (i), but there exist some nonlinear examples φ, such that by the analogue method mentioned in this theorem, the existence, and uniqueness can be proved for those. For example $\varphi(x) = \sin(x)$.

4. Applications

In this section, for efficiency of our theorem, some examples are introduced. For Examples 4.1 and 4.2, [5] is used. Maleknejad et al. presented some examples that the existence of their solutions can be established using their theorem. Generally, Examples 4.1 and 4.2 are introduced for the first time in this work. On the other hand, for Example 4.3, [12] is applied. In Chapter 6 of this reference, the existence theorems for Volterra integral equations with weakly singular kernels is discussed. Example 4.1 is extracted from this chapter.

Example 4.1. Consider the following linear Volterra integral equation:

$$u(x) = \ln\left(x^2\right) + \int_0^x txu(t)dt, \quad (x, t \in [0, 1]). \tag{4.1}$$

We have

$$\begin{aligned}|F(x,t,u)-F(x,t,v)| &= |(tx)u-(tx)v| = |(tx)(u-v)| \\ &\leq |tx||u-v| = \left|\left(\frac{|tx|}{\mu}\right)(\mu|u-v|)\right|,\end{aligned} \tag{4.2}$$

where $\sqrt{3}/3 \leq \mu < 1$. Now, we put $p(x,t) = (tx)/\mu$ and $\phi(t) = \mu t$. Because $\sup_{x\in[0,1]}\int_0^1 p^2(x,t)dt = 1/3\mu^2 \leq 1$, then by applying the result obtained in Theorem 3.1, we deduce that (4.1) has a unique solution in Banach space $C([0,1],\mathbb{R})$.

Example 4.2. Consider the following nonlinear Volterra integral equation:

$$u(x) = \sin\left(\frac{1}{1+x}\right) + \frac{x}{9}\int_0^x \frac{\cos(x^2t)}{(1+xt)^2}\arctan(u(t))dt, \quad (x,t\in[0,1]). \tag{4.3}$$

We write

$$\begin{aligned}|F(x,t,u)-F(x,t,v)| &= \left|\frac{x\cos(x^2t)}{9(1+xt)^2}(\arctan(u)-\arctan(v))\right| \\ &\leq \left|\frac{x\cos(x^2t)}{(1+xt)^2}\right|\left|\frac{u-v}{9}\right|.\end{aligned} \tag{4.4}$$

Take $p(x,t) = x\cos(x^2t)/(1+xt)^2$ and $\phi(t) = t/9$. Since $\sup_{x\in[0,1]}\int_0^1 p^2(x,t)dt \leq 1$, then (4.3) has a unique solution in $C([0,1],\mathbb{R})$.

Example 4.3 (see [12]). Consider the following singular Volterra integral equation

$$u(x) = f(x) + \lambda\int_0^x (x-t)^{-\alpha}u(t)dt, \quad (x,t\in[0,T]), \tag{4.5}$$

where $0 \leq \lambda < 1$ and $0 < \alpha < 1/2$. Then,

$$|F(x,t,u)-F(x,t,v)| = \left|\lambda(u-v)(x-t)^{-\alpha}\right| \leq |\lambda||u-v||(x-t)|^{-\alpha}. \tag{4.6}$$

Put $p(x,t) = (x-t)^{-\alpha}$ and $\phi(t) = \lambda t$. We have

$$\sup_{x\in[0,T]}\int_0^T p^2(x,t)dt = \sup_{x\in[0,T]}\int_0^T |(x-t)|^{-2\alpha}dt = \frac{T^{1-2\alpha}}{1-2\alpha}. \tag{4.7}$$

It follows that if $T^{1-\alpha} \leq (1-2\alpha)^{1/2}$, then (4.5) has a unique solution in complete metric space $C([0,T],\mathbb{R})$.

Remark 4.4. The unique solution $u \in C([0,1],\mathbb{R})$ of the Volterra integral (4.5) is given by

$$u(x) = E_{1-\alpha}\left(\lambda\Gamma(1-\alpha)x^{1-\alpha}\right)u_0, \quad (x \in [0,1]), \tag{4.8}$$

where $f(x) = u_0$ and

$$E_{\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\beta)}, \quad (\beta > 0) \tag{4.9}$$

denotes the Mittag-Leffler function. The Mittag-Leffler function was introduced early in the 20th century by the Swedish mathematician whose name it bears. Additional properties and applications can be found, for example, in Erdélyi [13] and, especially, in the survey paper by Mainardi and Gorenflo [14].

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