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Research Article

Practical Stability in terms of Two Measures for Impulsive Differential Equations with "Supremum"

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The object of investigations is a system of impulsive differential equations with "supremum." These equations are not widely studied yet, and at the same time they are adequate mathematical model of many real world processes in which the present state depends significantly on its maximal value on a past time interval. Practical stability for a nonlinear system of impulsive differential equations with "supremum" is defined and studied. It is applied Razumikhin method with piecewise continuous scalar Lyapunov functions and comparison results for scalar impulsive differential equations. To unify a variety of stability concepts and to offer a general framework for the investigation of the stability theory, the notion of stability in terms of two measures has been applied to both the given system and the comparison scalar equation. An example illustrates the usefulness of the obtained sufficient conditions.

1. Introduction

One of the main qualitative problem in the theory of differential equations is stability. In some real world situations, it is desired that the state of a system may be mathematically unstable; however, the system may oscillate sufficiently close to the desired state, so that its performance is deemed acceptable. For example, an aircraft may oscillate around a mathematically, unstable path, yet its performance may be acceptable. In this case, it is appropriate to be used the so-called practical stability, which is defined and studied for various types of differential equations in [1–8]. To unify a variety of stability concepts and to offer a general framework for the investigation of the stability theory, the notion of stability in terms of two measures has been proved to be very powerful (see, e.g., [9] and cited therein references). One of the very useful methods of investigation of stability of solutions is Lyapunov-Razumikhin method combined with a comparison method. In fact, application of Lyapunov functions allows us to consider a scalar equation and to study stability properties of its solution. Sometimes, its solution is not stable in Lyapunov sense, applying regular norm. It requires using two measures not only for the given equation but also for the comparison equation.

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in control theory correspond to the maximal deviation of the regulated quantity (see [10]). Such kind of problems could be adequately modeled by differential equations that contain the maxima operator. Note that various conditions for stability for differential equations with "maxima" are obtained in [1, 11–14].

In the paper, practical stability for a nonlinear system of impulsive differential equations involving the maximal value of the unknown function over a past time interval is studied. It is applied Razumikhin method with piecewise continuous scalar Lyapunov functions and comparison results for scalar impulsive differential equations. In this paper, differently than the existing up-to-date results, two different measures are applied to both the given system and the comparison scalar equation. Sufficient conditions for practical stability in the entire space (Theorem 2.10) as well as in a ball (Theorem 2.11) are obtained. The case of a ball requires more desired conditions than the global case. The main accent of the paper is consideration of

(i) a new type of functional differential equations which contain the supremum of the unknown function over a past time interval;

(ii) the practical stability, which is more stronger than the stability;

(iii) two pairs of different measures for both the comparison scalar equation and the given system.

An example illustrates the usefulness of the obtained sufficient conditions.

2. Main Results

Let $\mathbb{R}^n$ be n-dimensional Euclidean space with a norm $\|\cdot\|$, and $\mathbb{R}_+ = [0,\infty)$.

Let $\{\tau_k\}_1^\infty$ be a sequence of fixed points in $\mathbb{R}_+$ such that $\tau_{k+1} > \tau_k$ and $\lim_{k\to\infty}\tau_k = \infty$. Let $r > 0$ be a fixed constant.

Consider the system of nonlinear impulsive differential equations with "supremum":

$$\begin{aligned} x' &= f\left(t, x(t), \sup_{s\in[t-r,t]} x(s)\right) \quad \text{for } t \ge t_0,\ t \ne \tau_k, \\ x(\tau_k + 0) &= I_k(x(\tau_k - 0)) \quad \text{for } k = 1,2,\ldots, \end{aligned} \tag{2.1}$$

with initial condition

$$x(t) = \phi(t) \quad \text{for } t \in [t_0 - r, t_0], \tag{2.2}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,2,3,\ldots$, $t_0 \in \mathbb{R}_+$, and $\phi : [t_0-r, t_0] \to \mathbb{R}^n$.

Note that for $x : [t-r,t] \to \mathbb{R}^n,\ x = (x_1, x_2, \dots, x_n)$ we denote

$$\sup_{s\in[t-r,t]} x(s) = \left(\sup_{s\in[t-r,t]} x_1(s), \sup_{s\in[t-r,t]} x_2(s), \dots, \sup_{s\in[t-r,t]} x_n(s) \right). \tag{2.3}$$

Denote by $\mathrm{PC}(X,Y)$ $(X \subset \mathbb{R}, Y \subset \mathbb{R}^n)$ the set of all functions $u : X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $\tau_k \in X$ and which are continuous from the left at the points $\tau_k \in X$, $u(\tau_k) = u(\tau_k - 0)$.

In our further investigations, we will assume that for any initial function $\phi \in \mathrm{PC}([t_0 - r, t_0], \mathbb{R}^n)$ the solution of the initial value problem for system of impulsive differential equations with "supremum" (2.1) and (2.2) exists on $[t_0 - r, \infty)$. Note that differential equations with maxima as well as impulsive differential equations with supremum are not well studied yet. Although these equations are delayed functional differential equations, not all results known in the literature for delay differential equations are applied to them. The main reason is the presence of maximum function which is very nonlinear one. Only some partial results for existence are obtained in the continuous case (see [15, 16]).

Let $X \subset \mathbb{R}_+$. Denote by $Z(X)$ the set of all integers k such that $\tau_k \in X$.

We will define the following sets of measures:

$$\begin{gathered}
\Gamma = \left\{ h \in C([-r,\infty) \times \mathbb{R}^n, \mathbb{R}_+) : \min_{x\in\mathbb{R}^n} h(t,x) = 0 \text{ for each } t \in [-r,\infty) \right\}, \\
\overline{\Gamma} = \left\{ h \in C(\mathbb{R}, \mathbb{R}_+) : \min_{u\in\mathbb{R}} h(u) = 0 \right\}, \\
\widetilde{\Gamma} = \left\{ h^* \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+) : \min_{u\in\mathbb{R}} h^*(t,u) = 0 \text{ for each } t \in \mathbb{R}_+, \right. \\
\left. h^*(t,u_1) \le h^*(t,u_2) \text{ for } |u_1| \le |u_2|,\ t \in \mathbb{R}_+ \right\}.
\end{gathered} \tag{2.4}$$

Remark 2.1. Note that any norm in $\mathbb{R}^n$ is a function from Γ and any norm in $\mathbb{R}$ is from both classes $\widetilde{\Gamma}$ and $\overline{\Gamma}$. For example, the function $h^*(t,u) = e^{-t}|u| \in \widetilde{\Gamma}$.

Let $h_0 \in \Gamma$, $t_0 \in \mathbb{R}_+$, and $\phi \in \mathrm{PC}([t_0 - r, t_0], \mathbb{R}^n)$. We will use the following notation:

$$H_0(t_0, \phi) = \sup_{s\in[t_0-r,t_0]} h_0(s, \phi(s)). \tag{2.5}$$

Let $\rho > 0$ be a fixed number and $h \in \Gamma$. Define

$$\begin{aligned}
S(h,\rho) &= \{(t,x) \in [-r,\infty) \times \mathbb{R}^n : h(t,x) < \rho\}, \\
\overline{S}(h,\rho) &= \{(t,x) \in [-r,\infty) \times \mathbb{R}^n : h(t,x) \le \rho\}.
\end{aligned} \tag{2.6}$$

We will introduce the definition of a practical stability for impulsive differential equations with "supremum," based on the ideas of stability in terms of two measures considered in [8, 9].

Definition 2.2. Let the functions $h, h_0 \in \Gamma$ and the positive constants $\lambda,\ A$ be given. The system of impulsive differential equations with "supremum" (2.1) is said to be

($S1$) practically stable with respect to (λ, A) in terms of both measures (h_0, h) if there exists $t_0 \geq 0$ such that for any $\phi \in \mathrm{PC}([t_0 - r, t_0], \mathbb{R}^n)$ inequality $H_0(t_0, \phi) < \lambda$ implies $h(t, x(t; t_0, \phi)) < A$ for $t \geq t_0$, where the function H_0 is defined by (2.5), and $x(t; t_0, \phi)$ is a solution of (2.1) and (2.2);

($S2$) uniformly practically stable with respect to (λ, A) in terms of both measures (h_0, h) if ($S1$) is satisfied for all $t_0 \in \mathbb{R}_+$.

Remark 2.3. In the case $r = 0$ and $h_0(t, x) = h(t, x) = \|x\|$, Definition 2.2 reduces to a definition of *practical stability* of impulsive differential equations, studied in [4].

In the case $r = 0$, $I_k(x) \equiv x$, $k = 1, 2, \ldots$, and $h_0(t, x) = h(t, x) = \|x\|$, the above-given definitions reduce to definitions for the corresponding types of practical stability of ordinary differential equations, given in the books [4, 6].

In our further investigations, we will use the initial value problem for the comparison scalar impulsive differential equation:

$$u' = g(t, u), \quad t \geq t_0,\ t \neq \tau_k,$$
$$u(\tau_k + 0) = \xi_k(u(\tau_k)), \quad k = 1, 2, \ldots, \tag{2.7}$$

$$u(t_0) = u_0, \tag{2.8}$$

where $u, u_0 \in \mathbb{R}$, $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\xi_k : \mathbb{R} \to \mathbb{R}$, and $k = 1, 2, \ldots$.

In our further investigations, we will assume that for any initial point $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ the solution of scalar impulsive equation (2.7) exists on $[t_0, \infty)$, $t_0 \geq 0$. For some existence results, see the book of Bainov and Simeonov [17].

Note that the definition for practical stability in terms of two measures for scalar impulsive differential equation (2.7) is given by the following definition.

Definition 2.4. Let the functions $h^* \in \widetilde{\Gamma}, h_0^* \in \overline{\Gamma}$ and the positive constants $\lambda,\ A$ be given. The scalar impulsive differential equation (2.7) is said to be

($S3$) practically stable with respect to (λ, A) in terms of both measures (h_0^*, h^*) if there exists $t_0 \geq 0$ such that for any $u_0 \in \mathbb{R}$ the inequality $h_0^*(u_0) < \lambda$ implies $h^*(t, u(t; t_0, u_0)) < A$ for $t \geq t_0$, where $u(t; t_0, u_0)$ is a solution of (2.7) and (2.8);

($S4$) uniformly practically stable with respect to (λ, A) in terms of both measures (h_0^*, h^*) if ($S4$) is satisfied for all $t_0 \in \mathbb{R}_+$.

Remark 2.5. In the case $h^*(t, x) = \|x\|$ and $h_0^*(x) = \|x\|$, Definition 2.4 reduces to a definition for practical stability of zero solution of differential equations (see [6]).

We will give an example of a scalar differential equation which is not practically stable, but at the same time it is practically stable in terms of two measures.

Example 2.6. Consider scalar differential equation $u' = u$. The zero solution of this equation is not practically stable. At the same time, if we choose measures $h_0^*(u) = |u|$ and $h^*(t,u) = e^{-t}|u|$, then the same scalar equation is uniformly practically stable in terms of two measures.

We will study the connection between practical stability in terms of two measures of the scalar impulsive differential equation (2.7) and the corresponding practical stability in terms of two measures for the system of impulsive differential equations with "supremum" (2.1).

Introduce the following notations:

$$G_k = \{t \in [-r,\infty) : t \in (\tau_k, \tau_{k+1})\}, \quad k = 1,2,\ldots, \qquad \mathcal{G} = \bigcup_{k=1}^{\infty} G_k. \tag{2.9}$$

We will introduce the class Λ of piecewise continuous Lyapunov functions which will be used to investigate the practical stability of impulsive differential equations with "supremum."

Definition 2.7. We will say that the function $V(t,x) : \Delta \times \Omega \to \mathbb{R}_+$, $\Delta \subset [-r,\infty)$, $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, belongs to class Λ if

(1) $V(t,x)$ is a continuous function in $(\Delta \cap \mathcal{G}) \times \Omega$ and $V(t,0) \equiv 0$ for $t \in \Delta$,

(2) for every $k \in Z(\Delta)$ and $x \in \Omega$, there exist the finite limits

$$V(\tau_k, x) = V(\tau_k - 0, x) = \lim_{t \uparrow \tau_k} V(t,x), \qquad V(\tau_k + 0, x) = \lim_{t \downarrow \tau_k} V(t,x), \tag{2.10}$$

(3) function $V(t,x)$ is Lipschitz with respect to its second argument in the set $\Delta \times \Omega$.

Let $V(t,x) : \Delta\times\Omega \to \mathbb{R}_+$, $V \in \Lambda$. For any $t \in \Delta\cap\mathcal{G}$ and any function $\psi \in PC([t-r,t],\Omega)$, we will define a derivative of the function V along a trajectory of the solution of (2.1) as follows:

$$D_{(2.1)}V(t,\psi) = \lim_{\epsilon \to 0} \sup \frac{1}{\epsilon}\left[V\left(t+\epsilon, \psi(t) + \epsilon f\left(t, \psi(t), \sup_{s\in[-r,0]} \psi(t+s)\right)\right) - V(t,\psi(t))\right]. \tag{2.11}$$

Consider the following sets:

$$\begin{aligned} K &= \{a \in C(\mathbb{R}_+, \mathbb{R}_+) : a(r) \text{ is strictly increasing and } a(0) = 0\}, \\ \mathcal{K} &= \{a \in C(\mathbb{R}_+, \mathbb{R}_+) : a(r) \text{ is strictly increasing and } a(s) \geq s, a(0) = 0\}. \end{aligned} \tag{2.12}$$

In the further investigations, we will use the following comparison result.

Lemma 2.8 (Hristova [12]). *Let the following conditions be fulfilled.*

(1) *The functions* $f \in PC([t_0,T] \times \Omega \times \Omega, \mathbb{R}^n)$ *and* $I_k \in C(\Omega,\Omega)$ *for* $k \in Z([t_0,T))$, *where* $\Omega \subset \mathbb{R}^n$, *and* $t_0, T : 0 \leq t_0 < T < \infty$ *are constants.*

(2) *The function* $\phi \in PC([t_0 - r, t_0], \Omega)$.

(3) *The initial value problem* (2.1) *and* (2.2) *has a solution* $x(t) = x(t; t_0, \phi)$, *such that* $x(t) \in \Omega$ *on* $[t_0 - r, T]$.

(4) *The functions* $g \in PC([t_0,T] \times \mathbb{R}_+, \mathbb{R}_+)$, $g(t,0) \equiv 0$ *for* $t \in [t_0,T]$ *and* $\xi_k \in \mathcal{K}$, $k \in Z((t_0,T))$.

(5) *For any initial point* $u_0 \in \mathbb{R}_+$, *the initial value problem for the scalar impulsive differential equation* (2.7) *has a maximal solution* $u^*(t) = u^*(t; t_0, u_0)$, *which is defined for* $t \in [t_0,T]$.

(6) *The function* $V : [t_0 - r, T] \times \Omega \to \mathbb{R}_+$, $V \in \Lambda$ *is such that*

(i) *for any number* $t \in [t_0,T] : t \neq \tau_k$, $k \in Z((t_0,T))$ *and any function* $\psi \in PC([t - r, t], \Omega)$ *such that* $V(t,\psi(t)) \geq V(t+s, \psi(t+s))$ *for* $s \in [-r, 0)$, *the inequality*

$$D_{(2.1)} V(t, \psi(t)) \leq g(t, V(t, \psi(t))) \tag{2.13}$$

holds.

(ii) $V(\tau_k + 0, I_k(x)) \leq \xi_k(V(\tau_k, x))$, $k \in Z((t_0,T))$, $x \in \Omega$.

Then, the inequality $\sup_{s\in[-r,0]} V(t_0 + s, \phi(t_0 + s)) \leq u_0$ *implies the inequality* $V(t, x(t)) \leq u^*(t)$ *for* $t \in [t_0,T]$.

Remark 2.9. Lemma 2.8 is valid when $T = \infty$, that is, for $t \in [t_0, \infty)$.

We will obtain sufficient conditions for practical stability in terms of two measures for impulsive differential equations with "supremum." We will use Lyapunov functions from class Λ. The proof is based on Razumikhin method and a comparison method employing scalar impulsive differential equations.

In the case when the Lyapunov function satisfies globally the desired conditions, we obtain the following result.

Theorem 2.10. *Let the following conditions be fulfilled.*

(1) *The function* $f \in PC\,[\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $f(t,0,0) \equiv 0$.

(2) *The functions* $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ *and* $I_k(0) = 0$ *for* $k \in Z(\mathbb{R}_+)$.

(3) *The functions* $h_0, h \in \Gamma$ *and* $h_0^* \in \overline{\Gamma}$, $h^* \in \widetilde{\Gamma}$.

(4) *There exists a function* $V(t,x) : [-r, \infty) \times \mathbb{R}^n \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that*

(i) $b(h(t,x)) \leq h^*(t, V(t,x))$ *and* $h_0^*(V(t,x)) \leq a(h_0(t,x))$ *for* $(t,x) \in [-r,\infty) \times \mathbb{R}^n$, *where* $a, b \in K$;

(ii) *for any number* $t \in \mathbb{R}_+ : t \neq \tau_k, k \in Z(\mathbb{R}^+)$ *and any function* $\psi \in PC([t-r,t],\mathbb{R}^n)$ *such that* $V(t,\psi(t)) > V(t+s,\psi(t+s))$ *for* $s \in [-r,0)$*, the inequality*

$$D_{(2.1)}V(t,\psi(t)) \leq g(t,V(t,\psi(t))) \tag{2.14}$$

holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *and* $g(t,0) \equiv 0$;

(iii) $V(\tau_k+0, I_k(x)) \leq \xi_k(V(\tau_k,x))$*, for* $x \in \mathbb{R}^n, k \in Z(\mathbb{R}^+)$*, where* $\xi_k \in \mathcal{K}$.

Then, the (uniform) practical stability with respect to $(a(\lambda), b(A))$ *in terms of both measures* (h_0^*, h^*) *of scalar impulsive differential equation* (2.7) *implies (uniform) practical stability with respect to* (λ, A) *in terms of both measures* (h_0, h) *of system of impulsive differential equations with "supremum"* (2.1) *where the positive constants* λ, A *are given.*

Proof. Let scalar impulsive differential equation (2.7) be practically stable in both measures (h_0^*, h^*) with respect to $(a(\lambda), b(A))$. Therefore, there exists a point $t_0 \geq 0$ such that $h_0^*(u_0) < a(\lambda)$ implies

$$h^*(t, u(t;t_0,u_0)) < b(A) \quad \text{for } t \geq t_0, \tag{2.15}$$

where $u(t;t_0,u_0)$ is a solution of (2.7) and (2.8).

Choose a function $\phi \in \mathrm{PC}\ ([t_0-r,t_0],\mathbb{R}^n)$ such that

$$H_0(t_0,\phi) < \lambda, \tag{2.16}$$

and let $x(t;t_0,\phi)$ be a solution of (2.1) with initial condition (2.2).

Let $u_0 = \sup_{s\in[-r,0]} V(t_0+s,\phi(t_0+s))$. From Lemma 2.8 for $\Delta = [-r,\infty)$ and $\Omega = \mathbb{R}^n$, it follows the validity of the inequality

$$V(t,x(t;t_0,\phi)) \leq u^*(t;t_0,u_0) \quad \text{for } t \geq t_0. \tag{2.17}$$

From condition 4(i), we obtain for all $s \in [-r,0]$

$$h_0^*(V(t_0+s,\phi(t_0+s))) \leq a(h_0(t_0+s,\phi(t_0+s))) < a(\lambda). \tag{2.18}$$

From inequalities (2.18) and $h_0^*(\sup_{s\in[-r,0]} V(t_0+s,\phi(t_0+s))) \leq \sup_{s\in[-r,0]} h_0^*(V(t_0+s,\phi(t_0+s)))$, we obtain

$$h_0^*(u_0) < a(\lambda). \tag{2.19}$$

From condition 4(i), and inequalities (2.15), (2.17), and (2.19), we get for $t \geq t_0$

$$b(h(t,x(t;t_0,\phi))) \leq h^*(t,V(t,x(t;t_0,\phi))) \leq h^*(t,u^*(t;t_0,u_0)) < b(A) \tag{2.20}$$

or

$$h(t, x(t; t_0, \phi)) < A. \tag{2.21}$$

□

In the case when Lyapunov function does not satisfy globally the conditions 4(ii) and 4(iii) of Theorem 2.10, we obtain the following sufficient conditions.

Theorem 2.11. *Let the following conditions be fulfilled.*

(1) *The conditions* (1) *and (2) of Theorem 2.10 are satisfied.*

(2) *The functions* h_0, $h \in \Gamma$, $h_0^* \in \overline{\Gamma}$, $h^* \in \widetilde{\Gamma}$; *there exist positive constants* λ, A *and a function* $\Psi \in K$, $\Psi(x) \leq x$ *such that* $h(t,x) \leq \Psi(h_0(t,x))$ *for* $(t,x) \in S(h_0, \lambda)$, *and* $h(\tau_k, x) < A$ *implies* $h(\tau_k, I_k(x)) < A$ *for* $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.

(3) *There exists a function* $V(t,x) : \overline{S}(h, A) \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that*

  (i) $b(h(t,x)) \leq h^*(t, V(t,x))$ *and* $h_0^*(V(t,x)) \leq a(h_0(t,x))$ *for* $(t,x) \in \overline{S}(h, A)$, *where* $a, b \in K$;

  (ii) *for any number* $t \in \mathbb{R}_+ : t \neq \tau_k, k \in Z(\mathbb{R}_+)$ *and any function* $\psi \in PC([t-r, t], \mathbb{R}^n)$: $(t, \psi(t)) \in S(h, A)$ *such that* $V(t, \psi(t)) > V(t+s, \psi(t+s))$ *for* $s \in [-r, 0)$, *the inequality*

$$D_{(2.1)} V(t, \psi(t)) \leq g(t, V(t, \psi(t))) \tag{2.22}$$

  holds, where $g \in PC(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}_+)$ *and* $g(t, 0) \equiv 0$;

  (iii) $V(\tau_k + 0, I_k(x)) \leq \xi_k(V(\tau_k, x))$ *for* $(\tau_k, x) \in S(h, A), k \in Z(\mathbb{R}_+)$, *where* $\xi_k \in \mathcal{K}$.

Then, (uniform) practical stability with respect to $(a(\lambda), b(A))$ *in terms of both measures* (h_0^*, h^*) *of scalar impulsive differential equation* (2.7) *implies (uniform) practical stability with respect to* (λ, A) *in terms of both measures* (h_0, h) *of system of impulsive differential equations with "supremum"* (2.1).

Proof. Let scalar impulsive differential equation (2.7) be practically stable with respect to $(a(\lambda), b(A))$ in terms of both measures (h_0^*, h^*). Therefore, there exists a point $t_0 \geq 0$ such that $h_0^*(u_0) < a(\lambda)$ implies

$$h^*(t, u(t; t_0, u_0)) < b(A) \quad \text{for } t \geq t_0, \tag{2.23}$$

where $u(t; t_0, u_0)$ is a solution of (2.7) and (2.8).

Choose a function $\phi \in \mathrm{PC}\ ([t_0 - r, t_0], \mathbb{R}^n)$ such that

$$H_0(t_0, \phi) < \lambda, \tag{2.24}$$

and let $x(t; t_0, \phi)$ be a solution of (2.1) with initial condition (2.2).

We will prove that

$$h(t, x(t; t_0, \phi)) < A \tag{2.25}$$

holds for $t \geq t_0$.

From inclusion $(t, \phi(t)) \in S(h_0, \lambda)$ for $t \in [t_0 - r, t_0]$ and conditions (2) and 3(i), it follows that

$$h(s, \phi(s)) \leq \Psi(h_0(s, \phi(s))) \leq \Psi(H_0(t_0, \phi)) < \Psi(\lambda) \leq \lambda < A, \quad s \in [t_0 - r, t_0], \tag{2.26}$$

that is, inequality (2.25) holds on $[t_0 - r, t_0]$.

Assume that (2.25) does not hold for $t > t_0$.

Consider the following two cases.

Case 1. Let there exists a point $t^* > t_0$, $t^* \neq \tau_k, k \in Z((t_0, \infty))$ such that

$$h(t^*, x(t^*; t_0, \phi)) = A, \quad h(t, x(t; t_0, \phi)) < A \quad \text{for } t \in [t_0 - r, t^*). \tag{2.27}$$

Let $u_0 = \sup_{s \in [-r,0]} V(t_0 + s, \phi(t_0 + s))$. From Lemma 2.8 for the function $V(t, x)$ defined on the set $\{(t, x) \in [t_0, t^*] \times \mathbb{R}^n : h(t, x) \leq A\}$, it follows the validity of the inequality

$$V(t, x(t; t_0, \phi)) \leq u^*(t; t_0, u_0) \quad \text{for } t \in [t_0, t^*]. \tag{2.28}$$

From condition 3(i), we obtain

$$h_0^*(V(t_0 + s, \phi(t_0 + s))) \leq a(h_0(t_0 + s, \phi(t_0 + s))) < a(\lambda), \quad s \in [-r, 0] \tag{2.29}$$

or

$$h_0^*(u_0) < a(\lambda). \tag{2.30}$$

From inequalities (2.23), (2.28), and (2.30), the choice of the point t^*, and condition 3(i), we get

$$b(A) = b(h(t^*, x(t^*; t_0, \phi))) \leq h^*(t, V(t^*, x(t^*; t_0, \phi))) \leq h^*(t, u^*(t^*; t_0, u_0)) < b(A). \tag{2.31}$$

The obtained contradiction proves the validity of (2.25) for $t > t_0$.

Case 2. Let there exists a number $k \in Z((t_0, \infty))$ such that $h(t, x(t; t_0, \phi)) < A$ for $t \in [t_0 - r, \tau_k)$ and $h(\tau_k, x(\tau_k; t_0, \phi)) = A$. Then, as in Case 1 for $t^* = \tau_k$, we obtain a contradiction. The obtained contradictions prove the validity of (2.25) for $t > t_0$. □

Remark 2.12. Note that if in condition (2) inequality $h(\tau_k, x) < A$ implies $h(\tau_k, I_k(x)) \leq A$ for $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$, then the claim of Theorem 2.11 holds if functions $\xi \in \mathcal{K}, \xi(x) > x$, and $k \in Z(\mathbb{R}_+)$.

Corollary 2.13. *Let the following conditions be fulfilled.*

(1) *Conditions* (1) *and* (2) *of Theorem 2.10 are satisfied.*

(2) *The functions* $h_0, h \in \Gamma$, *and there exists a positive constant* A *such that* $h(\tau_k, x) < A$ *implies* $h(\tau_k, I_k(x)) < A$ *for* $x \in \mathbb{R}^n$, $k \in Z(\mathbb{R}_+)$.

(3) *There exists a function* $V(t,x) : [-r,\infty) \times \mathbb{R}^n \to \mathbb{R}_+$ *with* $V \in \Lambda$ *such that*

(i) $b(h(t,x)) \le V(t,x) \le a(h_0(t,x))$ *for* $(t,x) \in \overline{S}(h,A)$ *where* $a, b \in K$;

(ii) *for any number* $t \in \mathbb{R}_+ : t \ne \tau_k, k \in Z(\mathbb{R}_+)$ *and any function* $\varphi \in PC([t-r,t],\mathbb{R}^n)$: $(t,\varphi(t)) \in S(h,A)$ *such that* $V(t,\varphi(t)) > V(t+s,\varphi(t+s))$ *for* $s \in [-r,0)$, *the inequality*

$$D_{(2.1)}V(t,\varphi(t)) \le 0 \tag{2.32}$$

holds;

(iii) $V(\tau_k + 0, I_k(x)) \le V(\tau_k, x)$ *for* $(\tau_k, x) \in S(h,A), k \in Z(\mathbb{R}_+)$.

Then, the system of impulsive differential equations with "supremum" (2.1) *is uniformly practically stable with respect to* (λ, A) *in terms of both measures* (h_0, h).

Proof. The proof of Corollary 2.13 follows from the one of Theorem 2.10 for $g(t,x) \equiv 0$ and $\xi(x) \equiv x$. In this case, the scalar equation (2.7) is uniformly practically stable in terms of measures $h_0^*(u) = |u|$, and $h^*(t,u) = |u|$. □

3. Applications

Consider the following system of impulsive differential equations with "supremum":

$$\begin{gathered} x'(t) = y(t)\left(x^2(t) + y^2(t)\right)\sin^2 t + \sup_{s\in[t-r,t]} x(s), \\ y'(t) = -x(t)\left(x^2(t) + y^2(t)\right)\sin^2 t + \sup_{s\in[t-r,t]} y(s), \quad t \ge t_0,\ t \ne k, \\ x(k+0) = ax(k), \qquad y(k+0) = by(k), \end{gathered} \tag{3.1}$$

with initial conditions

$$x(t) = \phi_1(t - t_0), \quad y(t) = \phi_2(t - t_0) \quad \text{for } t \in [t_0 - r, t_0], \tag{3.2}$$

where $x, y \in \mathbb{R}$, $r > 0$ is small enough constant, $t_0 \ge 0$, and $a, b \in (1,2)$. Without loss of generality, we could assume that $t_0 < 1$.

Let $h_0(t,x,y) = |x| + \sqrt{2}|y|$, $h(t,x,y) = e^{-3t}(x^2 + y^2)$, $h_0^*(u) = |u|$, and $h^*(t,u) = e^{-3t}|u|$. Consider $V : \mathbb{R}^2 \to \mathbb{R}_+$, $V(x,y) = x^2 + 2y^2$. It is easy to check condition 3(i) of Theorem 2.10 for functions $a(s) = s^2 \in K$ and $b(s) = s \in K$.

Let $t \in \mathbb{R}_+, t \neq k, k = 1,2,\ldots$ and $\psi \in \mathrm{PC}([t-r,t],\mathbb{R}^2)$, $\psi = (\psi_1, \psi_2)$ be such that

$$\psi_1^2(t) + 2\psi_2^2(t) > \psi_1^2(t+s) + 2\psi_2^2(t+s) \quad \text{for } s \in [-r,0) \tag{3.3}$$

or $V(\psi_1(t), \psi_2(t)) > V(\psi_1(t+s), \psi_2(t+s))$.

Let $i = 1,2$. If there exists a point $\eta \in [t-r,t]$ such that $\sup_{s\in[t-r,t]}\psi_i(s) = \psi_i(\eta)$, then $(\sup_{s\in[t-r,t]}\psi_i(s))^2 = (\psi_i(\eta))^2 \leq \sup_{s\in[t-r,t]}(\psi_1^2(s) + 2\psi_2^2(s)) = \psi_1^2(t) + 2\psi_2^2(t)$.

The above inequality is also true if $\sup_{s\in[t-r,t]}\psi_i(s) > \psi_i(\eta)$ for all $\eta \in [-r,t]$; that is, there exists $k \in (t-r,t)$ such that $\sup_{s\in[t-r,t]}\psi_i(s) = \psi_i(k+0)$.

Then, for $i = 1,2$, we obtain

$$\begin{aligned} \psi_i(t) \sup_{s\in[t-r,t]} \psi_i(s) &\leq |\psi_i(t)| \left| \sup_{s\in[t-r,t]} \psi_i(s) \right| = \sqrt{(\psi_i(t))^2} \sqrt{\left(\sup_{s\in[t-r,t]} \psi_i(s) \right)^2} \\ &\leq \psi_1^2(t) + 2\psi_2^2(t) = V(\psi_1(t), \psi_2(t)). \end{aligned} \tag{3.4}$$

Therefore, if inequality (3.3) is fulfilled, then we have

$$\begin{aligned} D_{(3.1)} V(\psi_1(t), \psi_2(t)) &= \left(\psi_1(t) \sup_{s\in[t-r,t]} \psi_1(s) + 2\psi_2(t) \sup_{s\in[t-r,t]} \psi_2(s) \right) \\ &\leq 3V(\psi_1(t), \psi_2(t)). \end{aligned} \tag{3.5}$$

For any k, we obtain

$$V(ax, by) = \left(a^2x^2 + 2b^2y^2\right) \leq c^2\left(x^2 + 2y^2\right) = c^2 V(x,y), \tag{3.6}$$

where $c = \max(a,b) > 1$.

Now, consider the initial value problem for the scalar comparison impulsive differential equation

$$u' = 3u \quad \text{for } t \neq k, \qquad u(k+0) = c^2 u(k), \qquad u(t_0) = u_0. \tag{3.7}$$

The solution of the above initial value problem for impulsive differential equation is $u(t) = (\prod_{i=1}^{k}(c^2-1))u_0 e^{3(t-t_0)}$ for $t \in [k, k+1), k = 1,2,\ldots$. Let numbers $0 < \lambda < \sqrt{A}$ be given and $|u_0| < \lambda^2$. Then,

$$h^*(t, u(t)) = e^{-3t}|u| = \left(\prod_{i=1}^{k} \left(c^2 - 1\right) \right) |u_0| e^{-3t_0} \leq |u_0| < \lambda^2 \leq A, \tag{3.8}$$

that is, scalar comparison equation is uniformly practically stable in terms of measures (h_0^*, h^*). Therefore, according to Theorem 2.10, the system of impulsive differential equations with "supremum" (3.1) is uniformly practically stable in terms of two measures, that is, the inequality $\sup_{s\in[-r,0]}(|\phi_1(s)| + 2|\phi_2(s)|) < \lambda$ implies $e^{-3t}(x^2(t;t_0,\phi) + 2y^2(t;t_0,\phi)) < A$ for $t \geq t_0$.

4. Conclusions

Two types of sufficient conditions for practical stability of impulsive differential equations with “supremum” are obtained. The global case and the case on a ball are considered. Both types of results are based on the application of Lyapunov piecewise continuous functions and comparison results for scalar impulsive differential equations. To unify a variety of stability concepts and to offer a general framework for the investigation of the stability theory, the notion of stability in terms of two measures has been applied to both the given system and the comparison scalar equation. Note that in studying stability of differential equations everywhere in the literature two different measures are applied only to the given equation. Also, in the particular case of identity impulsive functions, that is, $I_k(x) \equiv x, k = 1, 2, \ldots$, the obtained results reduce to results for practical stability of differential equations with “maxima” which are also new ones. The obtained results are generalizations of the results for practical stability of ordinary differential equations (see [4, 6]), results for impulsive differential equations (see [4]), results for differential difference equations, and results for impulsive differential difference equations (see [2, 5]).

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