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Research Article

Global Mild Solutions and Attractors for Stochastic Viscous Cahn-Hilliard Equation

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This paper is devoted to the study of mild solutions for the initial and boundary value problem of stochastic viscous Cahn-Hilliard equation driven by white noise. Under reasonable assumptions we first prove the existence and uniqueness result. Then, we show that the existence of a stochastic global attractor which pullback attracts each bounded set in appropriate phase spaces.

1. Introduction

This paper is devoted to the existence of mild solutions and global asymptotic behavior for the following stochastic viscous Cahn-Hilliard equation:

$$d((1-\alpha)u-\alpha\Delta u)+\left(\Delta^2 u-\Delta f(u)\right)dt = dW, \quad (x,t)\in G\times(t_0,\infty), \tag{1.1}$$

subjected to homogeneous Dirichlet boundary conditions

$$u(x,t)=0, \qquad (x,t)\in \partial G\times[t_0,\infty), \tag{1.2}$$

in dimension $n = 1, 2$ or 3, where $G = \prod_{i=1}^{n}(0, L_i)$ in R^n, and $\alpha \in [0,1]$ is a parameter, f is a polynomial of odd degree with a positive leading coefficient

$$f(x) = \sum_{k=1}^{2p-1} a_k x^k, \ a_{2p-1} > 0. \tag{1.3}$$

In deterministic case, the model was first introduced by Novick-Cohen [1] to describe the dynamics of viscous first order phase transitions, which has been extensively studied in the past decades. The existence of global solutions and attractors are well known; moreover, the global attractor $\mathcal{A}_\alpha$ of the system has the same finite Hausdorff dimension for different parameter values α. One can also show that $\mathcal{A}_\alpha$ is continuous as α varies in $[0,1]$. See [2] for details and [1] for recent development.

While the deterministic model captures more intrinsic nature of phase transitions in binary, it ignores some random effects such as thermal fluctuations which are present in any material. In recent years, there appeared many interesting works on stochastic Cahn-Hilliard equations. Cardon-Weber [3] proved the existence of solution as well as its density for a class of stochastic Cahn-Hilliard equations with additive noise using an appropriate convolution semigroup (in the sense of that in [4]) posed on cubic domains. The authors in [5] derived the existence for a generalized stochastic Cahn-Hilliard equation in general convex or Lipschitz domains. The main novelty was the derivation of space-time Hölder estimates for the Greens kernel of the stochastic problem, by using the domains geometry, which can be very useful in many other circumstances. In [6], the asymptotic behavior for a generalized Cahn-Hilliard equation was studied, which can also act as a very good toy model for treating the stochastic case.

Instead of deterministic viscous Cahn-hilliard equation, here, we consider the general stochastic equation (1.1) which is affected by a space-time white noise. In such a case, new difficulties appear, and the resulting stochastic model must be treated in a different way. Fortunately, the rapidly growing theory of random dynamical systems provides an appropriate tool. Crauel and Flandoli [7] (see also Schmalfuss [8]) introduced the concept of a random attractor as a proper generalization of the corresponding deterministic global attractor which turns out to be very helpful in the understanding of the long-time dynamics for stochastic differential equations. In this present work, we first establish some existence results on mild solutions. Then, by applying the abstract theory on stochastic attractors mentioned above, we show that the system has global attractors in appropriate phase spaces.

In case $\alpha = 0$, (1.1) reduces to the stochastic Cahn-Hilliard equation which was studied in [9], where the authors obtain the existence and uniqueness of the weak solutions to the initial and Neumann boundary value problem in some phase spaces under appropriate assumptions on noise. Here, we make slightly stronger assumptions on noise and prove existence and uniqueness of mild solutions with higher regularity. Furthermore, we show the existence of random attractors in appropriate phase spaces.

This paper is organized as follows. In Section 2, we first make some preliminary works, then we state our main results. In Section 3, we consider the solutions of the the linear part of the system (1.1)-(1.2) and stochastic convolution. Regularities of solutions will also be addressed in this part. Section 4 consists of some investigations on the Stochastic Lyapunov functional of the system. The proofs on the existence results for mild solutions and global attractors will be given in Sections 5 and 6, respectively. Finally, the last section stands as an appendix for some basic knowledge of random dynamical system(RDS).

2. Preliminaries and Main Results

In this section, we first make some preliminary works, then we state explicitly our main results.

2.1. Functional Spaces

Let $(\cdot,\cdot)$ and $|\cdot|$ denote respectively the inner product and norm of $H = L^2(G)$. We define the linear operator $A = -\Delta$ with domain $D(A) = H^2(G) \bigcap H_0^1(G)$. A is positive and selfadjoint. By spectral theory, we can define the powers A^s and spaces $H_s = D(A^{s/2})$ with norms $|u|_s = |A^{s/2}u|$ for real s. Note that $H_0 = L^2(G)$. It is well known that H_s is a subspace of $H^s(G)$ and $|\cdot|_s$ is on $H^s(G)$ a norm equivalent to the usual one. Moreover, we have the following Poincare inequality and interpolation inequality:

$$|u|_{s_1} \le \lambda_1^{-(s_2-s_1)/2}|u|_{s_2}, \quad \forall s_1, s_2 \in R,\ s_1 < s_2,\ \forall u \in H_{s_2}, \tag{2.1}$$

$$|u|_{\sigma s_1+(1-\sigma)s_2} \le |u|_{s_1}^{\sigma}|u|_{s_2}^{1-\sigma}, \quad \sigma \in [0,1], \tag{2.2}$$

where λ_1 is the first eigenvalue of A.

We can define $A^{-1} : H \to D(A)$ to be the Green's operator for A. Thus,

$$v = A^{-1}w \Longleftrightarrow Av = w. \tag{2.3}$$

By Rellich's Theorem, we know that A^{-1} is compact, and $A : D(A) \to H$ is a linear and bounded operator. Finally, we introduce the invertible operator $B_\alpha : H_s \to H_s$, $s \in \mathbb{R}$ defined by

$$B_\alpha := \alpha I + (1-\alpha)A^{-1}. \tag{2.4}$$

For each $\alpha \in (0,1]$ and $\beta \ge 0$, we know that $B_\alpha^\beta : H_s \to H_s$ is bounded and has a bounded inverse (see [10, 11]). We also define the operator $A_\alpha := B_\alpha^{-1}A$ with domain

$$D(A_\alpha) = \begin{cases} D(A) & \text{if } \alpha > 0, \\ D(A_0) = H_4. \end{cases} \tag{2.5}$$

By definition, it is clear that $D(A_\alpha^{s/2}) = H_s$ in case $\alpha > 0$.

Lemma 2.1. *For $\alpha > 0$, there exist M_1, M_2, and M_3 such that*

$$\alpha^{1/2}|v| \le |v|_{B_\alpha} \le M_1^{1/2}|v|, \quad v \in H, \tag{2.6}$$

$$\alpha^{1/2}|v|_1 \le |v|_{1,B_\alpha} \le M_2^{1/2}|v|_1, \quad v \in H_1, \tag{2.7}$$

$$\left(\frac{\lambda_1}{\alpha\lambda_1 + 1 - \alpha}\right)^{1/2}|v| \le |v|_{B_\alpha^{-1}} \le M_3^{1/2}|v|, \quad v \in H, \tag{2.8}$$

where

$$|v|_{B_\alpha} := (v, B_\alpha v)^{1/2},$$
$$|v|_{1,B_\alpha} := \left(A^{1/2}v, B_\alpha A^{1/2}v\right)^{1/2}, \tag{2.9}$$
$$|v|_{B_\alpha^{-1}} := \left(v, B_\alpha^{-1}v\right)^{1/2}.$$

Proof. Here, we only verify (2.8) is valid; the proofs of (2.6) and (2.7) can be found in [11]. Since $B_\alpha^{-1/2}$ is bounded, there exists $M_3 \geq 0$, such that $|B_\alpha^{-1/2}|^2 \leq M_3$. Then, for any $v \in H$, we have

$$\left(v, B_\alpha^{-1}v\right) = \left(B_\alpha^{-1/2}v, B_\alpha^{-1/2}v\right) = \left|B_\alpha^{-1/2}u\right|^2 \leq M_3|v|^2, \tag{2.10}$$

which completes the right part of (2.8).

Now, we proof the left part of (2.8) let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \tag{2.11}$$

denote the eigenvalues of A, repeated with the respective multiplicity, and the corresponding unit eigenvector is denoted by $\{w_k\}_{k=1}^{\infty}$, which forms an orthonormal basis for H. We have

$$\left(w_k, B_\alpha^{-1}w_k\right) = \frac{\lambda_k}{\alpha\lambda_k + 1 - \alpha} \geq \frac{\lambda_1}{\alpha\lambda_1 + 1 - \alpha}, \quad k \in (\mathbb{Z}^+). \tag{2.12}$$

Since $v \in H$, there exist $\{b_k\}_{k=1}^{\infty} \subset \mathbb{R}$, such that $v = \sum_{k=1}^{+\infty} b_k w_k$. Consequently,

$$\begin{aligned}\left(v, B_\alpha^{-1}v\right) &= \left(\sum_{k=1}^{+\infty} b_k w_k, B_\alpha^{-1}\sum_{k=1}^{+\infty} b_k w_k\right) = \sum_{k=1}^{+\infty}\left(b_k w_k, B_\alpha^{-1} b_k w_k\right) \\ &= \sum_{k=1}^{+\infty} \frac{\lambda_k}{\alpha\lambda_k + 1 - \alpha} b_k^2 \geq \frac{\lambda_1}{\alpha\lambda_1 + 1 - \alpha}\sum_{k=1}^{+\infty} b_k^2 \\ &= \frac{\lambda_1}{\alpha\lambda_1 + 1 - \alpha}|v|^2,\end{aligned} \tag{2.13}$$

which finishes the proof. □

2.2. Assumptions on the Noise

The stochastic process $W(t)$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, is a two-side in time Wiener process on H which is given by the expansions

$$W(t) = \sum_{k=0}^{\infty} \sqrt{\alpha_k}\beta_k(t) w_k, \tag{2.14}$$

where $\{w_k\}_{k=1}^{\infty}$ is a basis of H consisting of unit eigenvectors of A, $\{\alpha_k\}_{k=1}^{\infty}$ is a bounded sequence of nonnegative numbers, and

$$\beta_k(t) = \frac{1}{\sqrt{\alpha_k}}(W(t), w_k), \quad k \in \mathbb{N} \tag{2.15}$$

is a sequence of mutually independent real valued standard Brownian motions in a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

For convenience, we will define the *covariance operator* Q on H as follows:

$$Qw_k = \alpha_k w_k, \quad k \in \mathbb{N}. \tag{2.16}$$

The process $W(t)$ will be called as the *Q-Wiener process*. We need to impose on Q one of the following assumptions:

(**Q1**) $\mathrm{Tr}[B_\alpha^{-1-\delta} A^{-2+\delta} Q] < \infty$ (for some $0 < \delta \leq 1$),

(**Q1***) $\mathrm{Tr}[B_\alpha^{-2} A^{-1} Q] < \infty$, and $\mathrm{Tr}[B_\alpha^{-2} A^{-2} Q] \leq 2D$,

(**Q2**) $\mathrm{Tr}[B_\alpha^{-1-\delta} A^{-1+\delta} Q] < \infty$ (for some $0 < \delta \leq 1$), $\mathrm{Tr}[B_\alpha^{-2} Q] < \infty$, and $\mathrm{Tr}[B_\alpha^{-2} A^{-2} Q] \leq 2D$,

(**Q2***) $\mathrm{Tr}[B_\alpha^{-1-\delta} A^{-1+\delta} Q] < \infty$, $\mathrm{Tr}[B_\alpha^{-2} A^{\sigma} Q] < \infty$ (for some $0 < \delta \leq 1$ and $\sigma > 0$), and $\mathrm{Tr}[B_\alpha^{-2} A^{-2} Q] \leq 2D$,

where D is given in Section 4. It is obvious that

$$(\mathbf{Q2}^*) \Longrightarrow (\mathbf{Q2}), \qquad (\mathbf{Q1}^*) \Longrightarrow (\mathbf{Q1}). \tag{2.17}$$

2.3. Main Results

We will assume throughout the paper that the space dimension n and the integer p in (1.3) satisfy the following *growth condition*:

$$p = \begin{cases} \text{any positive integer}, & \text{if } n = 1 \text{ or } 2, \\ 2, & \text{if } n = 3. \end{cases} \tag{2.18}$$

Under the above assumptions on the noise, we can now put the original problem (1.1)-(1.2) in an abstract form

$$du + \left(A_\alpha u + B_\alpha^{-1} f(u)\right) dt = B_\alpha^{-1} A^{-1}\, dW, \tag{2.19}$$

with which we will also associate the following initial condition:

$$u(t_0) = u_0. \tag{2.20}$$

Note that since B_α^{-1} is bounded from H_s into itself for each $\alpha > 0$, (2.19) is qualitatively of second order in space for $\alpha > 0$ although it also has a nonlocal character. In contrast, for $\alpha = 0$

the equation is of fourth-order in space and local in character. Thus, $\alpha = 0$ is a singular limit for the equation.

Definition 2.2. Let $I := [t_0, t_0 + \tau)$ be an interval in $\mathbb{R}$. We say that a stochastic process $u(t, \omega; t_0, u_0)$ is a mild solution of the system (2.19)-(2.20) in H_s, if

$$u(\cdot, \omega; t_0, u_0) \in C(I;\ H_s), \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \tag{2.21}$$

moreover, it satisfies in H_s the following integral equation:

$$u(t, \omega; t_0, u_0) = e^{-A_\alpha(t-t_0)} v_0 - \int_{t_0}^{t} e^{-A_\alpha(t-s)} \left(B_\alpha^{-1} f(u) - \beta W_A(s) \right) ds + W_A(t), \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \tag{2.22}$$

where $W_A(t)$ is called stochastic convolution (see Section 3 for details), β is a positive constant chosen in Section 3 and $v_0 = u_0 - W_A(t_0)$.

The main results of the paper are contained in the following two theorems.

Theorem 2.3. *(i) Let $\alpha = 0$, and, the hypothesis* (**Q2**) *be satisfied. Then for every $u_0 \in H_2$, there is a unique maximally defined mild solution $u(t, \omega; t_0, u_0)$ of* (2.19)-(2.20) *in H_2 for all $t \in [t_0, \infty)$.*

(ii) Let $\alpha \in (0, 1]$, and, the hypothesis (**Q1**) *be satisfied. Then for every $u_0 \in H_1$, there is a unique maximally defined mild solution $u(t, \omega; t_0, u_0)$ of* (2.19)-(2.20) *in H_1 for all $t \in [t_0, \infty)$.*

Theorem 2.4. *(i) Let $\alpha = 0$, and, the hypothesis* (**Q2***) *be satisfied. Then the stochastic flow associated with* (2.19)-(2.20) *has a compact stochastic attractor $\mathcal{A}_0(\omega) \subset H_2$ at time 0, which pullback attracts every bounded deterministic set $B \subset H_2$.*

(ii) Let $\alpha \in (0, 1]$, and, the hypothesis (**Q1***) *be satisfied. Then the stochastic flow associated with* (2.19)-(2.20) *has a compact stochastic attractor $\mathcal{A}_\alpha(\omega) \subset H_1$ at time 0, which pullback attracts every bounded deterministic set $B \subset H_1$.*

3. Stochastic Convolution

Let $W_A(t)$ be the unique solution of linear equation

$$du + (A_\alpha + \beta) u dt = B_\alpha^{-1} A^{-1}\, dW, \tag{3.1}$$

where β is a positive constant to be further determined. Then, $W_A(t)$ is an ergodic and stationary process [9, 12] called the stochastic convolution. Moreover,

$$W_A(t) = \int_{-\infty}^{t} e^{-(t-s)(A_\alpha + \beta)} B_\alpha^{-1}\ A^{-1} dW(s). \tag{3.2}$$

Some regularity properties satisfied by $W_A(t)$ are given below.

Lemma 3.1. *Assume that* (**Q1**) *holds. Then, $\nabla W_A(t)$ has a version which is γ-Hölder continuous with respect to $(t, x) \in \mathbb{R} \times G$ for any $\gamma \in [0, \delta/2)$.*

Proof. We only consider the case $n = 3$. For the sake of simplicity, we also assume that $G = \prod_{i=1}^{3}(0,\pi)$. The eigenvectors of A can be given explicitly as follows:

$$w_k(x) = \left(\frac{2}{\pi}\right)^{3/2} \cos k_1 x_1 \cos k_2 x_2 \cos k_3 x_3, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{3.3}$$

with corresponding eigenvalues

$$\mu_k = k_1^2 + k_2^2 + k_3^2 = |k|^2, \quad k \in (\mathbb{Z}^+)^3, \tag{3.4}$$

where $k = (k_1, k_2, k_3)$ varies in $(\mathbb{Z}^+)^3$. Using (2.14), we find that

$$W_A(t,x) = \sum_{k\in(\mathbb{Z}^+)^3} \left(\sqrt{\alpha_k} \int_{-\infty}^{t} e^{-(t-s)(\eta_k+\beta)} \frac{1}{\alpha\mu_k + 1 - \alpha} d\beta_k(s) \right) w_k(x), \tag{3.5}$$

where $\eta_k = \mu_k^2/(\alpha\mu_k + 1 - \alpha)$, and hence,

$$\begin{aligned}
&\nabla W_A(t,x) - \nabla W_A(t,y) \\
&\quad = \sum_{k\in(\mathbb{Z}^+)^3} \left(\sqrt{\alpha_k} \int_{-\infty}^{t} e^{-(t-s)(\eta_k+\beta)(1/(\alpha\mu_k+1-\alpha))} d\beta_k(s) \right) (\nabla w_k(x) - \nabla w_k(y)), \\
&\mathbf{E}\left(\left| \nabla W_A(t,x) - \nabla W_A(t,y) \right|^2 \right) \\
&\quad \le \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} \int_{-\infty}^{t} e^{-2(t-s)(\eta_k+\beta)} ds \left| \nabla w_k(x) - \nabla w_k(y) \right|^2.
\end{aligned} \tag{3.6}$$

For any $\gamma \in [0,1]$, one trivially verifies that there is a constant $c_\gamma > 0$ independent of k such that for any $k \in (\mathbb{Z}^+)^3$ and $x, y \in G$

$$\left| \nabla w_k(x) - \nabla w_k(y) \right| \le c_\gamma \mu_k^{(1+\gamma)/2} |x - y|^\gamma. \tag{3.7}$$

Thus, we have

$$\begin{aligned}
\mathbf{E}\left(\left| \nabla W_A(t,x) - \nabla W_A(t,y) \right|^2 \right) &\le \frac{c_\gamma^2}{2} |x - y|^{2\gamma} \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} \eta_k^{-1} \mu_k^{1+\gamma} \\
&= \frac{c_\gamma^2}{2} |x - y|^{2\gamma} \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} \frac{\alpha\mu_k + 1 - \alpha}{\mu_k^2} \mu_k^{1+\gamma} \\
&= \frac{c_\gamma^2}{2} |x - y|^{2\gamma} \sum_{k\in(\mathbb{Z}^+)^3} \alpha_k \frac{\mu_k}{\alpha\mu_k + 1 - \alpha} \mu_k^{-2+\gamma}.
\end{aligned} \tag{3.8}$$

Now, let $t, s \in \mathbb{R}$. We may assume that $t \geq s$. Then,

$$\begin{aligned}
&\mathbf{E}\left(|\nabla W_A(t,x) - \nabla W_A(s,x)|^2\right) \\
&\quad = \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} \\
&\qquad \times \left(\int_s^t e^{-2(\eta_k+\beta)(t-\sigma)} d\sigma + \int_{-\infty}^s \left[e^{-(\eta_k+\beta)(t-\sigma)} - e^{-(\eta_k+\beta)(s-\sigma)}\right]^2 d\sigma\right) |\nabla w_k(x)|^2 \\
&\quad = \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} \frac{1}{2(\eta_k + \beta)} \left(1 - e^{-2(\eta_k+\beta)(t-s)}\right) \cdot |\nabla w_k(x)|^2.
\end{aligned} \tag{3.9}$$

Let $0 \leq \gamma \leq 1/2$, and let

$$c'_\gamma = \sup_{r_1, r_2 \geq 0} \frac{|e^{-r_1} - e^{-r_2}|}{|r_1 - r_2|^{2\gamma}}. \tag{3.10}$$

Since the function $g(r) = e^{-r}$ is a Lipschitzoneon $[0, \infty)$, we always have $c'_\gamma < \infty$. Observe that

$$\begin{aligned}
&\mathbf{E}\left(|\nabla W_A(t,x) - \nabla W_A(s,x)|^2\right) \\
&\quad \leq \frac{4^\gamma}{\pi^3} c'_\gamma |t-s|^{2\gamma} \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} (\eta_k + \beta)^{2\gamma-1} \mu_k. \\
&\quad \leq \frac{4^\gamma}{\pi^3} c'_\gamma |t-s|^{2\gamma} \sum_{k\in(\mathbb{Z}^+)^3} \frac{\alpha_k}{(\alpha\mu_k + 1 - \alpha)^2} \eta_k^{2\gamma-1} \mu_k \\
&\quad = \frac{4^\gamma}{\pi^3} c'_\gamma |t-s|^{2\gamma} \sum_{k\in(\mathbb{Z}^+)^3} \alpha_k \left(\frac{\mu_k}{\alpha\mu_k + 1 - \alpha}\right)^{2\gamma+1} \mu_k^{-2+2\gamma}.
\end{aligned} \tag{3.11}$$

By (**Q1**), we know that $\mathrm{Tr}[B_\alpha^{-1-\delta} A^{-2+\delta} Q] < \infty$ for some $0 < \delta \leq 1$. Therefore, by (3.8) and (3.11), one deduces that there exists a constant $c''_\gamma > 0$ such that

$$\mathbf{E}\left(\left|\nabla W_A(t,x) - \nabla W_A(s,y)\right|^2\right) \leq c''_\gamma \left(\left|x-y\right|^2 + |t-s|^2\right)^\gamma, \quad \forall (t,x), (s,y) \in \mathbb{R} \times G. \tag{3.12}$$

As $W_A(t,x) - W_A(s,y)$ is a Gaussian process, we find that for each $m \in \mathbb{Z}^+$, there is a constant $c_\gamma^m > 0$ such that

$$\mathbf{E}\left(\left|\nabla W_A(t,x) - \nabla W_A(s,y)\right|^{2m}\right) \leq c_\gamma^m \left(\left|x-y\right|^2 + |t-s|^2\right)^{m\gamma}. \tag{3.13}$$

Now, thanks to the well-known Kolmogorov test, one concludes that $W_A(t,x)$ is $(\gamma - 2/m)$-Hölder continuous in (t,x). Because $\gamma \in [0, 1/2]$ and $m \in \mathbb{Z}^+$ are arbitrary, we see that the conclusion of the lemma holds true. The proof is complete. □

Lemma 3.2. *Assume* (**Q2**) *holds. Then, for any* $M > 0$*, there exists a* β_0 *such that for all* $\beta \ge \beta_0$*,*

$$\mathbf{E}\left(|W_A(t)|_2^2\right) \le M. \tag{3.14}$$

Proof.

$$\begin{aligned}
\mathbf{E}\left(|\Delta W_A(t)|^2\right) &= \mathbf{E}\left(\sum_{k\in(\mathbb{Z}^+)^3}\sqrt{\alpha_k}\int_{-\infty}^{t} e^{-(\eta_k+\beta)(t-s)}\frac{1}{\alpha\mu_k+1-\alpha}d\beta_k(s)\Delta w_k(x)\right)^2 \\
&= \sum_{k\in(\mathbb{Z}^+)^3}\frac{\alpha_k}{(\alpha\mu_k+1-\alpha)^2}\int_{-\infty}^{t} e^{-2(\eta_k+\beta)(t-s)}ds|\Delta w_k(x)|^2 \\
&\le \sum_{k\in(\mathbb{Z}^+)^3}\frac{\alpha_k}{(\alpha\mu_k+1-\alpha)^2}\frac{1}{2(\eta_k+\beta)}|\Delta w_k(x)|^2 \\
&\le \frac{1}{2(\eta_1+\beta)}\sum_{k\in(\mathbb{Z}^+)^3}\alpha_k\left(\frac{\mu_k}{\alpha\mu_k+1-\alpha}\right)^2.
\end{aligned} \tag{3.15}$$

Since $\operatorname{Tr}[B_\alpha^{-2}Q] < \infty$, one can now easily choose a β large enough so that $\mathbf{E}(|\Delta W_A(t)|^2) \le M$, and the proof is complete. □

Similarly, we can verify the following basic fact.

Lemma 3.3. *Assume* (**Q2**) *holds. Then,* ΔW_A *has a version which is* γ*-Hölder continuous with respect to* $(t,x) \in \mathbb{R}\times G$ *for any* $\gamma \in [0,\delta/2)$*.*

Lemma 3.4. *Assume that* (**Q2***) *holds. Then, for any* $M > 0$*, there exists* β_0 *such that for all* $\beta \ge \beta_0$*,*

$$\mathbf{E}\left(|W_A(t)|_{2+\sigma}^2\right) \le M. \tag{3.16}$$

4. Stochastic dissipativeness in H_1

It is well known that in the deterministic case without forcing terms,

$$J(u) = \frac{1}{2}|\nabla u|^2 + \int_G F(u)dx \tag{4.1}$$

is a Lyapunov functional of the system (i.e. $(d/dt)J(u) \le 0$), where $F(u)$ is the primitive function of $f(u)$ which vanishes at zero. In this section, we will prove a similar property for the stochastic equation by adapting some argument in [9].

Assume that u satisfies (2.19)-(2.20). As usual, we may assume in advance that u is sufficiently regular so that all the computations can be performed rigorously. Applying the Itô formula to $J(u)$, we obtain

$$\begin{aligned} dJ(u) &= (J_u(u), du) + \frac{1}{2}\operatorname{Tr}\left[J_{uu}(u)B_\alpha^{-2}A^{-2}Q\right]dt \\ &= \left(J_u(u), B_\alpha^{-1}A^{-1}\,dW\right) - \left(J_u(u), B_\alpha^{-1}Au + B_\alpha^{-1}f(u)\right)dt + \frac{1}{2}\operatorname{Tr}\left[J_{uu}(u)B_\alpha^{-2}A^{-2}Q\right]dt, \end{aligned} \tag{4.2}$$

where J_u, J_{uu} denote, respectively, the first and second derivative of J. Since

$$J_u(u) = Au + f(u), \tag{4.3}$$

there exists $C_1 > 0$ such that for $\alpha = 0$,

$$\begin{aligned} \left(J_u(u),\ B_\alpha^{-1}Au + B_\alpha^{-1}f(u)\right) &= \left|Au + f(u)\right|_1^2 \geq \lambda_1^2\left|Au + f(u)\right|_{-1}^2 \\ &= \lambda_1^2\left(Au + f(u),\ u + A^{-1}f(u)\right) \\ &= \lambda_1^2\left(|u|_1^2 + \left|f(u)\right|_{-1}^2 + 2(f(u),\ u)\right) \\ &\geq d\lambda_1^2\left(|u|_1^2 + \int_G F(u)dx\right) - C_1 = d\lambda_1^2 J(u) - C_1, \end{aligned} \tag{4.4}$$

where $d = \min\{1,\ 4pa_{2p-1}\}$. And for $0 < \alpha \leq 1$,

$$\begin{aligned} \left(J_u(u),\ B_\alpha^{-1}Au + B_\alpha^{-1}f(u)\right) &= \left(Au + f(u), B_\alpha^{-1}Au + B_\alpha^{-1}f(u)\right) \\ &= \left|Au + f(u)\right|_{B_\alpha^{-1}}^2 \\ &\geq \frac{\lambda_1^2}{\alpha\lambda_1 + 1 - \alpha}\left|Au + f(u)\right|_{-1}^2 \\ &\geq \frac{d\lambda_1^2}{\alpha\lambda_1 + 1 - \alpha}J(u) - C_1, \end{aligned} \tag{4.5}$$

where we have used (2.8). Simple computations show that

$$J_{uu}(u) = A + f'(u), \tag{4.6}$$

and hence,

$$\begin{aligned}\operatorname{Tr}\left[J_{uu}(u)B_\alpha^{-2}A^{-2}Q\right] &= \operatorname{Tr}\left[AB_\alpha^{-2}A^{-2}Q\right] + \sum_{i=1}^{\infty}\left(D_i\int_G f'(u)w_i^2\,dx\right) \\ &= \operatorname{Tr}\left[B_\alpha^{-2}A^{-1}Q\right] + \sum_{i=1}^{\infty}\left(D_i\int_G f'(u)w_i^2\,dx\right),\end{aligned} \tag{4.7}$$

where $\{w_i\}_{i=1}^{\infty}$ is the orthonormal basis of H as in (2.14), and $D_i = \alpha_i/(\alpha\lambda_i + 1 - \alpha)^2$.

We infer from (3.3) that

$$|w_i|_{L\infty} \le C_2, \tag{4.8}$$

where $C_2 > 0$ depends only on G. Therefore,

$$\left|\int_G f'(u)w_i^2\,dx\right| \le C_2^2\int_G \left|f'(u)\right|dx. \tag{4.9}$$

Set C_3 such that

$$\left|f'(s)\right| \le 2(2p-1)a_{2p-1}s^{2p-2} + C_3, \quad s \in \mathbb{R}, \tag{4.10}$$

then

$$\begin{aligned}\left|\int_G f'(u)w_i^2\,dx\right| &\le C_2^2\left(2(2p-1)a_{2p-1}\int_G u^{2p-2}\,dx + C_3|G|\right) \\ &\le \frac{1}{4p}a_{2p-1}\int_G u^{2p}\,dx + C_4,\end{aligned} \tag{4.11}$$

where C_4 depends on f, p, and G. Let C_5 satisfy

$$F(s) \ge \frac{1}{4p}a_{2p-1}s^{2p} - \frac{C_5}{|G|}, \quad s \in \mathbb{R}, \tag{4.12}$$

then

$$\left|\int_G f'(u)w_i^2\,dx\right| \le J(u) + C_4 + C_5. \tag{4.13}$$

Finally,

$$\operatorname{Tr}\left[J_{uu}(u)B_\alpha^{-2}A^{-2}Q\right] \le \operatorname{Tr}\left[B_\alpha^{-2}A^{-1}Q\right] + \operatorname{Tr}\left[B_\alpha^{-2}A^{-2}Q\right](J(u) + C_4 + C_5). \tag{4.14}$$

Since

$$\mathbf{E}\Big(J_u(u), B_\alpha^{-1}A^{-1}\,dW\Big) = 0, \tag{4.15}$$

we have from (4.2) that

$$\frac{d}{dt}\mathbf{E}(J(u)) = \mathbf{E}\Big(J_u(u), -B_\alpha^{-1}A(u) - B_\alpha^{-1}(u)\Big) + \frac{1}{2}\mathbf{E}\Big(\mathrm{Tr}\Big[J_{uu}(u)B_\alpha^{-2}A^{-2}Q\Big]\Big). \tag{4.16}$$

Further, by (4.4), (4.5) and (4.14), it holds that

$$\begin{aligned}&\left(\frac{d}{dt}\right)\mathbf{E}(J(u))\\ &\quad\le -\left(D - \frac{1}{2}\mathrm{Tr}\Big[B_\alpha^{-2}A^{-2}Q\Big]\right)\mathbf{E}(J(u)) + \mathrm{Tr}\Big[B_\alpha^{-2}A^{-1}Q\Big] + \mathrm{Tr}\Big[B_\alpha^{-2}A^{-2}Q\Big](C_4 + C_5) + C_1,\end{aligned} \tag{4.17}$$

where $D = \min\{d\lambda_1^2, d\lambda_1^2/(\alpha\lambda_1 + 1 - \alpha)\}$. This is precisely what we promised.

Now, by directly applying the classical Gronwall Lemma, we have the following lemma.

Lemma 4.1. *Let W be a H-valued Q-Wiener process with*

$$\mathrm{Tr}\Big[B_\alpha^{-2}A^{-1}Q\Big] < +\infty, \qquad \mathrm{Tr}\Big[B_\alpha^{-2}A^{-2}Q\Big] \le 2D, \tag{4.18}$$

and let $u(t)$ be the mild solution to (2.19)-(2.20). *Then,*

$$\mathbf{E}(J(u(t))) \le \mathbf{E}(J(u_0)) + C_Q, \quad t \in [t_0, \infty), \tag{4.19}$$

where

$$C_Q = \frac{\mathrm{Tr}\big[B_\alpha^{-2}A^{-1}Q\big] + \mathrm{Tr}\big[B_\alpha^{-2}A^{-2}Q\big](C_4 + C_5) + C_1}{D - (1/2)\,\mathrm{Tr}\big[B_\alpha^{-2}A^{-2}Q\big]}. \tag{4.20}$$

As a consequence, we immediately obtain the following basic result.

Corollary 4.2. *Let W be a H-valued Q-Wiener process with*

$$\mathrm{Tr}\Big[B_\alpha^{-2}A^{-1}Q\Big] < +\infty, \quad \mathrm{Tr}\Big[B_\alpha^{-2}A^{-2}Q\Big] \le 2D. \tag{4.21}$$

Then, there exists a continuous nonnegative function $\Psi(r)$ such that for any solution $u(t)$ of (2.19)-(2.20), *one has*

$$\mathbf{E}\Big(|u(t)|_1^2\Big) \le \Psi\Big(\mathbf{E}\Big(|u_0|_1^2\Big)\Big), \quad \forall t \in [t_0, \infty). \tag{4.22}$$

5. The Existence and Unique of Global Mild Solutions

In this section, we study the existence and unique of global mild solutions of the problem (2.19)-(2.20). The basic idea is to transform the original problem into a nonautonomous one by using the simple variable change below:

$$v(t) = u(t) - W_A(t). \tag{5.1}$$

We observe that $v(t)$ satisfies the following system:

$$\begin{gathered} \frac{dv}{dt} + \left(A_\alpha - \beta\right)v + B_\alpha^{-1} f(v + W_A) = 0, \\ v(t_0) = u_0 - W_A(t_0). \end{gathered} \tag{5.2}$$

Let

$$G(v,t) = -B_\alpha^{-1} f(v + W_A) + \beta W_A, \qquad v_0 = u_0 - W_A(t_0). \tag{5.3}$$

Then, (5.2) reads

$$\begin{gathered} \frac{dv}{dt} + A_\alpha v = G(v,t), \\ v(t_0) = v_0. \end{gathered} \tag{5.4}$$

To prove Theorem 2.3, it suffices to establish some corresponding existence results for the nonautonomous system (5.4).

Definition 5.1. Let $I := [t_0, t_0 + \tau)$ be an interval in $\mathbb{R}$. We say that a stochastic process $v(t, \omega; t_0, v_0)$ is a mild solution of the system (5.4) in H_s, *if*

$$v(\cdot, \omega; t_0, v_0) \in C(I;\ H_s), \quad \mathbf{P}\text{-a.s. } \omega \in \Omega, \tag{5.5}$$

and satisfies in H_s the following integral equation:

$$v(t, \omega; t_0, v_0) = e^{-A_\alpha(t-t_0)} v_0 - \int_{t_0}^{t} e^{-A_\alpha(t-s)} \Big(B_\alpha^{-1} f(u) - \beta W_A(s)\Big) ds, \quad \mathbf{P}\text{-a.s. } \omega \in \Omega. \tag{5.6}$$

Theorem 5.2. *Let $\alpha = 0$. Suppose that the Hypothesis* **(Q2)** *is satisfied.*

Then, for every $u_0 \in H_2$, there is a unique globally defined mild solution $v(t, \omega; t_0, v_0)$ of (5.4) *in H_2 with*

$$v(t, \omega; t_0, v_0) \in C([t_0, \infty);\ H_2) \cap C_{\text{loc}}^{0,1-r}((t_0,); H_{4r}) \cap C((t_0, \infty); H_4), \tag{5.7}$$

for all $0 \le r < 1$.

Proof. We only consider the case where $n = 3$. First, it is easy to verify that $\mathbf{P}$-a.s.

$$G(v,t) \in C_{\mathrm{Lip};\,\gamma}(H_2 \times [t_0,\infty), H). \tag{5.8}$$

Indeed, by Lemma 3.3, we see that $W_A(t) \in H_2$ is γ-Hölder continuous with respect to $t \in \mathbb{R}$ $\mathbf{P}$-a.s. Recall that f is a polynomial of degree $2p-1$ with $p = 2$ (in case $n = 3$). One deduces that there exist C_1, $C_2(\omega) > 0$ such that

$$\begin{aligned} |(G(v_1,t_1) - G(v_2,t_2))| &\le C_1(|v_1 - v_2|_2 + |W_A(t_1) - W_A(t_2)|_2) \\ &\le C_2(\omega)\left(|v_1 - v_2|_2 + |t_1 - t_2|^{\gamma}\right), \quad \mathbf{P}\text{-a.s.} \end{aligned} \tag{5.9}$$

It then follows from [11, Lemma 47.4] that there is a unique maximally defined mild solution v of (5.4) in H_2 on $[t_0, T)$ satisfying $\mathbf{P}$-a.s.

$$\begin{gathered} v(t,\omega;t_0,v_0) = e^{-A^2(t-t_0)}v_0 - \int_{t_0}^{t} e^{-A^2(t-s)}(Af(u(s)) - \beta W_A(s))ds, \\ v(t,\omega;t_0,v_0) \in C([t_0,T);\ H_2) \cap C^{0,1-r}_{\mathrm{loc}}((t_0,T);H_{4r}) \cap C((t_0,T);H_4), \end{gathered} \tag{5.10}$$

for all $0 \le r < 1$. Furthermore, we also know that v is a strong solution in H_2. Hence, it satisfies in the strong sense that

$$\frac{dv}{dt} + A^2 v + Af(u) - \beta W_A = 0, \qquad v(t_0) = v_0. \tag{5.11}$$

In what follows, we show $T = \infty$, thus proving the theorem.

Simple computations yields

$$\left|\Delta f(u)\right| \le \left|f'(u)\right|_{L^\infty}|\Delta u| + \left|f''(u)\right|_{L^\infty}|\nabla u|^2_{L^4}. \tag{5.12}$$

Since f is a polynomial of degree 3, there exist κ_1 and κ_2 such that

$$\left|f'(s)\right| \le \kappa_1\left(1 + |s|^2\right), \quad \left|f''(s)\right| \le \kappa_2(1 + |s|), \quad \forall s \in \mathbb{R}. \tag{5.13}$$

Therefore,

$$\begin{aligned} \left|f'(u)\right|_{L^\infty}|\Delta u| &\le \kappa_1\left(1 + |u|^2_{L^\infty}\right)|\Delta u| \\ &\le 2\kappa_1\left(1 + |v|^2_{L^\infty} + |W_A|^2_{L^\infty}\right)(|\Delta v| + |\Delta W_A|). \end{aligned} \tag{5.14}$$

By the Nirenberg-Gagiardo inequality, there exist $C_3,\ C_4,\ C_5 > 0$ such that

$$|u|_{L^\infty}^2 \le C_3|\Delta u|^2, \quad u \in H_2,$$
$$|u|_{L^\infty}^2 \le C_4\left|\Delta^2 u\right|^{1/3}|u|_{L^6}^{5/3}, \quad u \in H_4, \tag{5.15}$$
$$|\Delta u| \le C_5\left|\Delta^2 u\right|^{1/2}|\nabla u|^{1/2}, \quad u \in H_4.$$

Hence,

$$\begin{aligned}\left|f'(u)\right|_{L^\infty}|\Delta u| &\le 2\kappa_1\left(1+|v|_{L^\infty}^2+|W_A|_{L^\infty}^2\right)(|\Delta v|+|\Delta W_A|)\\ &\le 2\kappa_1\left(1+C_4\left|\Delta^2 v\right|^{1/3}|v|_{L^6}^{5/3}+C_3|\Delta W_A|^2\right)\left(C_5|\nabla v|^{1/2}\left|\Delta^2 v\right|^{1/2}+|\Delta W_A|\right).\end{aligned} \tag{5.16}$$

By **(Q2)** and Lemma 3.2, we know that for **P**-a.s. $\omega \in \Omega$, there exists an $R_1(\omega) > 0$ such that $|\Delta W_A(t)| \le R_1(\omega)$ (for all $t \in \mathbb{R}$). On the other hand, by Lemma 3.2 and Corollary 4.2, we find that **P**-a.s. v is bounded in H_1. Thus, for **P**-a.s. $\omega \in \Omega$, there exist $C_6(\omega),\ C_7(\omega) > 0$ such that

$$|v|_{L^6}^{5/3} \le C_6(\omega), \qquad |\nabla v|^{1/2} \le C_7(\omega), \tag{5.17}$$

where the continuous imbedding $H^1 \hookrightarrow L^6$ is used. Consequently, we have

$$\left|f'(u)\right|_{L^\infty}|\Delta u| \le C_8(\omega)\left(1+\left|\Delta^2 v\right|^{1/3}+R_1(\omega)\right)\left(\left|\Delta^2 v\right|^{1/2}+R_1(\omega)\right), \quad \mathbf{P}\text{-a.s. } \omega \in \Omega. \tag{5.18}$$

Similarly for **P**-a.s. $\omega \in \Omega$, one easily deduces that there exists $C_9(\omega) > 0$ such that

$$\left|f''(u)\right|_{L^\infty}|\Delta u| \le C_9(\omega)\left(1+\left|\Delta^2 v\right|^{1/6}+R_1(\omega)\right)\left(\left|\Delta^2 v\right|^{1/4}+R_1(\omega)\right). \tag{5.19}$$

It then follows from (5.12) that for **P**-a.s. $\omega \in \Omega$,

$$\begin{aligned}\left|\Delta f(u)\right| &\le C_8(\omega)\left(1+\left|\Delta^2 v\right|^{1/3}+R_1(\omega)\right)\left(\left|\Delta^2 v\right|^{1/2}+R_1(\omega)\right)\\ &\quad + C_9(\omega)\left(1+L_3\left|\Delta^2 v\right|^{1/6}+R_1(\omega)\right)\left(\left|\Delta^2 v\right|^{1/4}+R_1(\omega)\right)\\ &\le C_{10}(\omega)\left(1+\left|\Delta^2 v\right|^{5/6}\right).\end{aligned} \tag{5.20}$$

Now, taking the L^2 inner-product of equation (5.11) with $\Delta^2 v$, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\Delta v|^2+\left|\Delta^2 v\right|^2 &\le \left|\int_G \Delta f(u)\Delta^2 v dx\right|+\beta\left|\int_G W_A\Delta^2 v dx\right| \\ &\le \frac{1}{4}\left|\Delta^2 v\right|^2+\left|\Delta f(u)\right|^2+\frac{1}{4}\left|\Delta^2 v\right|^2+\beta^2|W_A|^2 \\ &\le \frac{1}{2}\left|\Delta^2 v\right|^2+\left|\Delta f(u)\right|^2+\beta^2\lambda_1^{-2}|\Delta W_A|^2.\end{aligned} \tag{5.21}$$

By (5.20), we deduce that **P**-a.s.

$$\frac{d}{dt}|\Delta v|^2+\left|\Delta^2 v\right|^2 \le C_{11}(\omega)\left(1+\left|\Delta^2 v\right|^{5/3}\right). \tag{5.22}$$

Furthermore, by Young's inequality and $|\Delta^2 v|^2 \ge \lambda_1^2|\Delta v|^2$, we know that there exists $C_{12}(\omega) > 0$ such that **P**-a.s.

$$\frac{d}{dt}|\Delta v|^2 \le -\frac{\lambda_1^2}{2}|\Delta v|^2+C_{12}(\omega). \tag{5.23}$$

Applying the gronwall lemma on (5.23), one gets

$$|\Delta v|^2 \le \frac{2C_{12}(\omega)}{\lambda_1^2}, \quad \mathbf{P}\text{-a.s. } \omega\in\Omega. \tag{5.24}$$

This implies that the weak solution solution v does not blow up in finite time in the space H_2. Hence, $T(v_0)=\infty$, for all $u_0 \in H_2$. □

Theorem 5.3. *Let $\alpha \in (0,1]$, and let Hypothesis **(Q1)** be satisfied. Then, for every $u_0 \in H_1$, there is a unique maximally defined mild solution $v(t,\omega;t_0,v_0)$ of* (5.4) *in H_1 for all $t \in [t_0,\infty)$ with*

$$v(t,\omega;t_0,v_0)\in C([t_0,\infty);H_1)\cap C_{\text{loc}}^{0,1-r}((t_0,\infty);H_{2r})\cap C((t_0,\infty);H_2), \tag{5.25}$$

for $0 \le r < 1$.

Proof. As noted above, $-A_\alpha$ is a positive selfadjoint linear operator on H with compact resolvent. The negative operator $-A_\alpha$ generate an analytic semigroup $e^{-A_\alpha t}$. It is easy to verify by Lemma 3.1 that **P**-a.s.

$$G(v,t)\in C_{\text{Lip};\,\gamma}(H_1\times[t_0,\infty),H). \tag{5.26}$$

It then follows from [11, Lemma 47.4] that there is a unique maximally defined mild solution v of (5.4) in H_1 on $[t_0,T)$ with

$$v(t,\omega;t_0,v_0)\in C([t_0,T);H_1)\cap C_{\text{loc}}^{0,1-r}((t_0,T);H_{2r})\cap C((t_0,T);H_2), \tag{5.27}$$

where $t_0 < T = T(v_0) \leq \infty$, and $0 \leq r < 1$. Furthermore, v is a strong solution in H_1 and hence solves (5.4) in the strong sense. To complete the proof of the theorem, there remains to check that $T(v_0) = \infty$.

Equation (5.4) is equivalent to

$$B_\alpha \frac{dv}{dt} + Av = -f(v + W_A) + \beta B_\alpha W_A, \qquad v(t_0) = v_0. \tag{5.28}$$

Multiplying (5.28) by Av, one gets

$$\frac{1}{2}\frac{d}{dt}|v|_{1,B_\alpha}^2 + |v|_2^2 + \left(f(v + W_A), Av\right) + \beta(B_\alpha W_A, Av) = 0. \tag{5.29}$$

We observe that

$$\begin{aligned}\left(f(v + W_A), Av\right) &= \int_G \nabla f(v + W_A)\nabla v\, dx \\ &= \int_G f'(v + W_A)|\nabla v|^2\, dx + \int_G f'(v + W_A)\nabla W_A \nabla v\, dx.\end{aligned} \tag{5.30}$$

We take C_1' and C_2' such that

$$\begin{aligned} f'(x) &\geq \frac{2p-1}{2} \quad a_{2p-1}x^{2p-2} - C_1', \\ \left|f'(x)\right| &\leq 2(2p-1)a_{2p-1}\ x^{2p-2} + C'_2, \end{aligned} \tag{5.31}$$

for all $x \in \mathbb{R}$. Then,

$$\begin{aligned}\left(f(v + W_A), Av\right) \geq{}& \frac{2p-1}{2}a_{2p-1}\int_G |v + W_A|^{2p-2}|\nabla v|^2 dx - C_1'\int_G |\nabla v|^2 dx \\ &- 2(2p-1)a_{2p-1}\int_G |v + W_A|^{2p-2}|\nabla v||\nabla W_A| dx - C_2'\int_G |\nabla v||\nabla W_A| dx \\ \geq{}& \frac{1}{4}(2p-1)a_{2p-1}\int_G |v + W_A|^{2p-2}|\nabla v|^2 dx - 2C_1'\int_G |\nabla v|^2 dx \\ &- C_3'\left(\int_G |\nabla W_A|^{2p} dx + \int_G |\nabla W_A|^2 dx\right),\end{aligned} \tag{5.32}$$

where we have used Hölder's inequality, Young's inequality, and the appropriate imbeddings $H^1(G) \hookrightarrow L^r(G)$ in dimension $n = 1$ *or* 2 and 3. We also know by (2.6) that there exists $\alpha \leq C_\alpha \leq M_1$ such that

$$(B_\alpha W_A, Av) \leq C_\alpha\left(\int_G |\nabla W_A|^2\, dx + \int_G |\nabla v|^2 dx\right). \tag{5.33}$$

Combining the last two inequalities together, we deduce that there exists constants C_4', $C_5' > 0$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|v|_{1,B_\alpha}^2 + |v|_2^2 + \frac{1}{4}(2p-1)a_{2p-1}\int_G |v + W_A|^{2p-2}|\nabla v|^2 dx \\ &\quad\leq 2C_4'\int_G |\nabla v|^2 dx + C_5'\left(\int_G |\nabla W_A|^{2p}\, dx + \int_G |\nabla W_A|^2 dx\right).\end{aligned} \tag{5.34}$$

In view of (2.7), there exists $\alpha \leq C_\alpha' \leq M_1$ such that

$$\begin{aligned}&\frac{1}{2}C_\alpha'\frac{d}{dt}|v|_1^2 + |v|_2^2 + \frac{1}{4}(2p-1)a_{2p-1}\int_G |v + W_A|^{2p-2}|\nabla v|^2 dx \\ &\quad\leq 2C_4'|v|_1^2 + C_5'\left(\int_G |\nabla W_A|^{2p}\, dx + \int_G |\nabla W_A|^2 dx\right).\end{aligned} \tag{5.35}$$

Using the gronwall lemma on (5.35), the following inequality holds:

$$|v|_1^2 \leq 2e^{4C_4'/C_\alpha'}C'_4|v_0|_1^2 \ + \ 2e^{4C_4'/C_\alpha'}\int_{t_0}^t C_5'\left(\int_G |\nabla W_A(s)|^{2p} dx + \int_G |\nabla W_A(s)|^2 dx\right)ds. \tag{5.36}$$

Lemma 3.1 guarantees that **P**-a.s.

$$\int_{t_0}^t \int_G |\nabla W_A(s)|^{2p}\, dx ds < +\infty, \qquad \int_{t_0}^t \int_G |\nabla W_A(s)|^2\, dx ds < +\infty. \tag{5.37}$$

This and (5.36) implies that the mild solution v does not blow up in finite time in the space H_1. It follows that $T(v_0) = \infty$. The proof is complete. □

Remark 5.4. The conclusions in Theorem 2.3 are readily implied in the above two theorems.

6. Attractors for Stochastic Viscous Cahn-Hilliard Equation

For convenience of the reader, some basic knowledge of RDS are summarized in the Appendix at the end of this paper.

6.1. Stochastic Flows

Thanks to Theorem 2.3, the mapping $u_0 \mapsto u(t, \omega; t_0, u_0)$ defines a stochastic flow $S_\alpha(t, s; \omega)$,

$$S_\alpha(t, s; \omega)u_0 = u(t, \omega; s, u_0), \quad \alpha \in [0, 1]. \tag{6.1}$$

Notice that **P**-a.s.

(i) $S_\alpha(t,s;\omega) = S_\alpha(t,r;\omega) \circ S_\alpha(r,s;\omega)$, for all $s \le r \le t$,

(ii) $S_0(t,s;\omega)$ is continuous in H_2, and $S_\alpha(t,s;\omega)$ is continuous in H_1 for $0 < \alpha \le 1$.

6.2. Compactness Properties of Stochastic Flow $S_\alpha(t,s;\omega)$

Lemma 6.1. *(i) Under Assumption* (**Q2***)*, the stochastic flow* $S_0(t,s;\omega)$ *is uniformly compact at time 0. More precisely, for all* $B \subset H_2$ *bounded and each* $t_0 < 0$, $S_0(0,t_0;\omega)B$ *is v relatively compact in* H_2.

(ii) Under Assumption (**Q1***)*, the flow* $S_\alpha(t,s;\omega)$, $0 < \alpha \le 1$, *is uniformly compact at time 0. More precisely, for all* $B \subset H_1$ *bounded and each* $t_0 < 0$, $S_\alpha(0,t_0;\omega)B$ *is* **P***-a.s. relatively compact in* H_1.

Proof. (i) Let $B \subset H_2$ be a given bounded deterministic set. By Lemma 3.4, we know that for **P**-a.s. $\omega \in \Omega$, there exists $R_2(\omega) > 0$, such that $|W_A(t)|_{2+\sigma} \le R_2(\omega)$, $t \in \mathbb{R}$. Define $\widehat{B} = B \cup B_{2+\sigma}(0, R_2(\omega))$, where $B_{2+\sigma}(0,R_2(\omega))$ denotes the open ball centered at 0 with radius $R_2(\omega)$ in $H_{2+\sigma}$. Then, $\widehat{B} \subset H_2$ is **P**-a.s. bounded, and

$$S_0(0,t_0;\omega)B \subset \left\{ e^{A^2 t_0} v_0 - \int_{t_0}^{0} e^{A^2 s} G(v(s),s)ds + W_A(0),\ v_0 \in \widehat{B} \right\} \subset N_1 + N_2 + N_3 + N_4, \tag{6.2}$$

P-a.s., where

$$\begin{aligned} N_1 &= e^{A^2 t_0} \widehat{B}, \\ N_2 &= \left\{ \int_{-\delta}^{0} e^{A^2 s} G(v(s),s)ds,\ v_0 \in \widehat{B} \right\}, \\ N_3 &= e^{-A^2 \delta} \left\{ \int_{t_0}^{-\delta} e^{A^2 (s+\delta)} G(v(s),s)ds,\ v_0 \in \widehat{B} \right\}, \\ N_4 &= B_{2+\sigma}(0, R_2(\omega)), \end{aligned} \tag{6.3}$$

and δ is an arbitrary constant satisfying $0 < \delta < -t_0$. □

Since for $t > 0$ fixed the operator $e^{-A^2 t}$ is compact, we see that N_1, N_3, and N_4 are relatively compact sets in H_2. Now, we show that **P**-a.s. $S_0(0,t_0;\omega)B$ is relatively compact. To this end, we first give an estimate on the Kuratowski measure of $N_2 \subset H_2$.

For $v_0 \in \widehat{B}$, one has

$$\left| \int_{-\delta}^{0} e^{A^2 (s-t_0)} G(v(s),s)ds \right|_2 = \left| \int_{-\delta}^{0} A e^{A^2 (s-t_0)} G(v(s),s)ds \right|. \tag{6.4}$$

Since A^2 is a positive sectorial operator on H, there exists a constant $M_A > 0$ such that

$$\left|Ae^{-A^2 t}\right|_{\mathcal{L}(H_2)} \le M_A t^{-1/2}, \quad \forall t \ge 0. \tag{6.5}$$

Recall that $G(v,t) \in C_{\mathrm{Lip};\gamma}(H_2 \times [t_0,+), H)$. So there is a $K_0(\omega) > 0$ such that $\mathbf{P}$-a.s.

$$|G(v,t)| \le K_0(\omega), \quad \forall (v,t) \in \widehat{B} \times [-\delta, 0]. \tag{6.6}$$

Therefore

$$\left|\int_{-\delta}^{0} e^{A^2 s} G(v(s),s)ds\right|_2 \le K_0(\omega) M_A \int_{-\delta}^{0} (-s)^{1/2}\, ds = \frac{1}{2} K_0(\omega) M_A \delta^{1/2}. \tag{6.7}$$

It follows that

$$\kappa(N_2) \le \ \mathrm{diam}_{H_2}(N_2) \le K_0 M_A \delta^{1/2}, \tag{6.8}$$

where $\kappa(\cdot)$ denotes the Kuratowski measure of noncompactness on H_2. Now since N_1, N_3, and N_4 are relatively compact sets in H_2 $\mathbf{P}$-a.s., we have

$$\kappa\Big(S_0(0,t_0;\omega)\widehat{B}\Big) \le \kappa(N_1) + \kappa(N_2) + \kappa(N_3) + \kappa(N_4) \le \kappa(N_2) \le K_0 M_A \delta^{1/2}. \tag{6.9}$$

Letting $\delta \to 0$, one immediately concludes that $\mathbf{P}$-a.s. $\kappa(S_0(0,t_0;\omega)\widehat{B}) = 0$, hence $S_0(0,t_0;\omega)\widehat{B}$ is relatively compact.

(ii) The proof of the compactness result for $S_\alpha(t,s;\omega)$ $(0 < \alpha \le 1)$ is fully analogous, and is thus omitted.

6.3. The Random Attractors

Now, we show that the system $S_\alpha(t,s;\omega)$ possesses a random attractor $\mathcal{A}_\alpha(\omega)$ for every $\alpha \in [0,1]$.

Proof of Theorem 2.4. We infer from the proofs of Theorem 5.2 and Lemma 6.1 that there exists $t(\omega) < 0$ such that for any $t_0 \le t(\omega)$, we can define an absorbing set for $S_0(t,t_0;\omega)$ at time 0 by

$$\mathfrak{B}_0 = \left\{ v : |\Delta v|^2 \le \frac{2C_{12}(\omega)}{\lambda_1^2} \right\} \cup B_{2+\sigma}(0, R_2(\omega)), \tag{6.10}$$

and for $S_\alpha(t,s;\omega)$ $(0 < \alpha \le 1)$, for any $t_0 < 0$ we can define an absorbing set for $S_\alpha(t,t_0;\omega)$ at time 0 by

$$\mathfrak{B}_\alpha = B_1(0, \Psi), \tag{6.11}$$

where $B_1(0,\Psi)$ denotes the open ball centered at 0 with radius Ψ in H_1. Now the conclusions of the theorem immediately follows from Proposition A.6 □

Appendix

Basic knowledge of RDS

In the Appendix, we present some notations of RDS, which are also introduced in [7, 13, 14].

Let (X,d) be a complete metric space, and let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space. We consider a family of mappings

$$\{S(t,s;\omega)\}_{t\geq s,\omega\in\Omega} : X \longrightarrow X, \tag{A.1}$$

satisfying $\mathbf{P}$-a.s.

(i) $S(t,s;\omega) = S(t,r;\omega)\circ S(r,s;\omega)$, for all $s \leq r \leq t$,

(ii) $S(t,s;\omega)$ is continuous in X, for all $s \leq t$.

Definition A.1. We say that $\mathcal{B}(t,\omega) \subset X$ is an absorbing set at time t, if $\mathbf{P}$-a.s.

(i) $\mathcal{B}(t,\omega)$ is bounded,

(ii) for all $B \subset X$ there exists s_B such that $S(t,s;\omega)B \subset \mathcal{B}(t,\omega)$, for all $s \leq s_B$.

Definition A.2. Given $t \in \mathbb{R}$ and $\omega \in \Omega$, we say that $\{S(t,s;\omega)\}_{t\geq s,\omega\in\Omega}$ is uniformly compact at time t if for all bounded set $B \subset X$, there exist s_B, such that $\mathbf{P}$-a.s.

$$\bigcup_{s\leq s_B} S(t,s;\omega)B \tag{A.2}$$

is relatively compact in X.

Definition A.3. Given $t \in \mathbb{R}$ and $\omega \in \Omega$, for any set $B \subset X$, we define the random omega limit set of a bounded set $B \subset X$ at time t as

$$\Omega_B(t,\omega) = \bigcap_{T\leq t} \overline{\bigcup_{s\leq T} S(t,s;\omega)B}. \tag{A.3}$$

Definition A.4. Let (X,d) be a metric space, and let $\{S(t,s;\omega)\}_{t\geq s,\omega\in\Omega}$ a family of operators that maps X into itself. We say that $\mathcal{A}(t,\omega)$ is a stochastic attractor if $\mathbf{P}$-a.s.

(i) $\mathcal{A}(t,\omega)$ is not empty and compact,

(ii) $S(\tau,s;\omega)\mathcal{A}(s,\omega) = \mathcal{A}(\tau,\omega)$ for all $\tau \geq s$,

(iii) for every bounded set $B \subset X$, $\lim_{t\to-\infty} d(S(t,s;\omega)B, \mathcal{A}(t,\omega)) = 0$.

Remark A.5. (i) In the stochastic case, it is not possible to construct the random attractor as the Ω-limit of the absorbing set (as done in the deterministic case). This is due to the fact that the Ω-limit set is taken from $-\infty$ and that the absorbing set is random.

(ii) Global attractor is connected.

Proposition A.6 (see [15]). *If there exists a random set absorbing every bounded deterministic set $B \subset X$ and $\{S(t,s;w)\}_{t\geq s,\omega\in\Omega}$ is uniformly compact at time t, then the RDS possesses a random attractor defined by*

$$\mathcal{A}(t,\omega) = \overline{\bigcup_{B\subset X} \Omega_B(t,\omega)}. \tag{A.4}$$

Remark A.7. In this paper, we write $\mathcal{A}(\omega)$ instead of $\mathcal{A}(0,\omega)$ for short.

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