Research Article

Multiple-Parameter Hamiltonian Approach for Higher Accurate Approximations of a Nonlinear Oscillator with Discontinuity

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We applied a new approach to obtain natural frequency of the nonlinear oscillator with discontinuity. He's Hamiltonian approach is modified for nonlinear oscillator with discontinuity for which the elastic force term is proportional to $\text{sgn}(u)$. We employed this method for higher-order approximate solution of the nonlinear oscillator equation. This property is used to obtain approximate frequency-amplitude relationship of a nonlinear oscillator with high accuracy. Many numerical results are given to prove the efficiency of the suggested technique.

1. Introduction

The study of nonlinear oscillator problems is of crucial importance not only in all areas of physics but also in engineering and other disciplines. It is of great importance to study analytically nonlinear oscillators to obtain approximate frequency-amplitude relationship because of their wide applications. Traditional perturbation method provides us with a simple approach to the determination of the frequency-amplitude relationship, but the results are valid only for special cases, that is, for weakly nonlinear systems or for the case when the amplitude is very small. In order to overcome the shortcomings arising in traditional perturbation methods, various alternative approaches have been proposed, for example, variational iteration method [1–3], homotopy perturbation method [4–7], Lindstedt-Poincare method [8], variational approach [9, 10], parameter-expanding method [11] and max-min approach [12], harmonic balance method [13], and Hamiltonian approach [14].

In the present study, the mentioned parameters are those undetermined values in the assumed solution. In the three-parameters technique, the motion is assumed as $u = A_0 \cos \omega t + A_1 \cos 3\omega t$, where ω, A_0, A_1 are the angular frequency of motion and Fourier coefficients, respectively. The three undetermined parameters are determined by using the governing equation of motion and the initial conditions imposed. The way for obtaining the parameters in He's Hamiltonian technique is quite different from that in the harmonic balance method. Therefore, the present technique is not the same as the harmonic balance method. Finally, the paper provides a lot of higher accurate results for the angular frequency ω of the motion.

2. Analysis

In this paper, we consider a general form of nonlinear oscillator

$$\ddot{u} + f(u) = 0 \tag{2.1}$$

with initial conditions $u(0) = A$ and $\dot{u}(0) = 0$.

The variational principle for (2.1) suggested by He [9] can be written as

$$J(u) = \int_0^{T/4} \left(\frac{\dot{u}^2}{2} - F(u) \right) dt, \tag{2.2}$$

where T is period of the nonlinear oscillator, $\partial F / \partial u = f$. In the functional (2.2), $\dot{u}^2/2$ is the kinetic energy, so that the functional (2.2) is the least Lagrangian action, from which we can write the Hamiltonian

$$H = \frac{\dot{u}^2}{2} + F(u) = \text{constant}. \tag{2.3}$$

In previous work [14], the trail function $u = A \cos \omega t$. In this paper, we consider a general trail function is of the form $u = \sum_{i=0}^{n} A_i \cos((2i + 1)\omega t)$. From (2.3), we have

$$\frac{\partial H}{\partial A_i} = 0. \tag{2.4}$$

Introducing a new function, $\widehat{H}(u)$ defined as

$$\widehat{H}(u) = \int_0^{T/4} \left(\frac{\dot{u}^2}{2} + F(u) \right) dt = \frac{1}{4} TH. \tag{2.5}$$

It is obvious that

$$\frac{\partial \widehat{H}}{\partial T} = \frac{H}{4}. \tag{2.6}$$

Equation (2.4) is, then, equivalent to the following one

$$\frac{\partial}{\partial A_i}\left(\frac{\partial \widehat{H}}{\partial T}\right) = 0 \tag{2.7a}$$

or

$$\frac{\partial}{\partial A_i}\left(\frac{\partial \widehat{H}}{\partial (1/w)}\right) = 0, \quad i = 0, 1, 2, \ldots, n. \tag{2.7b}$$

From system of (2.7a) and (2.7b), we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator up to higher order.

3. Approximation by Multiple-Parameter Hamiltonian Technique

Consider the following nonlinear oscillator with discontinuity

$$\ddot{u} + \text{sgn}(u) = 0 \tag{3.1}$$

with initial conditions $u(0) = A$ and $\dot{u}(0) = 0$.

Here, we suppose that $f(u) = \text{sgn}(u)$ such that

$$\text{sgn}(u) = \begin{cases} -1, & u < 0, \\ 1, & u \geq 0. \end{cases} \tag{3.2}$$

Therefore,

$$f(u) = \begin{cases} -1, & u < 0, \\ 1, & u > 0. \end{cases} \tag{3.3}$$

Its variational formulation can be written as

$$J(u) = \int_0^{T/4}\left(-\frac{\dot{u}^2}{2} + u\right)dt + \int_{T/4}^{T/2}\left(-\frac{\dot{u}^2}{2} - u\right)dt, \tag{3.4}$$

and $\widehat{H}(u)$ can be written in the form as

$$\widehat{H}(u) = \int_0^{T/4}\left(\frac{\dot{u}^2}{2} + u\right)dt + \int_{T/4}^{T/2}\left(\frac{\dot{u}^2}{2} - u\right)dt. \tag{3.5}$$

Using $u = \sum_{i=0}^{n} A_i \cos((2i + 1)wt)$ as an approximate solution. Here, we consider second-, third-, and higher-order Hamiltonian approach. The Hamiltonian approach applied by He [14] and obtained the frequency-amplitude relationship $w_{1(\text{app})} = 1.128379/\sqrt{A}$ for the first-order approximation of $u = A_0 \cos wt$.

To obtain more accurate results assume that the solution can be expressed as

$$u = A_0 \cos \omega t + A_1 \cos 3\omega t. \tag{3.6}$$

According to the initial conditions:

$$A = A_0 + A_1. \tag{3.7}$$

Inserting (3.6) in (3.5), we obtain

$$\widehat{H}(u) = \int_0^{T/4} \left(\frac{1}{2}(A_0\omega \sin \omega t + 3A_1\omega \sin 3\omega t)^2 + (A_0 \cos \omega t + A_1 \cos 3\omega t) \right) dt$$

$$+ \int_{T/4}^{T/2} \left(\frac{1}{2}(A_0\omega \sin \omega t + 3A_1\omega \sin 3\omega t)^2 - (A_0 \cos \omega t + A_1 \cos 3\omega t) \right) dt,$$

$$\widehat{H}(u) = \int_0^{\pi/2} \left(\frac{\omega}{2}(A_0 \sin t + 3A_1 \sin 3t)^2 + \frac{1}{\omega}(A_0 \cos t + A_1 \cos 3t) \right) dt \tag{3.8}$$

$$+ \int_{\pi/2}^{\pi} \left(\frac{\omega}{2}(A_0 \sin t + 3A_1 \sin 3t)^2 - \frac{1}{\omega}(A_0 \cos t + A_1 \cos 3t) \right) dt,$$

$$\widehat{H}(u) = \frac{6A_0 - 2A_1}{3\omega} + \frac{\pi\omega(A_0^2 + 9A_1^2)}{4},$$

setting

$$\frac{\partial}{\partial A_0} \left(\frac{\partial \widehat{H}}{\partial(1/\omega)} \right) = 2 - \frac{\omega^2 \pi A_0}{2} = 0,$$

$$\frac{\partial}{\partial A_1} \left(\frac{\partial \widehat{H}}{\partial(1/\omega)} \right) = -\frac{2}{3} - \frac{9\omega^2 \pi A_1}{2} = 0. \tag{3.9}$$

After some mathematical simplification by using Mathematica built-in utilities, we achieved

$$A_0 = 1.038461A, \qquad A_1 = -0.038461A. \tag{3.10}$$

We obtained the following frequency-amplitude relationship for nonlinear oscillator with discontinuity:

$$\omega_{2(\text{app})} = \frac{1.107286}{\sqrt{A}}. \tag{3.11}$$

Since the accuracy of the obtained results in three-parameters technique is not so high, the four parameters technique is introduced as follows:

$$u = A_0 \cos \omega t + A_1 \cos 3\omega t + A_2 \cos 5\omega t, \tag{3.12}$$

where A_0, A_1, A_2, ω are four undetermined parameters. According to the initial conditions:

$$A = A_0 + A_1 + A_2,$$

$$\frac{\partial}{\partial A_0}\left(\frac{\partial \widehat{H}}{\partial(1/\omega)}\right) = 2 - \frac{\omega^2 \pi A_0}{2} = 0,$$

$$\frac{\partial}{\partial A_1}\left(\frac{\partial \widehat{H}}{\partial(1/\omega)}\right) = -\frac{2}{3} - \frac{9\omega^2 \pi A_1}{2} = 0, \tag{3.13}$$

$$\frac{\partial}{\partial A_2}\left(\frac{\partial \widehat{H}}{\partial(1/\omega)}\right) = \frac{2}{5} - \frac{25\omega^2 \pi A_2}{2} = 0.$$

After some mathematical simplification by using Mathematica built-in utilities, we achieved

$$A_0 = 1.029905A, \qquad A_1 = -0.038144A, \qquad A_2 = 0.008239A. \tag{3.14}$$

We obtained the following frequency-amplitude relationship for nonlinear oscillator with discontinuity

$$\omega_{3(\text{app})} = \frac{1.111876}{\sqrt{A}}. \tag{3.15}$$

which is very close to the exact one [1, 14, 15]

$$\omega_{\text{Exact}} = \frac{\pi}{2\sqrt{2A}} = \frac{1.110721}{\sqrt{A}}. \tag{3.16}$$

4. Discussion and Conclusion

The present method is an extremely simple method, leading to high accuracy of the obtained results. The main merit of the method is that the obtained results are valid for the whole solution domain. In this study, we obtained the relative error as 0.309% for the second-order approximation while the other researchers [1, 8] obtained the relative error as 1.8%. The reason for the difference in the relative error is that the other researchers take less precision in the decimal numbers during calculations. We reached 0.103%, 0.046%, 0.024%, 0.014%, 0.009%, and 0.006% relative errors for the third, fourth, fifth, sixth, seventh, and eighth approximate periods, respectively. However, the calculations of the higher-order approximation are not presented. One can obtain higher-order accuracy by extending the idea given in this paper. Comparison of the multiple-parameter Hamiltonian solution with numerical solution obtained by Mathematica built-in utilities is given in Figure 1.
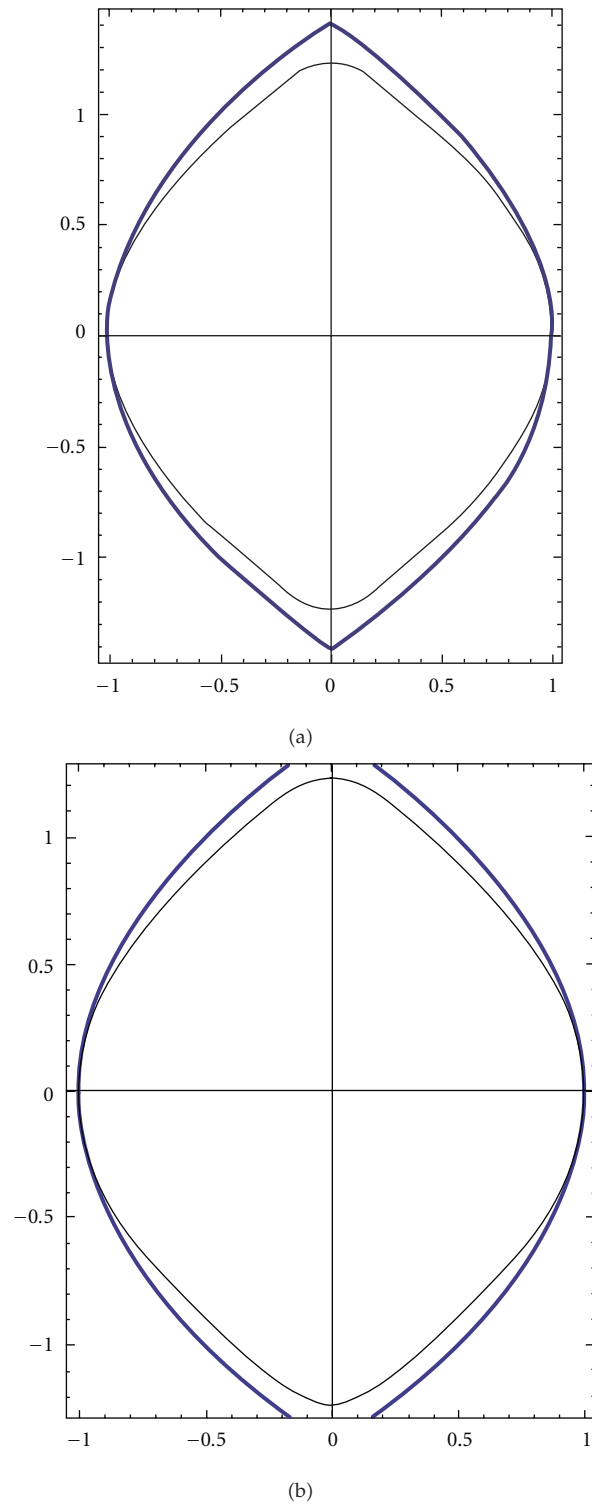
(a)

(b)

Figure 1: Comparison for the u versus $\dot{u}$ trajectory for the case of $A = 1$; bold line represents the numerical solution; and black line represents the multiple-parameter Hamiltonian approach (a) six-parameter (b) eight-parameter.

Acknowledgments

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