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Research Article

Common Fixed Point Theorems for a Pair of Weakly Compatible Mappings in Fuzzy Metric Spaces

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We prove some common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using the new property and give some examples. Our results improve and generalize the main results of Mihet in (Mihet, 2010) and many fixed point theorems in fuzzy metric spaces.

1. Introduction and Preliminaries

The notion of fuzzy sets was introduced by Zadeh [1] in 1965. Since that time a substantial literature has developed on this subject; see, for example, [2–4]. Fixed point theory is one of the most famous mathematical theories with application in several branches of science, especially in chaos theory, game theory, nonlinear programming, economics, theory of differential equations, and so forth. The works noted in [5–10] are some examples from this line of research.

Fixed point theory in fuzzy metric spaces has been developed starting with the work of Heilpern [11]. He introduced the concept of fuzzy mappings and proved some fixed point theorems for fuzzy contraction mappings in metric linear space, which is a fuzzy extension of the Banach's contraction principle. Subsequently several authors [12–20] have studied existence of fixed points of fuzzy mappings. Butnariu [21] also proved some useful fixed point results for fuzzy mappings. Badshah and Joshi [22] studied and proved a common fixed point theorem for six mappings on fuzzy metric spaces by using notion of semicompatibility and reciprocal continuity of mappings satisfying an implicit relation.

For the reader's convenience we recall some terminologies from the theory of fuzzy metric spaces, which will be used in what follows.

Definition 1.1 (Schweizer and Sklar [23]). A continuous t-norm is a binary operation $*$ on $[0,1]$ satisfying the following conditions:

(i) $*$ is commutative and associative;

(ii) $a * 1 = a$ for all $a \in [0,1]$;

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a,b,c,d \in [0,1])$;

(iv) the mapping $* : [0,1] \times [0,1] \to [0,1]$ is continuous.

Example 1.2. The following examples are classical examples of a continuous t-norms.

(TL) (the Lukasiewicz t-norm). A mapping $T_L : [0,1] \times [0,1] \to [0,1]$ which defined through

$$T_L(a,b) = \max\{a + b - 1, 0\}. \tag{1.1}$$

(TP) (the product t-norm). A mapping $T_P : [0,1] \times [0,1] \to [0,1]$ which defined through

$$T_P(a,b) = ab. \tag{1.2}$$

(TM) (the minimum t-norm). A mapping $T_M : [0,1] \times [0,1] \to [0,1]$ which defined through

$$T_M(a,b) = \min\{a,b\}. \tag{1.3}$$

In 1975, Kramosil and Michalek [4] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [24].

Definition 1.3 (Kramosil and Michalek [4]). A fuzzy metric space is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,1]$ such that the following axioms hold:

(KM-1) $M(x,y,0) = 0$ for all $x,y \in X$;

(KM-2) $M(x,y,t) = 1$ for all $x,y \in X$ where $t > 0 \Leftrightarrow x = y$;

(KM-3) $M(x,y,t) = M(y,x,t)$ for all $x,y \in X$;

(KM-4) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left continuous for all $x,y \in X$;

(KM-5) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ for all $x,y,z \in X$ and for all $s,t > 0$.

We will refer to these spaces as KM-fuzzy metric spaces.

Lemma 1.4 (Grabiec [15]). *For every $x,y \in X$, the mapping $M(x,y,\cdot)$ is nondecreasing on* $[0,\infty]$.

George and Veeramani [2, 25] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek.

Definition 1.5 (George and Veeramani [2, 25]). A fuzzy metric space is a triple $(X, M, *)$ where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,1]$ and the following conditions are satisfied for all $x, y \in X$ and $t, s > 0$:

(GV-1) $M(x, y, t) > 0$;

(GV-2) $M(x, y, t) = 1 \Leftrightarrow x = y$;

(GV-3) $M(x, y, t) = M(y, x, t)$;

(GV-4) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous;

(GV-5) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

From (GV-1) and (GV-2), it follows that if $x \neq y$, then $0 < M(x, y, t) < 1$ for all $t > 0$. In what follows, fuzzy metric spaces in the sense of George and Veeramani will be called GV-fuzzy metric spaces.

From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. Several authors have contributed to the development of this theory, for instance [26–29].

Example 1.6. Let (X, d) be a metric space, $a * b = T_M(a, b)$ and, for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}. \tag{1.4}$$

Then $(X, M, *)$ is a GV-fuzzy metric space, called standard fuzzy metric space induced by (X, d).

Definition 1.7. Let $(X, M, *)$ be a (KM- or GV-) fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if

$$\lim_{n \to \infty} M(x_n, x, t) = 1 \tag{1.5}$$

for all $t > 0$.

Definition 1.8. Let $(X, M, *)$ be a (KM- or GV-) fuzzy metric space. A sequence $\{x_n\}$ in X is said to be G-Cauchy sequence if

$$\lim_{n \to \infty} M(x_n, x_{n+m}, t) = 1 \tag{1.6}$$

for all $t > 0$ and $m \in \mathbb{N}$.

Definition 1.9. A fuzzy metric space $(X, M, *)$ is called G-complete if every G-Cauchy sequence converges to a point in X.

Lemma 1.10 (Schweizer and Sklar [23]). *If $(X, M, *)$ is a KM-fuzzy metric space and $\{x_n\}$, $\{y_n\}$ are sequences in X such that*

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} y_n = y, \tag{1.7}$$

then

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t) \tag{1.8}$$

for every continuity point t of $M(x, y, \cdot)$.

Definition 1.11 (Jungck and Rhoades [30]). Let X be a nonempty set. Two mappings $f, g : X \to X$ are said to be weakly compatible if $fgx = gfx$ for all x which $fx = gx$.

In 1995, Subrahmanyam [31] gave a generalization of Jungck's [32] common fixed point theorem for commuting mappings in the setting of fuzzy metric spaces. Even if in the recent literature weaker conditions of commutativity, as weakly commuting mappings, compatible mappings, R-weakly commuting mappings, weakly compatible mappings and several authors have been utilizing, the existence of a common fixed point requires some conditions on continuity of the maps, G-completeness of the space, or containment of ranges.

The concept of E.A. property in metric spaces has been recently introduced by Aamri and El Moutawakil [33].

Definition 1.12 (Aamri and El Moutawakil [33]). Let f and T be self-mapping of a metric space (X, d). We say that f and T satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \tag{1.9}$$

for some $t \in X$.

The class of E.A. mappings contains the class of noncompatible mappings.

In a similar mode, it is said that two self-mappings of f and T of a fuzzy metric space $(X, M, *)$ satisfy E.A. property, if there exists a sequence $\{x_n\}$ in X such that fx_n and gx_n converge to t for some $t \in X$ in the sense of Definition 1.7.

The concept of E.A. property allows to replace the completeness requirement of the space with a more natural condition of closeness of the range.

Recently, Mihet [34] proved two common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using E.A. property.

Let Φ be class of all mappings $\varphi : [0,1] \to [0,1]$ satisfying the following properties:

(φ1) φ is continuous and nondecreasing on $[0,1]$;

(φ2) $\varphi(x) > x$ for all $x \in (0,1)$.

Theorem 1.13 (see [34, Theorem 2.1]). *Let $(X, M, *)$ be a KM-fuzzy metric space satisfying the following property:*

$$\forall x, y \in X,\ x \neq y, \quad \exists t > 0 : 0 < M(x, y, t) < 1, \tag{1.10}$$

and let f, g be weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,

$$\begin{aligned} M(fx, fy, t) \geq \varphi(\min\{&M(gx, gy, t), M(fx, gx, t),\\ &M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}) \end{aligned} \tag{1.11}$$

for all $x, y \in X$ where $t > 0$. If f and g satisfy E.A. property and the range of g is a closed subspace of X, then f and g have a unique common fixed point.

Theorem 1.14 (see [34, Theorem 3.1]). *Let $(X, M, *)$ be a GV-fuzzy metric space and f, g weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$ and some $s > 0$,*

$$\begin{aligned} M(fx, fy, s) \geq \varphi(\min\{&M(gx, gy, s), M(fx, gx, s),\\ &M(fy, gy, s), M(fy, gx, s), M(fx, gy, s)\}) \end{aligned} \tag{1.12}$$

for all $x, y \in X$. If f and g satisfy E.A. property and the range of g is a closed subspace of X, then f and g have a unique common fixed point.

We obtain that Theorems 1.13 and 1.14 require special condition, that is, the range of g is a closed subspace of X. Sometimes, the range of g maybe is not a closed subspace of X. Therefore Theorems 1.13 and 1.14 cannot be used for this case.

The aim of this work is to introduce the new property which is so called "common limit in the range" for two self-mappings f, g and give some examples of mappings which satisfy this property. Moreover, we establish some new existence of a common fixed point theorem for generalized contractive mappings in fuzzy metric spaces both in the sense of Kramosil and Michalek and in the sense of George and Veeramani by using new property and give some examples. Ours results does not require condition of closeness of range and so our theorems generalize, unify, and extend many results in literature.

2. Common Fixed Point in KM and GV-Fuzzy Metric Spaces

We first introduce the concept of new property.

Definition 2.1. Suppose that (X, d) is a metric space and $f, g : X \to X$. Two mappings f and g are said to satisfy the *common limit in the range of g* property if

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \tag{2.1}$$

for some $x \in X$.

In what follows, the common limit in the range of g property will be denoted by the (CLRg) property.

Next, we show examples of mappings f and g which are satisfying the (CLRg) property.

Example 2.2. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \to X$ by $fx = x/4$ and $gx = 3x/4$ for all $x \in X$. We consider the sequence $\{x_n\} = \{1/n\}$. Since

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = 0 = g0, \tag{2.2}$$

therefore f and g satisfy the (CLRg) property.

Example 2.3. Let $X = [0, \infty)$ be the usual metric space. Define $f, g : X \to X$ by $fx = x + 1$ and $gx = 2x$ for all $x \in X$. Consider the sequence $\{x_n\} = \{1 + 1/n\}$. Since

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = 2 = g1, \tag{2.3}$$

therefore f and g satisfy the (CLRg) property.

In a similar mode, two self-mappings f and g of a fuzzy metric space $(X, M, *)$ satisfy the (CLRg) property, if there exists a sequence $\{x_n\}$ in X such that fx_n and gx_n converge to gx for some $x \in X$ in the sense of Definition 1.7.

Theorem 2.4. *Let $(X, M, *)$ be a KM-fuzzy metric space satisfying the following property:*

$$\forall x, y \in X, \ x \neq y, \quad \exists t > 0 : 0 < M(x, y, t) < 1, \tag{2.4}$$

and let f, g be weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,

$$\begin{aligned} M(fx, fy, t) \geq \varphi(\min\{&M(gx, gy, t), M(fx, gx, t), \\ &M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}) \end{aligned} \tag{2.5}$$

for all $x, y \in X$, where $t > 0$. If f and g satisfy the (CLRg) property, then f and g have a unique common fixed point.

Proof. Since f and g satisfy the (CLRg) property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = gx \tag{2.6}$$

for some $x \in X$. Let t be a continuity point of $(X, M, *)$. Then

$$\begin{aligned} M(fx_n, fx, t) \geq \varphi(\min\{&M(gx_n, gx, t), M(fx_n, gx_n, t), \\ &M(fx, gx, t), M(fx, gx_n, t), M(fx_n, gx, t)\}) \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$. By making $n \to \infty$, we have

$$\begin{aligned} M(gx, fx, t) &\geq \varphi(\min\{M(gx, gx, t), M(gx, gx, t), \\ &\qquad M(fx, gx, t), M(fx, gx, t), M(gx, gx, t)\}) \\ &= \varphi(\min\{1, 1, M(gx, fx, t), M(gx, fx, t), 1\}) \\ &= \varphi(M(gx, fx, t)) \end{aligned} \tag{2.8}$$

for every $t > 0$. We claim that $gx = fx$. If not, then

$$\exists t_0 > 0 : 0 < M(gx, fx, t_0) < 1. \tag{2.9}$$

It follows from the condition of ($\varphi 2$) that $\varphi(M(gx, fx, t_0)) > M(gx, fx, t_0)$, which is a contradiction. Therefore $gx = fx$.

Next, we let $z := fx = gx$. Since f and g are weakly compatible mappings, $fgx = gfx$ which implies that

$$fz = fgx = gfx = gz. \tag{2.10}$$

We claim that $fz = z$. Assume not, then by (2.4), it implies that $0 < M(fz, z, t_1) < 1$ for some $t_1 > 0$. By condition of ($\varphi 2$), we have $\varphi(M(fz, z, t_1)) > M(fz, z, t_1)$. Using condition (2.5) again, we get

$$\begin{aligned} M(fz, z, t) &= M(fz, fx, t) \\ &\geq \varphi(\min\{M(gz, gx, t), M(fz, gz, t), M(fx, gx, t), M(fx, gz, t), M(fz, gx, t)\}) \\ &= \varphi(\min\{M(gz, gx, t), 1, 1, M(fx, gz, t), M(fz, gx, t)\}) \\ &= \varphi(\min\{M(fz, fx, t), 1, 1, M(fx, fz, t), M(fz, fx, t)\}) \\ &= \varphi(\min\{M(fz, fx, t), 1, 1, M(fz, fx, t), M(fz, fx, t)\}) \\ &= \varphi(M(fz, fx, t)) \\ &= \varphi(M(fz, z, t)) \end{aligned} \tag{2.11}$$

for all $t > 0$, which is a contradiction. Hence $fz = z$, that is, $z = fz = gz$. Therefore z is a common fixed point of f and g.

For the uniqueness of a common fixed point, we suppose that w is another common fixed point in which $w \neq z$. It follows from condition (2.4) that there exists $t_2 > 0$ such that

$0 < M(w,z,t_2) < 1$. Since $M(w,z,t_2) \in (0,1)$, we have $\varphi(M(w,z,t_2)) > M(w,z,t_2)$ by virtue of (φ_2). From (2.5), we have

$$\begin{aligned} M(z,w,t) &= M(fz,fw,t) \\ &\geq \varphi(\min\{M(gz,gw,t), M(fz,gz,t), \\ &\qquad M(fw,gw,t), M(fw,gz,t), M(fz,gw,t)\}) \\ &= \varphi(\min\{M(z,w,t), 1, 1, M(w,z,t), M(z,w,t)\}) \\ &= \varphi(M(z,w,t)) \end{aligned} \tag{2.12}$$

for all $t > 0$, which is a contradiction. Therefore, it must be the case that $w = z$ which implies that f and g have a unique a common fixed point. This finishes the proof. □

Next, we will give example which cannot be used [34, Theorem 2.1]. However, we can apply Theorem 2.4 for this case.

Example 2.5. Let $X = (0,\infty)$ and, for each $x, y \in X$ and $t > 0$,

$$M(x,y,t) = \frac{\min\{x,y\}}{\max\{x,y\}}. \tag{2.13}$$

It is well known (see [2]) that (X, M, T_p) is a GV-fuzzy metric space. If the mappings $f, g : X \to X$ are defined on X through $fx = x^{1/4}$ and $gx = x^{1/2}$, then the range of g is $(0,\infty)$ which is not a closed subspace of X. So Theorem 2.1 of Mihet in [34] cannot be used for this case. It is easy to see that the mappings f and g satisfy the (CLRg) property with a sequence $\{x_n\} = \{1 + 1/n\}$. Therefore all hypothesis of the above theorem holds, with $\varphi(t) = t$ for $t \in [0,1]$. Their common fixed point is $x = 1$.

Corollary 2.6 ([34, Theorem 2.1]). *Let $(X, M, *)$ be a KM-fuzzy metric space satisfying the following property:*

$$\forall x, y \in X,\ x \neq y, \quad \exists t > 0 : 0 < M(x,y,t) < 1, \tag{2.14}$$

and let f, g be weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,

$$\begin{aligned} M(fx,fy,t) \geq \varphi(\min\{&M(gx,gy,t), M(fx,gx,t), \\ &M(fy,gy,t), M(fy,gx,t), M(fx,gy,t)\}) \end{aligned} \tag{2.15}$$

for all $x, y \in X$, where $t > 0$. If f and g satisfy E.A. property and the range of g is a closed subspace of X, then f and g have a unique common fixed point.

Proof. Since f and g satisfy E.A. property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \tag{2.16}$$

for some $u \in X$. It follows from gX being a closed subspace of X that $u = gx$ for some $x \in X$ and then f and g satisfy the (CLRg) property. By Theorem 2.4, we get that f and g have a unique common fixed point. □

Corollary 2.7. *Let $(X, M, *)$ be a KM-fuzzy metric space satisfying the following property:*

$$\forall x, y \in X,\ x \neq y, \quad \exists t > 0 : 0 < M(x, y, t) < 1, \tag{2.17}$$

and let f, g be weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,

$$M(fx, fy, t) \geq \varphi(M) \tag{2.18}$$

for all $x, y \in X$, where $t > 0$ and

$$M \in \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}. \tag{2.19}$$

If f and g satisfy the (CLRg) property, then f and g have a unique common fixed point.

Proof. As φ is nondecreasing and

$$M \geq \min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\} \tag{2.20}$$

for $M \in \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}$, we have

$$\varphi(M) \geq \varphi(\min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}). \tag{2.21}$$

So inequality (2.18) implies that

$$\begin{aligned} M(fx, fy, t) \geq \varphi(\min\{&M(gx, gy, t), M(fx, gx, t), \\ &M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}). \end{aligned} \tag{2.22}$$

By Theorem 2.4, we get f and g have a unique common fixed point. □

If $(X, M, *)$ is a fuzzy metric space in the sense of George and Veeramani, then some of the hypotheses in the preceding theorem can be relaxed.

Theorem 2.8. *Let $(X, M, *)$ be a GV-fuzzy metric space and f, g weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,*

$$\begin{aligned} M(fx, fy, t) \geq \varphi(\min\{&M(gx, gy, t), M(fx, gx, t), \\ &M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}) \end{aligned} \tag{2.23}$$

for all $x, y \in X$, where $t > 0$. If f and g satisfy the (CLRg) property, then f and g have a unique common fixed point.

Proof. It follows from f and g satisfying the (CLRg) property that we can find a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx \tag{2.24}$$

for some $x \in X$.

Let t be a continuity point of $(X, M, *)$. Then

$$\begin{aligned} M(fx_n, fx, t) \geq \varphi(\min\{&M(gx_n, gx, t), M(fx_n, gx_n, t),\\ &M(fx, gx, t), M(fx, gx_n, t), M(fx_n, gx, t)\}) \end{aligned} \tag{2.25}$$

for all $n \in \mathbb{N}$. By taking the limit as n tends to infinity in (2.25), we have

$$\begin{aligned} M(gx, fx, t) &\geq \varphi(\min\{M(gx, gx, t), M(gx, gx, t),\\ &\qquad M(fx, gx, t), M(fx, gx, t), M(gx, gx, t)\})\\ &= \varphi(\min\{1, 1, M(gx, fx, t), M(gx, fx, t), 1\})\\ &= \varphi(M(gx, fx, t)) \end{aligned} \tag{2.26}$$

for every $t > 0$. Now, we show that $gx = fx$. If $gx \neq fx$, then from (GV-1) and (GV-2),

$$0 < M(gx, fx, t) < 1 \tag{2.27}$$

for all $t > 0$. From condition of $(\varphi 2)$, $\varphi(M(gx, fx, t)) > M(gx, fx, t)$, which is a contradiction. Hence $gx = fx$.

Similarly in the proof of Theorem 2.4, by denoting a point $fx(= gx)$ by z. Since f and g are weakly compatible mappings, $fgx = gfx$ which implies that $fz = gz$.

Next, we will show that $fz = z$. We will suppose that $fz \neq z$. By (GV-1) and (GV-2), it implies that $0 < M(fz, z, t) < 1$ for all $t > 0$. By $(\varphi 2)$, we know that $\varphi(M(fz, z, t)) > M(fz, z, t)$. It follows from condition (2.23) that

$$\begin{aligned} M(fz, z, t) &= M(fz, fx, t)\\ &\geq \varphi(\min\{M(gz, gx, t), M(fz, gz, t),\\ &\qquad M(fx, gx, t), M(fx, gz, t), M(fz, gx, t)\})\\ &= \varphi(\min\{M(gz, gx, t), 1, 1, M(fx, gz, t), M(fz, gx, t)\})\\ &= \varphi(\min\{M(fz, fx, t), 1, 1, M(fx, fz, t), M(fz, fx, t)\})\\ &= \varphi(\min\{M(fz, fx, t), 1, 1, M(fz, fx, t), M(fz, fx, t)\})\\ &= \varphi(M(fz, fx, t))\\ &= \varphi(M(fz, z, t)) \end{aligned} \tag{2.28}$$

for all $t > 0$, which is contradicting the above inequality. Therefore $fz = z$, and then $z = fz = gz$. Consequently, f and g have a common fixed point that is z.

Finally, we will prove that a common fixed point of f and g is unique. Let us suppose that w is a common fixed point of f and g in which $w \neq z$. It follows from condition of (GV-1) and (GV-2) that for every $t > 0$, we have $M(w, z, t) \in (0, 1)$ which implies that $\varphi(M(w, z, t)) > M(w, z, t)$. On the other hand, we know that

$$\begin{aligned} M(z, w, t) &= M(fz, fw, t) \\ &\geq \varphi(\min\{M(gz, gw, t), M(fz, gz, t), \\ &\qquad M(fw, gw, t), M(fw, gz, t), M(fz, gw, t)\}) \\ &= \varphi(\min\{M(z, w, t), 1, 1, M(w, z, t), M(z, w, t)\}) \\ &= \varphi(M(z, w, t)) \end{aligned} \tag{2.29}$$

for all $t > 0$, which is contradiction. Hence we conclude that $w = z$. It finishes the proof of this theorem. □

Corollary 2.9 ([34, Theorem 3.1]). *Let $(X, M, *)$ be a GV-fuzzy metric space and f, g weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,*

$$\begin{aligned} M(fx, fy, t) \geq \varphi(\min\{&M(gx, gy, t), M(fx, gx, t), \\ &M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}) \end{aligned} \tag{2.30}$$

for all $x, y \in X$, where $t > 0$. If f and g satisfy E.A. property and the range of g is a closed subspace of X, then f and g have a unique common fixed point.

Proof. Since f and g satisfy E.A. property, there exists a sequence $\{x_n\}$ in X satisfies

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \tag{2.31}$$

for some $u \in X$. It follows from gX being a closed subspace of X that there exists $x \in X$ in which $u = gx$. Therefore f and g satisfy the (CLRg) property. It follows from Theorem 2.8 that there exists a unique common fixed point of f and g. □

Corollary 2.10. *Let $(X, M, *)$ be a GV-fuzzy metric space and f, g weakly compatible self-mappings of X such that, for some $\varphi \in \Phi$,*

$$M(fx, fy, t) \geq \varphi(M) \tag{2.32}$$

for all $x, y \in X$, where $t > 0$ and

$$M \in \{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}. \tag{2.33}$$

If f and g satisfy the (CLRg) property, then f and g have a unique common fixed point.

Proof. Since φ is nondecreasing and

$$M \geq \min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}, \tag{2.34}$$

where

$$M \in \{M(gx, gy, \mathrm{t}), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}, \tag{2.35}$$

we get

$$\varphi(M) \geq \varphi(\min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}). \tag{2.36}$$

Now, we know that inequality (2.32) implies that

$$\begin{aligned} M(fx, fy, t) \geq \varphi(\min\{&M(gx, gy, t), M(fx, gx, t), \\ &M(fy, gy, t), M(fy, gx, t), M(fx, gy, t)\}). \end{aligned} \tag{2.37}$$

It follows from Theorem 2.8 that f and g have a unique common fixed point. □

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