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Research Article

On Nonoscillation of Advanced Differential Equations with Several Terms

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Existence of positive solutions for advanced equations with several terms $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0$, $h_k(t) \geq t$ is investigated in the following three cases: (a) all coefficients a_k are positive; (b) all coefficients a_k are negative; (c) there is an equal number of positive and negative coefficients. Results on asymptotics of nonoscillatory solutions are also presented.

1. Introduction

This paper deals with nonoscillation properties of scalar advanced differential equations. Advanced differential equations appear in several applications, especially as mathematical models in economics; an advanced term may, for example, reflect the dependency on anticipated capital stock [1, 2].

It is not quite clear how to formulate an initial value problem for such equations, and existence and uniqueness of solutions becomes a complicated issue. To study oscillation, we need to assume that there exists a solution of such equation on the halfline. In the beginning of 1980s, sufficient oscillation conditions for first-order linear advanced equations with constant coefficients and deviations of arguments were obtained in [3] and for nonlinear equations in [4]. Later oscillation properties were studied for other advanced and mixed differential equations (see the monograph [5], the papers [6–12] and references therein). Overall, these publications mostly deal with sufficient oscillation conditions; there are only few results [7, 9, 12] on existence of positive solutions for equations with several advanced terms and variable coefficients, and the general nonoscillation theory is not complete even for first-order linear equations with variable advanced arguments and variable coefficients of the same sign.

The present paper partially fills up this gap. We obtain several nonoscillation results for advanced equations using the generalized characteristic inequality [13]. The main method of this paper is based on fixed point theory; thus, we also state the existence of a solution in certain cases.

In the linear case, the best studied models with advanced arguments were the equations of the types

$$\dot{x}(t) - a(t)x(h(t)) + b(t)x(t) = 0,$$
$$\dot{x}(t) - a(t)x(t) + b(t)x(g(t)) = 0, \tag{1.1}$$

where $a(t) \geq 0$, $b(t) \geq 0$, $h(t) \geq t$, and $g(t) \geq t$.

Let us note that oscillation of higher order linear and nonlinear equations with advanced and mixed arguments was also extensively investigated, starting with [14]; see also the recent papers [15–19] and references therein.

For equations with an advanced argument, the results obtained in [20, 21] can be reformulated as Theorems A–C below.

Theorem A (see [20]). *If a, b, and h are equicontinuous on $[0,\infty)$, $a(t) \geq 0$, $b(t) \geq 0$, $h(t) \geq t$, and $\limsup_{t\to\infty}[h(t) - t] < \infty$, then the advanced equation*

$$\dot{x}(t) + a(t)x(h(t)) + b(t)x(t) = 0 \tag{1.2}$$

has a nonoscillatory solution.

In the present paper, we extend Theorem A to the case of several deviating arguments and coefficients (Theorem 2.10).

Theorem B (see [20]). *If a, b, and h are equicontinuous on $[0,\infty)$, $a(t) \geq 0$, $b(t) \geq 0$, $h(t) \geq t$, $\limsup_{t\to\infty}[h(t) - t] < \infty$, and*

$$\limsup_{t\to\infty} \int_t^{h(t)} a(s) \exp\left\{ \int_s^{h(s)} b(\tau)d\tau \right\} ds < \frac{1}{e}, \tag{1.3}$$

then the advanced equation

$$\dot{x}(t) - a(t)x(h(t)) - b(t)x(t) = 0 \tag{1.4}$$

has a nonoscillatory solution.

Corollary 2.3 of the present paper extends Theorem B to the case of several coefficients $a_k \geq 0$ and advanced arguments h_k (generally, $b(t) \equiv 0$); if

$$\int_t^{\max_k h_k(t)} \sum_{i=1}^{m} a_i(s)ds \leq \frac{1}{e}, \tag{1.5}$$

then the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(h_k(t)) = 0 \tag{1.6}$$

has an eventually positive solution. To the best of our knowledge, only the opposite inequality (with $\min_k h_k(t)$ rather than $\max_k h_k(t)$ in the upper bound) was known as a sufficient oscillation condition. Coefficients and advanced arguments are also assumed to be of a more general type than in [20]. Comparison to equations with constant arguments deviations, and coefficients (Corollary 2.8) is also outlined.

For advanced equations with coefficients of different sign, the following result is known.

Theorem C (see [21]). *If $0 \le a(t) \le b(t)$ and $h(t) \ge t$, then there exists a nonoscillatory solution of the equation*

$$\dot{x}(t) - a(t)x(h(t)) + b(t)x(t) = 0. \tag{1.7}$$

This result is generalized in Theorem 2.13 to the case of several positive and negative terms and several advanced arguments; moreover, positive terms can also be advanced as far as the advance is not greater than in the corresponding negative terms.

We also study advanced equations with positive and negative coefficients in the case when positive terms dominate rather than negative ones; some sufficient nonoscillation conditions are presented in Theorem 2.15; these results are later applied to the equation with constant advances and coefficients. Let us note that analysis of nonoscillation properties of the mixed equation with a positive advanced term

$$\dot{x}(t) + a(t)x(h(t)) - b(t)x\bigl(g(t)\bigr) = 0, \quad h(t) \ge t,\ g(t) \le t,\ a(t) \ge 0,\ b(t) \ge 0 \tag{1.8}$$

was also more complicated compared to other cases of mixed equations with positive and negative coefficients [21].

In nonoscillation theory, results on asymptotic properties of nonoscillatory solutions are rather important; for example, for equations with several delays and positive coefficients, all nonoscillatory solutions tend to zero if the integral of the sum of coefficients diverges; under the same condition for negative coefficients, all solutions tend to infinity. In Theorems 2.6 and 2.11, the asymptotic properties of nonoscillatory solutions for advanced equations with coefficients of the same sign are studied.

The paper is organized as follows. Section 2 contains main results on the existence of nonoscillatory solutions to advanced equations and on asymptotics of these solutions: first for equations with coefficients of the same sign, then for equations with both positive and negative coefficients. Section 3 involves some comments and open problems.

2. Main Results

Consider first the equation

$$\dot{x}(t) - \sum_{k=1}^{m} a_k(t) x(h_k(t)) = 0, \tag{2.1}$$

under the following conditions:

(a1) $a_k(t) \geq 0$, $k = 1, \ldots, m$, are Lebesgue measurable functions locally essentially bounded for $t \geq 0$,

(a2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \geq t$, $k = 1, \ldots, m$.

Definition 2.1. A locally absolutely continuous function $x : [t_0, \infty) \to R$ is called *a solution of problem* (2.1) if it satisfies (2.1) for almost all $t \in [t_0, \infty)$.

The same definition will be used for all further advanced equations.

Theorem 2.2. *Suppose that the inequality*

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{ \int_t^{h_k(t)} u(s) ds \right\}, \quad t \geq t_0 \tag{2.2}$$

has a nonnegative solution which is integrable on each interval $[t_0, b]$, *then* (2.1) *has a positive solution for* $t \geq t_0$.

Proof. Let $u_0(t)$ be a nonnegative solution of inequality (2.2). Denote

$$u_{n+1}(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{ \int_t^{h_k(t)} u_n(s) ds \right\}, \quad n = 0, 1, \ldots, \tag{2.3}$$

then

$$u_1(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{ \int_t^{h_k(t)} u_0(s) ds \right\} \leq u_0(t). \tag{2.4}$$

By induction we have $0 \leq u_{n+1}(t) \leq u_n(t) \leq u_0(t)$. Hence, there exists a pointwise limit $u(t) = \lim_{n \to \infty} u_n(t)$. By the Lebesgue convergence theorem, we have

$$u(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{ \int_t^{h_k(t)} u(s) ds \right\}. \tag{2.5}$$

Then, the function

$$x(t) = x(t_0) \exp\left\{ \int_{t_0}^{t} u(s) ds \right\} \quad \text{for any } x(t_0) > 0 \tag{2.6}$$

is a positive solution of (2.1). □

Corollary 2.3. *If*

$$\int_{t}^{\max_k h_k(t)} \sum_{i=1}^{m} a_i(s)ds \leq \frac{1}{e}, \quad t \geq t_0, \tag{2.7}$$

then (2.1) *has a positive solution for* $t \geq t_0$.

Proof. Let $u_0(t) = e\sum_{k=1}^{m} a_k(t)$, then u_0 satisfies (2.2) at any point t where $\sum_{k=1}^{m} a_k(t) = 0$. In the case when $\sum_{k=1}^{m} a_k(t) \neq 0$, inequality (2.7) implies

$$\begin{aligned}
&\frac{u_0(t)}{\sum_{k=1}^{m} a_k(t)\exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\}} \\
&\quad\geq \frac{u_0(t)}{\sum_{k=1}^{m} a_k(t)\exp\left\{\int_t^{\max_k h_k(t)} u_0(s)ds\right\}} \\
&\quad= \frac{e\sum_{k=1}^{m} a_k(t)}{\sum_{k=1}^{m} a_k(t)\exp\left\{e\int_t^{\max_k h_k(t)} \sum_{i=1}^{m} a_i(s)ds\right\}} \\
&\quad\geq \frac{e\sum_{k=1}^{m} a_k(t)}{\sum_{k=1}^{m} a_k(t)e} = 1.
\end{aligned} \tag{2.8}$$

Hence, $u_0(t)$ is a positive solution of inequality (2.2). By Theorem 2.2, (2.1) has a positive solution for $t \geq t_0$. □

Corollary 2.4. *If there exists* $\sigma > 0$ *such that* $h_k(t) - t \leq \sigma$ *and* $\int_0^\infty \sum_{k=1}^{m} a_k(s)ds < \infty$, *then* (2.1) *has an eventually positive solution.*

Corollary 2.5. *If there exists* $\sigma > 0$ *such that* $h_k(t) - t \leq \sigma$ *and* $\lim_{t\to\infty} a_k(t) = 0$, *then* (2.1) *has an eventually positive solution.*

Proof. Under the conditions of either Corollary 2.4 or Corollary 2.5, obviously there exists $t_0 \geq 0$ such that (2.7) is satisfied. □

Theorem 2.6. *Let* $\int^\infty \sum_{k=1}^{m} a_k(s)ds = \infty$ *and* x *be an eventually positive solution of* (2.1), *then* $\lim_{t\to\infty} x(t) = \infty$.

Proof. Suppose that $x(t) > 0$ for $t \geq t_1$, then $\dot{x}(t) \geq 0$ for $t \geq t_1$ and

$$\dot{x}(t) \geq \sum_{k=1}^{m} a_k(t)x(t_1), \quad t \geq t_1, \tag{2.9}$$

which implies

$$x(t) \geq x(t_1)\int_{t_1}^{t} \sum_{k=1}^{m} a_k(s)ds. \tag{2.10}$$

Thus, $\lim_{t\to\infty} x(t) = \infty$. □

Consider together with (2.1) the following equation:

$$\dot{x}(t) - \sum_{k=1}^{m} b_k(t) x(g_k(t)) = 0, \tag{2.11}$$

for $t \geq t_0$. We assume that for (2.11) conditions (a1)-(a2) also hold.

Theorem 2.7. *Suppose that $t \leq g_k(t) \leq h_k(t)$, $0 \leq b_k(t) \leq a_k(t)$, $t \geq t_0$, and the conditions of Theorem 2.2 hold, then* (2.11) *has a positive solution for $t \geq t_0$.*

Proof. Let $u_0(t) \geq 0$ be a solution of inequality (2.2) for $t \geq t_0$, then this function is also a solution of this inequality if $a_k(t)$ and $h_k(t)$ are replaced by $b_k(t)$ and $g_k(t)$. The reference to Theorem 2.2 completes the proof. □

Corollary 2.8. *Suppose that there exist $a_k > 0$ and $\sigma_k > 0$ such that $0 \leq a_k(t) \leq a_k$, $t \leq h_k(t) \leq t + \sigma_k$, $t \geq t_0$, and the inequality*

$$\lambda \geq \sum_{k=1}^{m} a_k e^{\lambda \sigma_k} \tag{2.12}$$

has a solution $\lambda \geq 0$, then (2.1) *has a positive solution for $t \geq t_0$.*

Proof. Consider the equation with constant parameters

$$\dot{x}(t) - \sum_{k=1}^{m} a_k x(t + \sigma_k) = 0. \tag{2.13}$$

Since the function $u(t) \equiv \lambda$ is a solution of inequality (2.2) corresponding to (2.13), by Theorem 2.2, (2.13) has a positive solution. Theorem 2.7 implies this corollary. □

Corollary 2.9. *Suppose that $0 \leq a_k(t) \leq a_k$, $t \leq h_k(t) \leq t + \sigma$ for $t \geq t_0$, and*

$$\sum_{k=1}^{m} a_k \leq \frac{1}{e\sigma}, \tag{2.14}$$

then (2.1) *has a positive solution for $t \geq t_0$.*

Proof. Since $\sum_{k=1}^{m} a_k \leq 1/e\sigma$, the number $\lambda = 1/\sigma$ is a positive solution of the inequality

$$\lambda \geq \left(\sum_{k=1}^{m} a_k \right) e^{\lambda \sigma}, \tag{2.15}$$

which completes the proof. □

Consider now the equation with positive coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(h_k(t)) = 0. \tag{2.16}$$

Theorem 2.10. *Suppose that $a_k(t) \geq 0$ are continuous functions bounded on $[t_0, \infty)$ and h_k are equicontinuous functions on $[t_0, \infty)$ satisfying $0 \leq h_k(t) - t \leq \delta$, then* (2.16) *has a nonoscillatory solution.*

Proof. In the space $C[t_0, \infty)$ of continuous functions on $[t_0, \infty)$, consider the set

$$M = \left\{ u \mid 0 \leq u \leq \sum_{k=1}^{m} a_k(t) \right\}, \tag{2.17}$$

and the operator

$$(Hu)(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{ -\int_{t}^{h_k(t)} u(s) ds \right\}. \tag{2.18}$$

If $u \in M$, then $Hu \in M$.

For the integral operator

$$(Tu)(t) := \int_{t}^{h_k(t)} u(s) ds, \tag{2.19}$$

we will demonstrate that TM is a compact set in the space $C[t_0, \infty)$. If $u \in M$, then

$$\|(Tu)(t)\|_{C[t_0,\infty)} \leq \sup_{t \geq t_0} \int_{t}^{t+\delta} |u(s)| ds \leq \sup_{t \geq t_0} \sum_{k=1}^{m} a_k(t) \delta < \infty. \tag{2.20}$$

Hence, the functions in the set TM are uniformly bounded in the space $C[t_0, \infty)$.

Functions h_k are equicontinuous on $[t_0, \infty)$, so for any $\varepsilon > 0$, there exists a $\sigma_0 > 0$ such that for $|t - s| < \sigma_0$, the inequality

$$|h_k(t) - h_k(s)| < \frac{\varepsilon}{2} \left(\sup_{t \geq t_0} \sum_{k=1}^{m} a_k(t) \right)^{-1}, \quad k = 1, \ldots, m \tag{2.21}$$

holds. From the relation

$$\int_{t_0}^{h_k(t_0)} - \int_{t}^{h_k(t)} = \int_{t_0}^{t} + \int_{t}^{h_k(t_0)} - \int_{t}^{h_k(t_0)} - \int_{h_k(t_0)}^{h_k(t)} = \int_{t_0}^{t} - \int_{h_k(t_0)}^{h_k(t)}, \tag{2.22}$$

we have for $|t-t_0| < \min\{\sigma_0, \varepsilon/2\sup_{t\ge t_0}\sum_{k=1}^m a_k(t)\}$ and $u\in M$ the estimate

$$\begin{aligned}
|(Tu)(t)-(Tu)(t_0)| &= \left|\int_{\mathrm{t}}^{h_k(t)} u(s) - \int_{t_0}^{h_k(t_0)} u(s)ds\right| \\
&\le \int_{t_0}^{t}|u(s)|ds + \int_{h_k(t_0)}^{h_k(t)}|u(s)|ds \\
&\le |t-t_0|\sup_{t\ge t_0}\sum_{k=1}^m a_k(t) + |h_k(t)-h_k(t_0)|\sup_{t\ge t_0}\sum_{k=1}^m a_k(t) \\
&< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{2.23}$$

Hence, the set TM contains functions which are uniformly bounded and equicontinuous on $[t_0,\infty)$, so it is compact in the space $C[t_0,\infty)$; thus, the set HM is also compact in $C[t_0,\infty)$.

By the Schauder fixed point theorem, there exists a continuous function $u\in M$ such that $u = Hu$, then the function

$$x(t) = \exp\left\{-\int_{t_0}^{t} u(s)ds\right\} \tag{2.24}$$

is a bounded positive solution of (2.16). Moreover, since u is nonnegative, this solution is nonincreasing on $[t_0,\infty)$. □

Theorem 2.11. *Suppose that the conditions of Theorem 2.10 hold,*

$$\int_{t_0}^{\infty}\sum_{k=1}^m a_k(s)ds = \infty, \tag{2.25}$$

and x is a nonoscillatory solution of (2.16), *then* $\lim_{t\to\infty}x(t) = 0$.

Proof. Let $x(t) > 0$ for $t\ge t_0$, then $\dot{x}(t)\le 0$ for $t\ge t_0$. Hence, $x(t)$ is nonincreasing and thus has a finite limit. If $\lim_{t\to\infty}x(t) = d > 0$, then $x(t) > d$ for any t, and thus $\dot{x}(t)\le -d\sum_{k=1}^m a_k(t)$ which implies $\lim_{t\to\infty}x(t) = -\infty$. This contradicts to the assumption that $x(t)$ is positive, and therefore $\lim_{t\to\infty}x(t) = 0$. □

Let us note that we cannot guarantee any (exponential or polynomial) rate of convergence to zero even for constant coefficients a_k, as the following example demonstrates.

Example 2.12. Consider the equation $\dot{x}(t) + x(h(t)) = 0$, where $h(t) = t^{t\ln t}$, $t\ge 3$, $x(3) = 1/\ln 3$. Then, $x(t) = 1/(\ln t)$ is the solution which tends to zero slower than t^{-r} for any $r > 0$.

Consider now the advanced equation with positive and negative coefficients

$$\dot{x}(t) - \sum_{k=1}^m\left[a_k(t)x(h_k(t)) - b_k(t)x(g_k(t))\right] = 0, \quad t\ge 0. \tag{2.26}$$

Theorem 2.13. *Suppose that $a_k(t)$ and $b_k(t)$ are Lebesgue measurable locally essentially bounded functions, $a_k(t) \ge b_k(t) \ge 0$, $h_k(t)$ and $g_k(t)$ are Lebesgue measurable functions, $h_k(t) \ge g_k(t) \ge t$, and inequality* (2.2) *has a nonnegative solution, then* (2.26) *has a nonoscillatory solution; moreover, it has a positive nondecreasing and a negative nonincreasing solutions.*

Proof. Let u_0 be a nonnegative solution of (2.2) and denote

$$u_{n+1}(t) = \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_n(s)ds\right\}\right), \quad t \ge t_0,\ n \ge 0. \tag{2.27}$$

We have $u_0 \ge 0$, and by (2.2),

$$\begin{aligned} u_0 &\ge \sum_{k=1}^{m} a_k(t)\exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} \\ &\ge \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_0(s)ds\right\}\right) = u_1(t). \end{aligned} \tag{2.28}$$

Since $a_k(t) \ge b_k(t) \ge 0$ and $t \le g_k(t) \le h_k(t)$, then $u_1(t) \ge 0$.

Next, let us assume that $0 \le u_n(t) \le u_{n-1}(t)$. The assumptions of the theorem imply $u_{n+1} \ge 0$. Let us demonstrate that $u_{n+1}(t) \le u_n(t)$. This inequality has the form

$$\begin{aligned} &\sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_n(s)ds\right\}\right) \\ &\quad\le \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_{n-1}(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_{n-1}(s)ds\right\}\right), \end{aligned} \tag{2.29}$$

which is equivalent to

$$\begin{aligned} &\sum_{k=1}^{m}\exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\}\left(a_k(t) - b_k(t)\exp\left\{-\int_{g_k(t)}^{h_k(t)} u_n(s)ds\right\}\right) \\ &\quad\le \sum_{k=1}^{m}\exp\left\{\int_t^{h_k(t)} u_{n-1}(s)ds\right\}\left(a_k(t) - b_k(t)\exp\left\{-\int_{g_k(t)}^{h_k(t)} u_{n-1}(s)ds\right\}\right). \end{aligned} \tag{2.30}$$

This inequality is evident for any $0 \le u_n(t) \le u_{n-1}(t)$, $a_k(t) \ge 0$, and $b_k \ge 0$; thus, we have $u_{n+1}(t) \le u_n(t)$.

By the Lebesgue convergence theorem, there is a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$ satisfying

$$u(t) = \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u(s)ds\right\}\right), \quad t \ge t_0, \tag{2.31}$$

$u(t) \geq 0$, $t \geq t_0$. Then, the function

$$x(t) = x(t_0) \exp\left\{\int_{t_0}^{t} u(s)ds\right\}, \quad t \geq t_0 \tag{2.32}$$

is a positive nondecreasing solution of (2.26) for any $x(t_0) > 0$ and is a negative nonincreasing solution of (2.26) for any $x(t_0) < 0$. □

Corollary 2.14. *Let $a_k(t)$ and $b_k(t)$ be Lebesgue measurable locally essentially bounded functions satisfying $a_k(t) \geq b_k(t) \geq 0$, and let $h_k(t)$ and $g_k(t)$ be Lebesgue measurable functions, where $h_k(t) \geq g_k(t) \geq t$. Assume in addition that inequality* (2.7) *holds. Then,* (2.26) *has a nonoscillatory solution.*

Consider now the equation with constant deviations of advanced arguments

$$\dot{x}(t) - \sum_{k=1}^{m}[a_k(t)x(t+\tau_k) - b_k(t)x(t+\sigma_k)] = 0, \tag{2.33}$$

where a_k, b_k are continuous functions, $\tau_k \geq 0$, $\sigma_k \geq 0$.

Denote $A_k = \sup_{t\geq t_0} a_k(t)$, $a_k = \inf_{t\geq t_0} a_k(t)$, $B_k = \sup_{t\geq t_0} b_k(t)$, $b_k = \inf_{t\geq t_0} b_k(t)$.

Theorem 2.15. *Suppose that $a_k \geq 0$, $b_k \geq 0$, $A_k < \infty$, and $B_k < \infty$.*

If there exists a number $\lambda_0 < 0$ such that

$$\sum_{k=1}^{m}\left(a_k e^{\lambda_0 \tau_k} - B_k\right) \geq \lambda_0, \tag{2.34}$$

$$\sum_{k=1}^{m}\left(A_k - b_k e^{\lambda_0 \sigma_k}\right) \leq 0, \tag{2.35}$$

then (2.33) *has a nonoscillatory solution; moreover, it has a positive nonincreasing and a negative nondecreasing solutions.*

Proof. In the space $C[t_0, \infty)$, consider the set $M = \{u \mid \lambda_0 \leq u \leq 0\}$ and the operator

$$(Hu)(t) = \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_{t}^{t+\tau_k} u(s)ds\right\} - b_k(t)\exp\left\{\int_{t}^{t+\sigma_k} u(s)ds\right\}\right). \tag{2.36}$$

For $u \in M$, we have from (2.34) and (2.35)

$$\begin{aligned}(Hu)(t) &\leq \sum_{k=1}^{m}\left(A_k - b_k e^{\lambda_0\sigma_k}\right) \leq 0,\\ (Hu)(t) &\geq \sum_{k=1}^{m}\left(a_k e^{\lambda_0\tau_k} - B_k\right) \geq \lambda_0.\end{aligned} \tag{2.37}$$

Hence, $HM \subset M$.

Consider the integral operator

$$(Tu)(t) := \int_t^{t+\delta} u(s)ds, \quad \delta > 0. \tag{2.38}$$

We will show that TM is a compact set in the space $C[t_0,\infty)$. For $u \in M$, we have

$$\|(Tu)(t)\|_{C[t_0,\infty)} \leq \sup_{t\geq t_0} \int_t^{t+\delta} |u(s)|ds \leq |\lambda_0|\delta. \tag{2.39}$$

Hence, the functions in the set TM are uniformly bounded in the space $C[t_0,\infty)$.

The equality $\int_{t_0}^{t_0+\delta} - \int_t^{t+\delta} = \int_{t_0}^{t} + \int_t^{t_0+\delta} - \int_t^{t_0+\delta} - \int_{t_0+\delta}^{t+\delta} = \int_{t_0}^{t} - \int_{t_0+\delta}^{t+\delta}$ implies

$$\begin{aligned} |(Tu)(t) - (Tu)(t_0)| &= \left| \int_t^{t+\delta} u(s) - \int_{t_0}^{t_0+\delta} u(s)ds \right| \\ &\leq \int_{t_0}^{t} |u(s)|ds + \int_{t_0+\delta}^{t+\delta} |u(s)|ds \leq 2|\lambda_0||t - t_0|. \end{aligned} \tag{2.40}$$

Hence, the set TM and so the set HM are compact in the space $C[t_0,\infty)$.

By the Schauder fixed point theorem, there exists a continuous function u satisfying $\lambda_0 \leq u \leq 0$ such that $u = Hu$; thus, the function

$$x(t) = x(t_0) \exp\left\{ \int_{t_0}^{t} u(s)ds \right\}, \quad t \geq t_0 \tag{2.41}$$

is a positive nonincreasing solution of (2.33) for any $x(t_0) > 0$ and is a negative nondecreasing solution of (2.26) for any $x(t_0) < 0$. □

Let us remark that (2.35) for any $\lambda_0 < 0$ implies $\sum_{k=1}^{m}(A_k - b_k) < 0$.

Corollary 2.16. *Let* $\sum_{k=1}^{m}(A_k - b_k) < 0$, $\sum_{k=1}^{m} A_k > 0$, *and for*

$$\lambda_0 = \frac{\ln\left(\sum_{k=1}^{m} A_k / \sum_{k=1}^{m} b_k\right)}{\max_k \sigma_k}, \tag{2.42}$$

the inequality

$$\sum_{k=1}^{m}\left(a_k e^{\lambda_0 \tau_k} - B_k\right) \geq \lambda_0 \tag{2.43}$$

holds, then (2.33) *has a bounded positive solution.*

Proof. The negative number λ_0 defined in (2.42) is a solution of both (2.34) and (2.35); by definition, it satisfies (2.35), and (2.43) implies (2.34). □

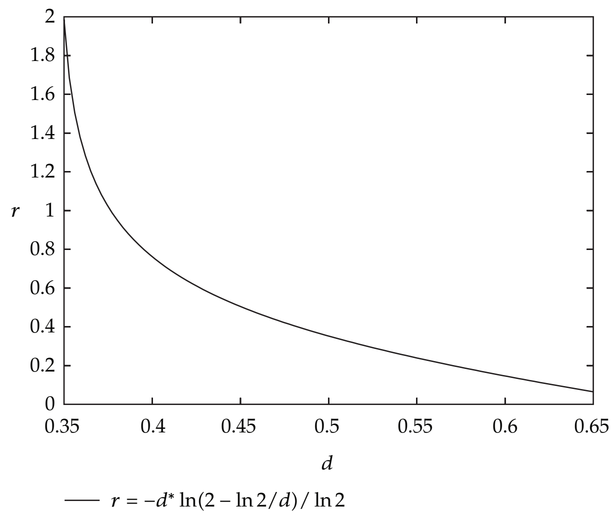

Figure 1: The domain of values (d, r) satisfying inequality (2.47). If the values of advances d and r are under the curve, then (2.44) has a positive solution.

Example 2.17. Consider the equation with constant advances and coefficients

$$\dot{x}(t) - ax(t+r) + bx(t+d) = 0, \tag{2.44}$$

where $0 < a < b$, $d > 0$, $r \geq 0$. Then, $\lambda_0 = (1/d)\ln(a/b)$ is the minimal value of λ for which inequality (2.35) holds; for (2.44), it has the form $a - be^{\lambda d} \leq 0$.

Inequality (2.34) for (2.44) can be rewritten as

$$f(\lambda) = ae^{\lambda r} - b - \lambda \geq 0, \tag{2.45}$$

where the function $f(x)$ decreases on $(-\infty, -\ln(ar)/r]$ if $\tau > 0$ and for any negative x if $r = 0$; besides, $f(0) < 0$. Thus, if $f(\lambda_1) < 0$ for some $\lambda_1 < 0$, then $f(\lambda) < 0$ for any $\lambda \in [\lambda_1, 0)$. Hence, the inequality

$$f(\lambda_0) = a\left(\frac{a}{b}\right)^{r/d} - b - \frac{1}{d}\ln\left(\frac{a}{b}\right) \geq 0 \tag{2.46}$$

is necessary and sufficient for the conditions of Theorem 2.15 to be satisfied for (2.44).

Figure 1 demonstrates possible values of advances d and r, such that Corollary 2.16 implies the existence of a positive solution in the case $1 = a < b = 2$. Then, (2.46) has the form $0.5^{r/d} \geq 2 - (\ln 2)/d$, which is possible only for $d > 0.5\ln 2 \approx 0.347$ and for these values is equivalent to

$$r \leq \frac{-d\ln(2 - \ln 2/d)}{\ln 2}. \tag{2.47}$$

3. Comments and Open Problems

In this paper, we have developed nonoscillation theory for advanced equations with variable coefficients and advances. Most previous nonoscillation results deal with either oscillation or constant deviations of arguments. Among all cited papers, only [8] has a nonoscillation condition (Theorem 2.11) for a partial case of (2.1) (with $h_k(t) = t + \tau_k$), which in this case coincides with Corollary 2.4. The comparison of results of the present paper with the previous results of the authors was discussed in the introduction.

Finally, let us state some open problems and topics for research.

(1) Prove or disprove:
if (2.1), with $a_k(t) \geq 0$, has a nonoscillatory solution, then (2.26) with positive and negative coefficients also has a nonoscillatory solution.

As the first step in this direction, prove or disprove that if $h(t) \geq t$ and the equation

$$\dot{x}(t) - a^{+}(t)x(h(t)) = 0 \tag{3.1}$$

has a nonoscillatory solution, then the equation

$$\dot{x}(t) - a(t)x(h(t)) = 0 \tag{3.2}$$

also has a nonoscillatory solution, where $a^{+}(t) = \max\{a(t), 0\}$.

If these conjectures are valid, obtain comparison results for advanced equations.

(2) Deduce nonoscillation conditions for (2.1) with oscillatory coefficients. Oscillation results for an equation with a constant advance and an oscillatory coefficient were recently obtained in [22].

(3) Consider advanced equations with positive and negative coefficients when the numbers of positive and negative terms do not coincide.

(4) Study existence and/or uniqueness problem for the initial value problem or boundary value problems for advanced differential equations.

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References

[1] D. M. Dubois, "Extension of the Kaldor-Kalecki model of business cycle with a computational anticipated capital stock," *Journal of Organisational Transformation and Social Change*, vol. 1, no. 1, pp. 63–80, 2004.

[2] A. Kaddar and H. Talibi Alaoui, "Fluctuations in a mixed IS-LM business cycle model," *Electronic Journal of Differential Equations*, vol. 2008, no. 134, pp. 1–9, 2008.

[3] G. Ladas and I. P. Stavroulakis, "Oscillations caused by several retarded and advanced arguments," *Journal of Differential Equations*, vol. 44, no. 1, pp. 134–152, 1982.

[4] Y. Kitamura and T. Kusano, "Oscillation of first-order nonlinear differential equations with deviating arguments," *Proceedings of the American Mathematical Society*, vol. 78, no. 1, pp. 64–68, 1980.
[5] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, vol. 190 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1995.
[6] N. T. Markova and P. S. Simeonov, "Oscillation criteria for first order nonlinear differential equations with advanced arguments," *Communications in Applied Analysis*, vol. 10, no. 2-3, pp. 209–221, 2006.
[7] R. P. Agarwal, S. R. Grace, and D. O'Regan, "On the oscillation of certain advanced functional differential equations using comparison methods," *Fasciculi Mathematici*, no. 35, pp. 5–22, 2005.
[8] X. Li and D. Zhu, "Oscillation and nonoscillation of advanced differential equations with variable coefficients," *Journal of Mathematical Analysis and Applications*, vol. 269, no. 2, pp. 462–488, 2002.
[9] Q. Meng and J. Yan, "Nonautonomous differential equations of alternately retarded and advanced type," *International Journal of Mathematics and Mathematical Sciences*, vol. 26, no. 10, pp. 597–603, 2001.
[10] L. Dai and C. Tian, "Oscillation criteria for advanced type differential equations," *Journal of Ocean University of Qingdao*, vol. 30, no. 2, pp. 364–368, 2000.
[11] I.-G. E. Kordonis and Ch. G. Philos, "Oscillation and nonoscillation in delay or advanced differential equations and in integrodifferential equations," *Georgian Mathematical Journal*, vol. 6, no. 3, pp. 263–284, 1999.
[12] B. Baculíková, R. P. Agarwal, T. Li, and J. Džurina, "Oscillation of third-order nonlinear functional differential equations with mixed arguments," to appear in *Acta Mathematica Hungarica*.
[13] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA, 1991.
[14] T. Kusano, "On even-order functional-differential equations with advanced and retarded arguments," *Journal of Differential Equations*, vol. 45, no. 1, pp. 75–84, 1982.
[15] R. Koplatadze, "On asymptotic behavior of solutions of n-th order Emden-Fowler differential equations with advanced argument," *Czechoslovak Mathematical Journal*, vol. 60(135), no. 3, pp. 817–833, 2010.
[16] S. Murugadass, E. Thandapani, and S. Pinelas, "Oscillation criteria for forced second-order mixed type quasilinear delay differential equations," *Electronic Journal of Differential Equations*, vol. 2010, no. 73, pp. 1–9, 2010.
[17] A. Zafer, "Interval oscillation criteria for second order super-half linear functional differential equations with delay and advanced arguments," *Mathematische Nachrichten*, vol. 282, no. 9, pp. 1334–1341, 2009.
[18] S. R. Grace, R. P. Agarwal, R. Pavani, and E. Thandapani, "On the oscillation of certain third order nonlinear functional differential equations," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 102–112, 2008.
[19] D. R. Anderson, "Oscillation of second-order forced functional dynamic equations with oscillatory potentials," *Journal of Difference Equations and Applications*, vol. 13, no. 5, pp. 407–421, 2007.
[20] L. Berezansky and Y. Domshlak, "Differential equations with several deviating arguments: Sturmian comparison method in oscillation theory. II," *Electronic Journal of Differential Equations*, vol. 2002, no. 31, pp. 1–18, 2002.
[21] L. Berezansky, E. Braverman, and S. Pinelas, "On nonoscillation of mixed advanced-delay differential equations with positive and negative coefficients," *Computers & Mathematics with Applications*, vol. 58, no. 4, pp. 766–775, 2009.
[22] X. Li, D. Zhu, and H. Wang, "Oscillation for advanced differential equations with oscillating coefficients," *International Journal of Mathematics and Mathematical Sciences*, no. 33, pp. 2109–2118, 2003.