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Research Article

Asymptotical Stability of Nonlinear Fractional Differential System with Caputo Derivative

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This paper deals with the stability of nonlinear fractional differential systems equipped with the Caputo derivative. At first, a sufficient condition on asymptotical stability is established by using a Lyapunov-like function. Then, the fractional differential inequalities and comparison method are applied to the analysis of the stability of fractional differential systems. In addition, some other sufficient conditions on stability are also presented.

1. Introduction

Fractional calculus is more than 300 years old, but it did not attract enough interest at the early stage of development. In the last three decades, fractional calculus has become popular among scientists in order to model various physical phenomena with anomalous decay, such as dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, and viscoelastic systems [1]. Recent advances in fractional calculus have been reported in [2].

Recently, stability of fractional differential systems has attracted increasing interest. In 1996, Matignon [3] firstly studied the stability of linear fractional differential systems with the Caputo derivative. Since then, many researchers have done further studies on the stability of linear fractional differential systems [4–11]. For the nonlinear fractional differential systems, the stability analysis is much more difficult and only a few are available.

Some authors [12, 13] studied the following nonlinear fractional differential system:

$$ {}_{C}D_{0,t}^{q}x(t) = f(t, x(t)), \tag{1.1} $$

with initial values $x(0) = x_0^{(0)}, \ldots, x^{(m-1)}(0) = x_0^{(m-1)}$, where $m-1 < q \le m$. They discussed the continuous dependence of solution on initial conditions and the corresponding structural stability by applying Gronwall's inequality. In [14] the authors dealt with the following fractional differential system:

$$\mathfrak{D}_{0,t}^{q} x(t) = f(t, x(t)), \tag{1.2}$$

where $0 < q \le 1$, $\mathfrak{D}_{0,t}^{q}$ denotes either the Caputo, or the Riemann-Liouville fractional derivative operator. They proposed fractional Lyapunov's second method and firstly extended the exponential stability of integer order differential systems to the Mittag-Leffler stability of fractional differential systems. Moreover, the pioneering work on the generalized Mittag-Leffler stability and the generalized fractional Lyapunov direct method was proposed in [15].

In this paper, we further study the stability of nonlinear fractional differential systems with Caputo derivative by utilizing a Lyapunov-like function. Taking into account the relation between asymptotical stability and generalized Mittag-Leffler stability, we are able to weaken the conditions assumed for the Lyapunov-like function. In addition, based on the comparison principle of fractional differential equations [16, 17], we also study the stability of nonlinear fractional differential systems by utilizing the comparison method. Our contribution in this paper is that we have relaxed the condition of the Lyapunov-like function and that we have further studied the stability. The present paper is organized as follows. In Section 2, some definitions and lemmas are introduced. In Section 3, sufficient conditions on asymptotical stability and generalized Mittag-Leffler stability are given. The comparison method is applied to the analysis of the stability of fractional differential systems in Section 4. Conclusions are included in the last section.

2. Preliminaries and Notations

Let us denote by $\mathbb{R}_+$ the set of nonnegative real numbers, by $\mathbb{R}$ the set of real numbers, and by $\mathbb{Z}_+$ the set of positive integer numbers. Let $0 < q < 1$ and set $C_q([t_0, T], \mathbb{R}) = \{f \in C((t_0, T], \mathbb{R}), (t - t_0)^q f(t) \in C([t_0, T], \mathbb{R})\}$, and $C_q([t_0, T] \times \Omega, \mathbb{R}) = \{f(t, x(t)) \in C((t_0, T] \times \Omega, \mathbb{R}), (t - t_0)^q f(t, x(t)) \in C([t_0, T] \times \Omega, \mathbb{R})\}$, where $C((t_0, t], \mathbb{R})$ denotes the space of continuous functions on the interval $(t_0, t]$.

Let us first introduce several definitions, results, and citations needed here with respect to fractional calculus which will be used later. As to fractional integrability and differentiability, the reader may refer to [18].

Definition 2.1. The fractional integral with noninteger order $q \ge 0$ of function $x(t)$ is defined as follows:

$$D_{t_0,t}^{-q} x(t) = \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t - \tau)^{q-1} x(\tau) d\tau, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The Riemann-Liouville derivative with order q of function $x(t)$ is defined as follows:

$$ {}_{\mathrm{RL}}D^{q}_{t_0,t}x(t) = \frac{1}{\Gamma(m-q)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{m-q-1}x(\tau)d\tau, \tag{2.2}$$

where $m-1 \le q < m$ and $m \in \mathbb{Z}_{+}$.

Definition 2.3. The Caputo derivative with noninteger order q of function $x(t)$ is defined as follows:

$$ {}_{C}D^{q}_{t_0,t}x(t) = \frac{1}{\Gamma(m-q)}\int_{t_0}^{t}(t-\tau)^{m-q-1}x^{(m)}(\tau)d\tau, \tag{2.3}$$

where $m-1 < q < m$ and $m \in \mathbb{Z}_{+}$.

Definition 2.4. The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{2.4}$$

where $\alpha > 0$, $z \in \mathbb{R}$. The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{2.5}$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$, $z \in \mathbb{R}$.

Clearly $E_{\alpha}(z) = E_{\alpha,1}(z)$. The following definitions are associated with the stability problem in the paper.

Definition 2.5. The constant x_{eq} is an equilibrium of fractional differential system $\mathfrak{D}^{q}_{t_0,t}x(t) = f(t,x)$ if and only if $f(t,x_{\mathrm{eq}}) = \mathfrak{D}^{q}_{t_0,t}x(t)|_{x(t)=x_{\mathrm{eq}}}$ for all $t > t_0$, where $\mathfrak{D}^{q}_{t_0,t}$ means either the Caputo or the Riemann-Liouville fractional derivative operator.

Throughout the paper, we always assume that $x_{\mathrm{eq}} = 0$.

Definition 2.6 (see [15]). The zero solution of $\mathfrak{D}^{q}_{t_0,t}x(t) = f(t,x(t))$ with order $q \in (0,1)$ is said to be stable if, for any initial value x_0, there exists an $\varepsilon > 0$ such that $\|x(t)\| \le \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to +\infty$.

Definition 2.7. Let $\mathbb{B} \subset \mathbb{R}^n$ be a domain containing the origin. The zero solution of $\mathfrak{D}^{q}_{t_0,t}x(t) = f(t,x(t))$ is said to be Mittag-Leffler stable if

$$\|x(t)\| \le \left\{m(x_0)E_q\left(-\lambda(t-t_0)^q\right)\right\}^b, \tag{2.6}$$

where t_0 is the initial time and x_0 is the corresponding initial value, $q \in (0,1)$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \subset \mathbb{R}^n$ with the Lipschitz constant $\mathcal{L}_0$.

Definition 2.8. Let $\mathbb{B} \subset \mathbb{R}^n$ be a domain containing the origin. The zero solution of $\mathfrak{D}^q_{t_0,t}x(t) = f(t, x(t))$ is said to be generalized Mittag-Leffler stable if

$$\|x(t)\| \leq \left\{m(x_0)(t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^b, \tag{2.7}$$

where t_0 is the initial time and x_0 is the corresponding initial value, $q \in (0, 1)$, $-q < \gamma \leq 1-q$, $\lambda \geq 0, b > 0, m(0) = 0, m(x) \geq 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \subset \mathbb{R}^n$ with the Lipschitz constant $\mathcal{L}_0$.

Remark 2.9. Mittag-Leffler stability and generalized Mittag-Leffler stability both belong to algebraical stability, which also imply asymptotical stability (see [15]).

Definition 2.10. A function $\alpha(r)$ is said to belong to class-$\mathcal{K}$ if $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous function such that $\alpha(0) = 0$ and it is strictly increasing.

Definition 2.11 (see [19]). The class-$\mathcal{K}$ functions $\alpha(r)$ and $\beta(r)$ are said to be with local growth momentum at the same level if there exist $r_1 > 0$, $k_i > 0$ $(i = 1, 2)$ such that $k_1\alpha(r) \geq \beta(r) \geq k_2\alpha(r)$ for all $r \in [0, r_1]$. The class-$\mathcal{K}$ functions $\alpha(r)$ and $\beta(r)$ are said to be with global growth momentum at the same level if there exist $k_i > 0$ $(i = 1, 2)$ such that $k_1\alpha(r) \geq \beta(r) \geq k_2\alpha(r)$ for all $r \in \mathbb{R}_+$.

It is useful to recall the following lemmas for our developments in the sequel.

Lemma 2.12 (see [20]). *Let $v, w \in C_{1-q}([t_0, T], \mathbb{R})$ be locally Hölder continuous for an exponent $0 < q < \nu \leq 1$, $h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and*

(i) ${}_{RL}D^q_{t_0,t}v(t) \leq h(t, v(t))$,

(ii) ${}_{RL}D^q_{t_0,t}w(t) \geq h(t, w(t))$, $t_0 < t \leq T$,

with nonstrict inequalities (i) and (ii), where $v_0 = \Gamma(q)v(t)(t-t_0)^{1-q}|_{t=t_0}$ *and* $w_0 = \Gamma(q)w(t)(t-t_0)^{1-q}|_{t=t_0}$. *Suppose further that h satisfies the standard Lipschitz condition*

$$h(t, x) - h(t, y) \leq \mathcal{L}(x - y), \quad x \geq y, \ \mathcal{L} > 0. \tag{2.8}$$

Then, $v_0 \leq w_0$ *implies* $v(t) \leq w(t)$, $t_0 < t \leq T$.

Remark 2.13. In Lemma 2.12, if we replace ${}_{\mathrm{RL}}D^q_{t_0,t}$ by ${}_{C}D^q_{t_0,t}$, but other conditions remain unchanged, then the same result holds.

Lemma 2.14 (see [16]). *Let $v, w \in C_{1-q}([t_0, T], \mathbb{R})$, $h \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ and*

(i) $v(t) \leq (v_0/\Gamma(q))(t-t_0)^{q-1} + (1/\Gamma(q))\int_{t_0}^t (t-s)^{q-1}h(s, v(s))ds$,

(ii) $w(t) \geq (w_0/\Gamma(q))(t-t_0)^{q-1} + (1/\Gamma(q))\int_{t_0}^t (t-s)^{q-1}h(s, w(s))ds$,

where $t_0 < t \leq T$, $v_0 = \Gamma(q)v(t)(t-t_0)^{1-q}|_{t=t_0}$, $w_0 = \Gamma(q)w(t)(t-t_0)^{1-q}|_{t=t_0}$, *and* $0 < q < 1$. *Assume that both inequalities are nonstrict and $h(t, x)$ is nondecreasing in x for each t. Further, suppose that h satisfies the standard Lipschitz condition*

$$h(t, x) - h(t, y) \leq \mathcal{L}(x - y), \quad x \geq y, \ \mathcal{L} > 0. \tag{2.9}$$

Then, $v_0 \leq w_0$ *implies* $v(t) \leq w(t)$, $t_0 < t \leq T$.

Remark 2.15. In Lemmas 2.12 and 2.14, T can be $+\infty$.

3. Stability of Nonlinear Fractional Differential Systems

Let us consider the following nonlinear fractional differential system [14, 15]:

$$ {}_C D_{t_0,t}^{q} x(t) = f(t, x(t)), \tag{3.1} $$

with the initial condition $x_0 = x(t_0)$, where $f : [t_0, \infty) \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and $\Omega \subset \mathbb{R}^n$ is a domain that contains the equilibrium point $x_{\text{eq}} = 0$, $0 < q < 1$. Here and throughout the paper, we always assume there exists a unique solution $x(t) \in C^1[t_0, \infty)$ to system (3.1) with the initial condition $x(t_0)$.

Recently, Li et al. [14, 15] investigated the Mittag-Leffler stability and the generalized Mittag-Leffler stability (the asymptotic stability) of system (3.1) by using the fractional Lyapunov's second method, where the following theorem has been presented.

Theorem 3.1. *Let $x_{eq} = 0$ be an equilibrium point of system* (3.1) *with $t_0 = 0$, and let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0, \infty) \times \mathbb{D} \to \mathbb{R}_+$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$ \alpha_1 \|x\|^a \le V(t, x(t)) \le \alpha_2 \|x\|^{ab}, \tag{3.2} $$

$$ {}_C D_{0,t}^{p} V(t, x(t)) \le -\alpha_3 \|x\|^{ab}, \tag{3.3} $$

where $t \ge 0$, $x \in \mathbb{D}$, $p \in (0, 1)$, and α_1, α_2, α_3, a, and b are arbitrary positive constants. Then $x_{eq} = 0$ is Mittag-Leffler stable (locally asymptotically stable). If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally Mittag-Leffler stable (globally asymptotically stable).

In the following, we give a new proof for Theorem 3.1.

Proof of Theorem 3.1. From (3.2) and (3.3), we can get

$$ {}_C D_{0,t}^{p} V(t, x(t)) \le -\frac{\alpha_3}{\alpha_2} V(t, x(t)). \tag{3.4} $$

Obviously, for the initial value $V(0, x(0))$, the linear fractional differential equation

$$ {}_C D_{0,t}^{p} V(t, x(t)) = -\frac{\alpha_3}{\alpha_2} V(t, x(t)) \tag{3.5} $$

has a unique solution $V(t, x(t)) = V(0, x(0)) E_p((-\alpha_3/\alpha_2) t^p)$.

Taking into account Remark 2.13 and the relationship between (3.4) and (3.5), we obtain

$$ V(t, x(t)) \le V(0, x(0)) E_p\left(-\frac{\alpha_3}{\alpha_2} t^p\right), \tag{3.6} $$

where $E_p((-\alpha_3/\alpha_2)t^p)$ is a nonnegative function [21]. Substituting (3.6) in (3.2) yields

$$\|x(t)\| \le \left[\frac{V(0,x(0))}{\alpha_1} E_p\left(-\frac{\alpha_3}{\alpha_2}t^p\right)\right]^{1/a}, \tag{3.7}$$

where $E_p((-\alpha_3/\alpha_2)t^p) \to 0$ $(t \to +\infty)$ from the asymptotic expansion of Mittag-Leffler function [22]. Hence the proof is completed. □

According to the above results, we have the following theorem.

Theorem 3.2. *Let $x_{eq} = 0$ be an equilibrium point of system (3.1), and let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a continuously differentiable function $V(t,x(t)) : [t_0,\infty) \times \mathbb{D} \to \mathbb{R}_+$ and class-$\mathcal{K}$ function α satisfying*

$$V(t,x(t)) \ge \alpha(\|x\|), \tag{3.8}$$

$$_C D^p_{t_0,t} V(t,x(t)) \le 0, \tag{3.9}$$

where $x \in \mathbb{D}$, $p \in (0,1)$. Then $x_{eq} = 0$ is locally stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally stable.

Proof. Proceeding the same way as that in the proof of Theorem 3.1, it follows from (3.9) that $V(t,x(t)) \le V(t_0,x(t_0))$. Again taking into account (3.8), one can get

$$\|x(t)\| \le \alpha^{-1}(V(t_0,x(t_0))), \tag{3.10}$$

where $t \ge t_0$. Therefore, the equilibrium point $x_{\text{eq}} = 0$ is stable. So the proof is finished. □

In the above two theorems, the stronger requirements on function V have been assumed to ensure the existence of $_C D^p_{t_0,t} V(t,x(t))$. This undoubtedly increases the difficulty in choosing the function $V(t,x(t))$. In fact, we can weaken the continuously differential function $V(t,x(t))$ as $V(t,x(t)) \in C_{1-p}([t_0,\infty) \times \mathbb{D}, \mathbb{R}_+)$. Here we give the corresponding results.

Theorem 3.3. *Let $x_{eq} = 0$ be an equilibrium point of system (3.1), and let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin, $V(t,x(t)) \in C_{1-p}([t_0,\infty) \times \mathbb{D}, \mathbb{R}_+)$. Assume there exists a class-$\mathcal{K}$ function α such that*

$$V(t,x(t)) \ge \alpha(\|x\|), \tag{3.11}$$

$$_{RL} D^p_{t_0,t} V(t,x(t)) \le 0, \tag{3.12}$$

where $t > t_0 \geq 0$, $x \in \mathbb{D}$, and $p \in (0,1)$. Then $x_{eq} = 0$ is locally asymptotically stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally asymptotically stable.

Proof. Note that the linear fractional differential equation

$$ {}_{\mathrm{RL}}D^{p}_{t_0,t} V(t, x(t)) = 0 \tag{3.13} $$

has a unique solution $V(t,x(t)) = (V_0/\Gamma(p))(t-t_0)^{p-1}$ for initial value $V_0 = \Gamma(p)V(t,x(t))(t-t_0)^{1-p}|_{t=t_0}$.

Taking into account Lemma 2.12 and the relationship between (3.12) and (3.13), we obtain

$$ V(t,x(t)) \leq \frac{V_0}{\Gamma(p)}(t-t_0)^{p-1}. \tag{3.14} $$

Substituting (3.14) into (3.11) gives

$$ \|x(t)\| \leq \alpha^{-1}\left(\frac{V_0}{\Gamma(p)}(t-t_0)^{p-1}\right) \longrightarrow 0 \quad (t \longrightarrow +\infty), \tag{3.15} $$

from the definition of class-$\mathcal{K}$. This completes the proof. □

Corollary 3.4. *Let $x_{eq} = 0$ be an equilibrium point of system* (3.1)*, let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin, and let $V(t,x(t)) \in C_{1-p}([t_0,\infty) \times \mathbb{D}, \mathbb{R}_+)$ be locally Lipschitz with respect to x. Assume $V(t,0) = 0$,*

$$ V(t,x(t)) \geq a\|x\|^b, \qquad {}_{RL}D^{p}_{t_0,t} V(t,x(t)) \leq 0, \tag{3.16} $$

where $t > t_0 \geq 0$, $x \in \mathbb{D}$, $p \in (0,1)$, and a, b are arbitrary positive constants. Then $x_{eq} = 0$ is generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally generalized Mittag-Leffler stable.

Proof. In Theorem 3.3, by replacing $\alpha(\|x\|)$ by $a\|x\|^b$, we can get

$$ \|x(t)\| \leq \left\{\frac{V_0}{a}(t-t_0)^{p-1}E_{p,p}\left(0 \cdot (t-t_0)^p\right)\right\}^{1/b}, \tag{3.17} $$

so the conclusion holds. □

Theorem 3.5. *Let $x_{eq} = 0$ be an equilibrium point of system* (3.1)*, let $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin, and let $V(t,x(t)) \in C_{1-p}([t_0,\infty) \times \mathbb{D}, \mathbb{R}_+)$ be locally Lipschitz with respect to x. Assume*

(i) *there exist class-$\mathcal{K}$ functions α_i $(i = 1, 2, 3)$ having global growth momentum at the same level and satisfying*

$$\begin{aligned} \alpha_1(\|x\|) &\le V(t, x(t)) \le \alpha_2(\|x\|), \\ {}_{RL}D^p_{t_0,t} V(t, x(t)) &\le -\alpha_3(\|x\|), \end{aligned} \tag{3.18}$$

(ii) *there exists $a > 0$ such that $\alpha_1(r)$ and r^a have global growth momentum at the same level,*

where $t > t_0 \ge 0$, $x \in \mathbb{D}$, and $p \in (0, 1)$. Then $x_{eq} = 0$ is locally generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{eq} = 0$ is globally generalized Mittag-Leffler stable.

Proof. It follows from condition (i) that there exists $k_1 > 0$ such that

$$\begin{aligned} {}_{\mathrm{RL}}D^p_{t_0,t} V(t, x(t)) &\le -\alpha_3(\|x\|) \\ &\le -k_1 \alpha_2(\|x\|) \\ &\le -k_1 V(t, x(t)). \end{aligned} \tag{3.19}$$

On the other hand, the linear fractional differential equation

$${}_{\mathrm{RL}}D^p_{t_0,t} V(t, x(t)) = -k_1 V(t, x(t)) \tag{3.20}$$

has a unique solution

$$V(t, x(t)) = \frac{V_0}{\Gamma(p)} (t - t_0)^{p-1} \cdot E_{p,p}\left(-k_1 (t - t_0)^p\right), \tag{3.21}$$

for the initial value $V_0 = \Gamma(p) V(t, x(t)) (t - t_0)^{1-p}|_{t=t_0}$.

Using (3.19), (3.20), and Lemma 2.12, we obtain

$$\alpha_1(\|x\|) \le V(t, x(t)) \le \frac{V_0}{\Gamma(p)} (t - t_0)^{p-1} \cdot E_{p,p}\left(-k_1 (t - t_0)^p\right), \tag{3.22}$$

where $E_{p,p}(-k_1(t - t_0)^p)$ is a nonnegative function [23, 24].

In addition, using condition (ii), one gets

$$(k_2 \|x\|)^a \le \alpha_1(\|x\|), \tag{3.23}$$

for all $x \in \mathbb{D}$, where $k_2 > 0$.

Substituting (3.23) into (3.22), we finally obtain

$$\|x(t)\| \le \left\{ \frac{V_0}{k_2^a \Gamma(p)} (t - t_0)^{p-1} E_{p,p}\left(-k_1 (t - t_0)^p\right) \right\}^{1/a}. \tag{3.24}$$

Hence, the zero solution of system (3.1) is locally generalized Mittag-Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x_{\text{eq}} = 0$ is globally generalized Mittag-Leffler stable. The proof is completed. □

Remark 3.6. The nonnegative function $E_{p,p}(-k_1(t-t_0)^p)$ tends to zero as t approaches infinity from the asymptotic expansion of two-parameter Mittag-Leffler function [22], so the zero solution of system (3.1) satisfying the conditions of Theorem 3.5 is also asymptotically stable.

4. The Comparison Results on the Stability

It is well known that the comparison method is an effective way in judging the stability of ordinary differential systems. In this section, we will discuss similar results on the stability of fractional differential systems by using the comparison method.

In what follows, we consider system (3.1) with $f(t,0) = 0$ and the scalar fractional differential equation

$$_{\text{RL}}D^{q}_{t_0,t}u(t) = g(t,u), \qquad u_0 = \Gamma(q)u(t)(t-t_0)^{1-q}\Big|_{t=t_0}, \tag{4.1}$$

where the initial value $u_0 \in \mathbb{R}_+$, $u(t) \in C_{1-q}([t_0,\infty),\mathbb{R})$, $g \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ is Lipschitz in u and $g(t,0) = 0$, $0 < q < 1$. Also, we assume there exists a unique solution $u(t)$ $(t \ge t_0)$ for system (4.1) with the initial value u_0.

Theorem 4.1. *For system* (3.1), *let $x_{eq} = 0$ be an equilibrium point of system* (3.1), *and let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exist a Lyapunov-like function $V \in C_{1-q}([t_0,\infty)\times\Omega,\mathbb{R}_+)$ and a class-$\mathcal{K}$ function α such that $V(t,0) = 0$, $V(t,x) \ge \alpha(\|x\|)$, and $V(t,x)$ satisfies the inequality*

$$_{RL}D^{q}_{t_0,t}V(t,x) \le g(t,V(t,x)), \quad (t,x) \in [t_0,\infty)\times\Omega. \tag{4.2}$$

Suppose further that $g(t,x)$ is nondecreasing in x for each t.

(i) *If the zero solution of* (4.1) *is stable, then the zero solution of system* (3.1) *is stable;*

(ii) *if the zero solution of* (4.1) *is asymptotically stable, then the zero solution of system* (3.1) *is asymptotically stable, too.*

Proof. Let $x(t) = x(t,t_0,x_0)$ denote the solution of system (3.1) with initial value $x_0 \in \Omega$. Along the solution curve $x(t)$, $V(t,x(t)$ can be written as $V(t)$ and

$$V(t) \le \frac{V_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,V(s))ds, \tag{4.3}$$

where $V_0 = \Gamma(q)V(t)(t-t_0)^{1-q}|_{t=t_0}$. Applying the fractional integral operator $D^{-q}_{t_0,t}$ to both sides of (4.1) leads to

$$u(t) = \frac{u_0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}g(s,u(s))ds. \tag{4.4}$$

Now, taking $u_0 = V_0$ and applying Lemma 2.14 to inequalities (4.3) and (4.4), one has $V(t) \le u(t)$, $t > t_0$.

(i) If the zero solution of (4.1) is stable, then for any initial value $u_0 \ge 0$, there exists $\epsilon > 0$ such that $|u(t)| < \epsilon$ for all $t > t_0$. Therefore, taking into account $V(t, x(t)) \ge \alpha(\|x\|)$, one gets

$$\alpha(\|x(t)\|) \le V(t,x) \le u(t) < \epsilon, \tag{4.5}$$

that is, $\|x(t)\| < \alpha^{-1}(\epsilon)$, and the zero solution of system (3.1) is stable.

(ii) One can directly derive

$$\alpha(\|x(t)\|) \le V(t,x) \le u(t) < \epsilon \tag{4.6}$$

from the same argument in (i). Then, taking the limit to both sides of (4.6) and combining with the definition of class-$\mathcal{K}$ function, one can obtain $\lim_{t\to+\infty}\|x(t)\| = 0$.

The proof is thus finished. □

Remark 4.2. In Theorem 4.1 and system (4.1), if we replace order q by $p \in (0,1)$, but other conditions remain unchanged, then the result in Theorem 4.1 still holds.

Especially, if the class-$\mathcal{K}$ function $\alpha(\|x\|)$ in Theorem 4.1 and $\|x\|^a$ have global growth momentum at the same level, then we can have similar comparison result on the generalized Mittag-Leffler stability as follows.

Theorem 4.3. *For system* (3.1), *let $x_{eq} = 0$ be an equilibrium of system* (3.1), *and let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin. Assume that there exists a Lyapunov-like function $V \in C_{1-q}([t_0,\infty) \times \Omega, \mathbb{R}_+)$ such that $V(t,0) = 0$, $V(t,x) \ge k\|x\|^a$, and $V(t,x)$ is locally Lipschitz in x and satisfies the inequality*

$$_{RL}D^q_{t_0,t}V(t,x) \le g(t, V(t,x)), \quad (t,x) \in [t_0,\infty) \times \Omega, \tag{4.7}$$

where $k > 0$, $a > 0$. Suppose further that $g(t,x)$ is nondecreasing in x for each t. Then the zero solution of system (3.1) *is also locally generalized Mittag-Leffler stable if the zero solution of* (4.1) *is locally generalized Mittag-Leffler stable. In addition, if the assumptions hold globally on $\mathbb{R}^n$, then the globally generalized Mittag-Leffler stability of zero solution of* (4.1) *implies the globally generalized Mittag-Leffler stability of zero solution of system* (3.1).

Proof. First, from Definition 2.8, if the zero solution of (4.1) is generalized Mittag-Leffler stable, then there exist $\lambda \ge 0$, $b > 0$, $-q < \gamma \le 1 - q$ such that

$$|u(t)| \le \left\{m(u_0)(t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^b, \tag{4.8}$$

where $m(0) = 0$, $m(x) \ge 0$ and $m(x)$ is locally Lipschitz in x with Lipschitz constant $\mathcal{L}_0$.

Taking $u_0 = V_0 = \Gamma(q)V(t,x)(t-t_0)^{1-q}|_{t=t_0}$ and noting that $V(t,x) \leq u(t)$ holds from Theorem 4.1, then taking into account (4.8) and $V(t,x) \geq k\|x\|^a$, we obtain

$$k\|x(t)\|^a \leq V(t,x) \leq \left\{m(u_0)(t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^b. \tag{4.9}$$

Furthermore,

$$\|x(t)\| \leq \left\{\frac{m\left(\Gamma(q)V(t,x(t_0))(t-t_0)^{1-q}\Big|_{t=t_0}\right)}{k^{1/b}} \cdot (t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^{b/a}. \tag{4.10}$$

Let $M(x) = m(\Gamma(q)V(t,x)(t-t_0)^{1-q}|_{t=t_0})/k^{1/b}$. Then it follows that

$$\|x(t)\| \leq \left\{M(x(t_0))(t-t_0)^{-\gamma}E_{q,1-\gamma}\left(-\lambda(t-t_0)^q\right)\right\}^{b/a}, \tag{4.11}$$

where $M(0) = m(\Gamma(q)V(t,0)(t-t_0)^{1-q}|_{t=t_0})/k^{1/b} = 0$ due to $V(t,0) = 0$. It is obvious that $M(x)$ is a nonnegative function from $m(x), V(t,x) \geq 0$ and $k > 0$. In addition, $M(x)$ is locally Lipschitz in x since $m(x)$ and $V(t,x)$ are locally Lipschitz in x. So, the zero solution of system (3.1) is generalized Mittag-Leffler stable. The proof is completed. □

5. Conclusion

In this paper, we have studied the stability of the zero solution of nonlinear fractional differential systems with the Caputo derivative and the commensurate order $0 < q < 1$ by using a Lyapunov-like function. Compared to [15], we weaken the continuously differential function $V(t,x)$ as $V(t,x) \in C_{1-p}([t_0,\infty)\times\mathbb{D},\mathbb{R}_+)$. Sufficient conditions on generalized Mittag-Leffler stability and asymptotical stability are derived. Meanwhile, comparison method is applied to the analysis of the stability of fractional differential systems by fractional differential inequalities.

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