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Research Article

Boundary-Value Problems for Weakly Nonlinear Delay Differential Systems

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Conditions are derived of the existence of solutions of nonlinear boundary-value problems for systems of n ordinary differential equations with constant coefficients and single delay (in the linear part) and with a finite number of measurable delays of argument in nonlinearity: $\dot{z}(t) = Az(t-\tau) + g(t) + \varepsilon Z(z(h_i(t), t, \varepsilon),\ t \in [a,b]$, assuming that these solutions satisfy the initial and boundary conditions $z(s) := \psi(s)$ if $s \notin [a,b]$, $\ell z(\cdot) = \alpha \in \mathbb{R}^m$. The use of a delayed matrix exponential and a method of pseudoinverse by Moore-Penrose matrices led to an *explicit* and *analytical* form of sufficient conditions for the existence of solutions in a given space and, moreover, to the construction of an iterative process for finding the solutions of such problems in a general case when the number of boundary conditions (defined by a linear vector functional ℓ) does not coincide with the number of unknowns in the differential system with a single delay.

1. Introduction

First, we derive some auxiliary results concerning the theory of differential equations with delay. Consider a system of linear differential equations with concentrated delay

$$\dot{z}(t) - A(t)z(h_0(t)) = g(t) \quad \text{if } t \in [a,b], \tag{1.1}$$

assuming that

$$z(s) := \psi(s) \quad \text{if } s \notin [a,b], \tag{1.2}$$

where A is an $n\times n$ real matrix and g is an n-dimensional real column-vector with components in the space $L_p[a,b]$ (where $p \in [1,\infty)$) of functions summable on $[a,b]$; the delay $h_0(t) \le t$ is a function $h_0 : [a,b] \to \mathbb{R}$ measurable on $[a,b]$; $\psi : \mathbb{R} \setminus [a,b] \to \mathbb{R}^n$ is a given function. Using the denotations

$$(S_{h_0} z)(t) := \begin{cases} z(h_0(t)) & \text{if } h_0(t) \in [a,b], \\ \theta & \text{if } h_0(t) \notin [a,b], \end{cases} \tag{1.3}$$

$$\psi^{h_0}(t) := \begin{cases} \theta & \text{if } h_0(t) \in [a,b], \\ \psi(h_0(t)) & \text{if } h_0(t) \notin [a,b], \end{cases} \tag{1.4}$$

where θ is an n-dimensional zero column-vector and assuming $t \in [a,b]$, it is possible to rewrite (1.1), (1.2) as

$$(Lz)(t) := \dot{z}(t) - A(t)(S_{h_0} z)(t) = \varphi(t), \quad t \in [a,b], \tag{1.5}$$

where φ is an n-dimensional column-vector defined by the formula

$$\varphi(t) := g(t) + A(t)\psi^{h_0}(t) \in L_p[a,b]. \tag{1.6}$$

We will investigate (1.5) assuming that the operator L maps a Banach space $D_p[a,b]$ of absolutely continuous functions $z : [a,b] \to \mathbb{R}^n$ into a Banach space $L_p[a,b]$ ($1 \le p < \infty$) of functions $\varphi : [a,b] \to \mathbb{R}^n$ summable on $[a,b]$; the operator S_{h_0} maps the space $D_p[a,b]$ into the space $L_p[a,b]$. Transformations (1.3), (1.4) make it possible to add the initial function $\psi(s)$, $s < a$ to nonhomogeneity generating an additive and homogeneous operation not depending on ψ and without the classical assumption regarding the continuous connection of solution $z(t)$ with the initial function $\psi(t)$ at the point $t = a$.

A solution of differential system (1.5) is defined as an n-dimensional column vector-function $z \in D_p[a,b]$, absolutely continuous on $[a,b]$, with a derivative $\dot{z} \in L_p[a,b]$ satisfying (1.5) almost everywhere on $[a,b]$.

Such approach makes it possible to apply well-developed methods of linear functional analysis to (1.5) with a linear and bounded operator L. It is well-known (see: [1, 2]) that a nonhomogeneous operator equation (1.5) with delayed argument is solvable in the space $D_p[a,b]$ for an arbitrary right-hand side $\varphi \in L_p[a,b]$ and has an n-dimensional family of solutions ($\dim \ker L = n$) in the form

$$z(t) = X(t)c + \int_a^b K(t,s)\varphi(s)\,ds \quad \forall c \in \mathbb{R}^n, \tag{1.7}$$

where the kernel $K(t,s)$ is an $n \times n$ Cauchy matrix defined in the square $[a,b] \times [a,b]$ being, for every fixed $s \le t$, a solution of the matrix Cauchy problem

$$(LK(\cdot,s))(t) := \frac{\partial K(t,s)}{\partial t} - A(t)(S_{h_0}K(\cdot,s))(t) = \Theta, \quad K(s,s) = I, \tag{1.8}$$

where $K(t,s) \equiv \Theta$ if $a \le t < s \le b$, Θ is $n \times n$ null matrix and I is $n \times n$ identity matrix. A fundamental $n \times n$ matrix $X(t)$ for the homogeneous ($\varphi \equiv \theta$) equation (1.5) has the form $X(t) = K(t,a)$, $X(a) = I$ [2]. Throughout the paper, we denote by Θ_s an $s \times s$ null matrix if $s \neq n$, by $\Theta_{s,p}$ an $s \times p$ null matrix, by I_s an $s \times s$ identity matrix if $s \neq n$, and by θ_s an s-dimensional zero column-vector if $s \neq n$.

A serious disadvantage of this approach, when investigating the above-formulated problem, is the necessity to find the Cauchy matrix $K(t,s)$ [3, 4]. It exists but, as a rule, can only be found numerically. Therefore, it is important to find systems of differential equations with delay such that this problem can be solved directly. Below we consider the case of a system with so-called single delay [5]. In this case, the problem of how to construct the Cauchy matrix is successfully solved *analytically* due to a delayed matrix exponential defined below.

1.1. A Delayed Matrix Exponential

Consider a Cauchy problem for a linear nonhomogeneous differential system with constant coefficients and with a single delay τ

$$\dot{z}(t) = Az(t-\tau) + g(t), \tag{1.9}$$

$$z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0], \tag{1.10}$$

with an $n \times n$ constant matrix A, $g : [0,\infty) \to \mathbb{R}^n$, $\psi : [-\tau, 0] \to \mathbb{R}^n$, $\tau > 0$ and an unknown vector-solution $z : [-\tau,\infty) \to \mathbb{R}^n$. Together with a nonhomogeneous problem (1.9), (1.10), we consider a related homogeneous problem

$$\begin{gathered} \dot{z}(t) = Az(t-\tau), \\ z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0]. \end{gathered} \tag{1.11}$$

Denote by e_τ^{At} a matrix function called a delayed matrix exponential (see [5]) and defined as

$$e_\tau^{At} := \begin{cases} \Theta & \text{if } -\infty < t < -\tau, \\ I & \text{if } -\tau \le t < 0, \\ I + A\dfrac{t}{1!} & \text{if } 0 \le t < \tau, \\ I + A\dfrac{t}{1!} + A^2\dfrac{(t-\tau)^2}{2!} & \text{if } \tau \le t < 2\tau, \\ \cdots & \\ I + A\dfrac{t}{1!} + \cdots + A^k\dfrac{(t-(k-1)\tau)^k}{k!} & \text{if } (k-1)\tau \le t < k\tau, \\ \cdots . & \end{cases} \tag{1.12}$$

This definition can be reduced to the following expression:

$$e_\tau^{At} = \sum_{n=0}^{[t/\tau]+1} A^n \,\frac{(t-(n-1)\tau)^n}{n!}, \tag{1.13}$$

where $[t/\tau]$ is the greatest integer function. The delayed matrix exponential equals the unit matrix I on $[-\tau, 0]$ and represents a fundamental matrix of a homogeneous system with single delay. Thus, the delayed matrix exponential solves the Cauchy problem for a homogeneous system (1.11), satisfying the unit initial conditions

$$z(s) = \varphi(s) \equiv e_\tau^{As} = I \quad \text{if } -\tau \le s \le 0, \tag{1.14}$$

and the following statement holds (see, e.g., [5], [6, Remark 1], [7, Theorem 2.1]).

Lemma 1.1. *A solution of a Cauchy problem for a nonhomogeneous system with single delay* (1.9), *satisfying a constant initial condition*

$$z(s) = \varphi(s) = c \in \mathbb{R}^n \quad \textit{if } s \in [-\tau, 0] \tag{1.15}$$

has the form

$$z(t) = e_\tau^{A(t-\tau)}c + \int_0^t e_\tau^{A(t-\tau-s)}g(s)ds. \tag{1.16}$$

The delayed matrix exponential was applied, for example, in [6, 7] to investigation of boundary value problems of diffferential systems with a single delay and in [8] to investigation of the stability of linear perturbed systems with a single delay.

1.2. Fredholm Boundary-Value Problem

Without loss of generality, let $a = 0$ and, with a view of the above, the problem (1.9), (1.10) can be transformed ($h_0(t) := t - \tau$) to an equation of the type (1.1) (see (1.5))

$$\dot{z}(t) - A(S_{h_0}z)(t) = \varphi(t), \quad t \in [0, b], \tag{1.17}$$

where, in accordance with (1.3),(1.4)

$$\begin{gathered}(S_{h_0}z)(t) = \begin{cases} z(t-\tau) & \text{if } t-\tau \in [0,b], \\ \theta & \text{if } t-\tau \notin [0,b], \end{cases} \\ \varphi(t) = \ g(t) + A\ \psi^{h_0}(t) \ \in \ L_p[0,b], \\ \psi^{h_0}(t) = \begin{cases} \theta & \text{if } t-\tau \in [0,b], \\ \psi(t-\tau) & \text{if } t-\tau \notin [0,b]. \end{cases}\end{gathered} \tag{1.18}$$

A general solution of problem (1.17) for a nonhomogeneous system with single delay and zero initial data has the form (1.7)

$$z(t) = X(t)c + \int_0^b K(t,s)\varphi(s)ds \quad \forall c \in \mathbb{R}^n, \tag{1.19}$$

where, as can easily be verified (in view of the above-defined delayed matrix exponential) by substituting into (1.17),

$$X(t) = e_\tau^{A(t-\tau)}, \qquad X(0) = e_\tau^{-A\tau} = I \tag{1.20}$$

is a normal fundamental matrix of the homogeneous system related to (1.17) (or (1.9)) with initial data $X(0) = I$, and the Cauchy matrix $K(t,s)$ has the form

$$\begin{gathered}K(t,s) = e_\tau^{A(t-\tau-s)} \quad \text{if } 0 \le s < t \le b, \\ K(t,s) \equiv \Theta \quad \text{if } 0 \le t < s \le b.\end{gathered} \tag{1.21}$$

Obviously

$$K(t,0) = e_\tau^{A(t-\tau)} = X(t), \qquad K(0,0) = e_\tau^{A(-\tau)} = X(0) = I, \tag{1.22}$$

and, therefore, the initial problem (1.17) for systems of ordinary differential equations with constant coefficients and single delay has an n-parametric family of linearly independent solutions (1.16).

Now, we will deal with a general boundary-value problem for system (1.17). Using the results [2, 9], it is easy to derive statements for a general boundary-value problem if the number m of boundary conditions does not coincide with the number n of unknowns in a differential system with single delay.

We consider a boundary-value problem

$$\begin{gathered} \dot{z}(t) - Az(t-\tau) = g(t), \quad t \in [0,b], \\ z(s) := \psi(s), \quad s \notin [0,b], \end{gathered} \tag{1.23}$$

assuming that

$$\ell z(\cdot) = \alpha \in \mathbb{R}^m, \tag{1.24}$$

or, using (1.18), its equivalent form

$$\begin{gathered} \dot{z}(t) - A(S_{h_0}z)(t) = \varphi(t), \quad t \in [0,b], \\ \ell z(\cdot) = \alpha \in \mathbb{R}^m, \end{gathered} \tag{1.25}$$

where α is an m-dimensional constant vector-column ℓ is an m-dimensional linear vector-functional defined on the space $D_p[0,b]$ of an n-dimensional vector-functions

$$\ell = \text{col}\ (\ell_1, \dots, \ell_m) : D_p[0,b] \longrightarrow \mathbb{R}^m, \qquad \ell_i : D_p[0,b] \longrightarrow \mathbb{R}, \quad i = 1, \dots, m, \tag{1.26}$$

absolutely continuous on $[0,b]$. Such problems for functional-differential equations are of Fredholm's type (see, e.g., [1, 2]). In order to formulate the following result, we need several auxiliary abbreviations. We set

$$Q := \ell X(\cdot) = \ell e_\tau^{A(\cdot-\tau)}. \tag{1.27}$$

We define an $n \times n$-dimensional matrix (orthogonal projection)

$$P_Q := I - Q^+ Q, \tag{1.28}$$

projecting space $\mathbb{R}^n$ to $\ker Q$ of the matrix Q.

Moreover, we define an $m \times m$-dimensional matrix (orthogonal projection)

$$P_{Q^*} := I_m - QQ^+, \tag{1.29}$$

projecting space $\mathbb{R}^m$ to $\ker Q^*$ of the transposed matrix $Q^* = Q^T$, where I_m is an $m \times m$ identity matrix and Q^+ is an $n \times m$-dimensional matrix pseudoinverse to the $m \times n$-dimensional matrix Q. Denote $d := \operatorname{rank} P_{Q^*}$ and $n_1 := \operatorname{rank} Q = \operatorname{rank} Q^*$. Since

$$\operatorname{rank} P_{Q^*} = m - \operatorname{rank} Q^*, \tag{1.30}$$

we have $d = m - n_1$.

We will denote by $P_{Q_d^*}$ an $d \times m$-dimensional matrix constructed from d linearly independent rows of the matrix P_{Q^*}. Denote $r := \operatorname{rank} P_Q$. Since

$$\operatorname{rank} P_Q = n - \operatorname{rank} Q, \tag{1.31}$$

we have $r = n - n_1$. By P_{Q_r} we will denote an $n \times r$-dimensional matrix constructed from r linearly independent columns of the matrix P_Q. Finally, we define

$$X_r(t) := X(t) P_{Q_r}, \tag{1.32}$$

and a generalized Green operator

$$(G\varphi)(t) := \int_0^b G(t,s)\varphi(s)ds, \tag{1.33}$$

where

$$G(t,s) := K(t,s) - e_\tau^{A(t-\tau)} Q^+ \ell K(\cdot,s) \tag{1.34}$$

is a generalized Green matrix corresponding to the boundary-value problem (1.25) (the Cauchy matrix $K(t,s)$ has the form (1.21)).

In [6, Theorem 4], the following result (formulating the necessary and sufficient conditions of solvability and giving representations of the solutions $z \in D_p[0,b]$, $\dot{z} \in L_p[0,b]$ of the boundary-value problem (1.25) in an *explicit analytical* form) is proved.

Theorem 1.2. *If $n_1 \leq \min(m,n)$, then:*

(i) *the homogeneous problem*

$$\begin{aligned} \dot{z}(t) - A(S_{h_0} z)(t) &= \theta, \quad t \in [0,b], \\ \ell z(\cdot) &= \theta_m \in \mathbb{R}^m \end{aligned} \tag{1.35}$$

corresponding to problem (1.25) *has exactly r linearly independent solutions*

$$z(t,c_r) = X_r(t)c_r = e_\tau^{A(t-\tau)} P_{Q_r} c_r \in D_p[0,b], \tag{1.36}$$

(ii) *nonhomogeneous problem* (1.25) *is solvable in the space $D_p[0,b]$ if and only if $\varphi \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ satisfy d linearly independent conditions*

$$P_{Q_d^*} \cdot \left(\alpha - \ell \int_0^b K(\cdot,s)\varphi(s)ds \right) = \theta_d, \tag{1.37}$$

(iii) *in that case the nonhomogeneous problem* (1.25) *has an r-dimensional family of linearly independent solutions represented in an analytical form*

$$z(t) = z_0(t, c_r) := X_r(t)c_r + \big(G\varphi\big)(t) + X(t)Q^{+}\alpha \quad \forall c_r \in \mathbb{R}^r. \tag{1.38}$$

2. Perturbed Weakly Nonlinear Boundary Value Problems

As an example of applying Theorem 1.2, we consider a problem of the branching of solutions $z : [0, b] \to \mathbb{R}^n$, $b > 0$ of systems of nonlinear ordinary differential equations with a small parameter ε and with a finite number of measurable delays $h_i(t)$, $i = 1, 2, \ldots, k$ of argument of the form

$$\dot{z}(t) = Az(t - \tau) + g(t) + \varepsilon Z(z(h_i(t)), t, \varepsilon), \quad t \in [0, b],\ h_i(t) \leq t, \tag{2.1}$$

satisfying the initial and boundary conditions

$$z(s) = \psi(s), \quad \text{if } s < 0, \qquad \ell z(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^m, \tag{2.2}$$

and such that its solution $z = z(t, \varepsilon)$, satisfying

$$\begin{aligned} &z(\cdot, \varepsilon) \in D_p[0, b], \\ &\dot{z}(\cdot, \varepsilon) \in L_p[0, b], \\ &z(t, \cdot) \in C[0, \varepsilon_0], \end{aligned} \tag{2.3}$$

for a sufficiently small $\varepsilon_0 > 0$, for $\varepsilon = 0$, turns into one of the generating solutions (1.38); that is, $z(t, 0) = z_0(t, c_r)$ for a $c_r \in \mathbb{R}^r$. We assume that the $n \times 1$ vector-operator Z satisfies

$$\begin{aligned} &Z(\cdot, t, \varepsilon) \in C^1\left[\|z - z_0\| \leq q\right], \\ &Z(z(h_i(t)), \cdot, \varepsilon) \in L_p[0, b], \\ &Z(z(h_i(t)), t, \cdot) \in C[0, \varepsilon_0], \end{aligned} \tag{2.4}$$

where $q > 0$ is sufficiently small. Using denotations (1.3), (1.4), and (1.6), it is easy to show that the perturbed nonlinear boundary value problem (2.1), (2.2) can be rewritten in the form

$$\dot{z}(t) = A(S_{h_0}z)(t) + \varepsilon Z((S_h z)(t), t, \varepsilon) + \varphi(t), \quad \ell z(\cdot) = \alpha,\ t \in [0, b]. \tag{2.5}$$

In (2.5), A is an $n \times n$ constant matrix, $h_0 : [0, b] \to \mathbb{R}$ is a single delay defined by $h_0(t) := t - \tau$, $\tau > 0$,

$$(S_h z)(t) = \operatorname{col}[(S_{h_1}z)(t), \ldots, (S_{h_k}z)(t)] \tag{2.6}$$

is an N-dimensional column vector, where $N = nk$, and φ is an n-dimensional column vector given by

$$\varphi(t) = g(t) + A\ \psi^{h_0}(t). \tag{2.7}$$

The operator S_h maps the space D_p into the space

$$L_p^N = \underbrace{L_p \times \cdots \times L_p}_{k\text{-times}}, \tag{2.8}$$

that is, $S_h : D_p \to L_p^N$. Using denotation (1.3) for the operator $S_{h_i} : D_p \to L_p$, $i = 1, \ldots, k$, we have the following representation:

$$(S_{h_i} z)(t) = \int_0^b \chi_{h_i}(t,s)\dot{z}(s)ds + \chi_{h_i}(t,0)z(0), \tag{2.9}$$

where

$$\chi_{h_i}(t,s) = \begin{cases} 1, & \text{if } (t,s) \in \Omega_i, \\ 0, & \text{if } (t,s) \notin \Omega_i \end{cases} \tag{2.10}$$

is the characteristic function of the set

$$\Omega_i := \{(t,s) \in [0,b] \times [0,b] : 0 \le s \le h_i(t) \le b\}. \tag{2.11}$$

Assume that the generating boundary value problem

$$\dot{z}(t) = A(S_{h_0} z)(t) + \varphi(t), \quad lz = \alpha, \tag{2.12}$$

being a particular case of (2.5) for $\varepsilon = 0$, has solutions for nonhomogeneities $\varphi \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ that satisfy conditions (1.37). In such a case, by Theorem 1.2, the problem (2.12) possesses an r-dimensional family of solutions of the form (1.38).

Problem 1. Below, we consider the following problem: derive the necessary and sufficient conditions indicating when solutions of (2.5) turn into solutions (1.38) of the boundary value problem (2.12) for $\varepsilon = 0$.

Using the theory of generalized inverse operators [2], it is possible to find conditions for the solutions of the boundary value problem (2.5) to be branching from the solutions of (2.5) with $\varepsilon = 0$. Below, we formulate statements, solving the above problem. As compared with an earlier result [10, page 150], the present result is derived in an *explicit analytical* form. The progress was possible by using the delayed matrix exponential since, in such a case, all the necessary calculations can be performed to the full.

Theorem 2.1 (necessary condition). *Consider the system* (2.1)*; that is,*

$$\dot{z}(t) = Az(t-\tau) + g(t) + \varepsilon Z(z(h_i(t)), t, \varepsilon), \quad t \in [0, b], \tag{2.13}$$

where $h_i(t) \leq t$, $i = 1, \ldots, k$, with the initial and boundary conditions (2.2)*; that is,*

$$z(s) = \psi(s), \quad \text{if } s < 0 < b, \qquad \ell z(\cdot) = \alpha \ \in \mathbb{R}^m, \tag{2.14}$$

and assume that, for nonhomogeneities

$$\varphi(t) = \ g(t) + A \ \psi^{h_0}(t) \ \in L_p[0, b], \tag{2.15}$$

and for $\alpha \in \mathbb{R}^m$, the generating boundary value problem

$$\dot{z}(t) = A(S_{h_0} z)(t) + \varphi(t), \quad \ell z(\cdot) = \alpha, \tag{2.16}$$

corresponding to the problem (1.25)*, has exactly an r-dimensional family of linearly independent solutions of the form* (1.38)*. Moreover, assume that the boundary value problem* (2.13)*,* (2.14) *has a solution $z(t, \varepsilon)$ which, for $\varepsilon = 0$, turns into one of solutions $z_0(t, c_r)$ in* (1.38) *with a vector-constant $c_r := c_r^0 \in \mathbb{R}^r$.*

Then, the vector c_r^0 satisfies the equation

$$F\left(c_r^0\right) := \int_0^b H(s) Z\Big((S_h z_0)\left(s, c_r^0\right), s, 0\Big) ds = \theta_d, \tag{2.17}$$

where

$$H(s) := P_{Q_d^*} \ell K(\cdot, s) = P_{Q_d^*} \ell e_\tau^{A(\cdot - \tau - s)}. \tag{2.18}$$

Proof. We consider the nonlinearity in system (2.13), that is, the term $\varepsilon Z(z(h_i(t)), t, \varepsilon)$ as an inhomogeneity, and use Theorem 1.2 assuming that condition (1.37) is satisfied. This gives

$$\int_0^b H(s) Z((S_h z)(s, \varepsilon), s, \varepsilon) ds = \theta_d. \tag{2.19}$$

In this integral, letting $\varepsilon \to 0$, we arrive at the required condition (2.17). □

Corollary 2.2. *For periodic boundary-value problems, the vector-constant $c_r \in \mathbb{R}^r$ has a physical meaning-it is the amplitude of the oscillations generated. For this reason,* (2.17) *is called an equation generating the amplitude [11]. By analogy with the investigation of periodic problems, it is natural to say* (2.17) *is an equation for generating the constants of the boundary value problem* (2.13)*,* (2.14)*.*

If (2.17) *is solvable, then the vector constant* $c_r^0 \in \mathbb{R}^r$ *specifies the generating solution* $z_0(t, c_r^0)$ *corresponding to the solution* $z = z(t, \varepsilon)$ *of the original problem such that*

$$\begin{gathered} z(\cdot, \varepsilon) : [0, b] \longrightarrow \mathbb{R}^n, \\ z(\cdot, \varepsilon) \in D_p[0, b], \\ \dot{z}(\cdot, \varepsilon) \in L_p[0, b], \\ z(t, \cdot) \in C[0, \varepsilon_0], \\ z(t, 0) = z_0\left(t, c_r^0\right). \end{gathered} \tag{2.20}$$

Also, if (2.17) *is unsolvable, the problem* (2.13), (2.14) *has no solution in the analyzed space. Note that, here and in what follows, all expressions are obtained in the real form and hence, we are interested in real solutions of* (2.17), *which can be algebraic or transcendental.*

Sufficient conditions for the existence of solutions of the boundary-value problem (2.13), (2.14) can be derived using results in [10, page 155] and [2]. By changing the variables in system (2.13), (2.14)

$$z(t, \varepsilon) = z_0\left(t, c_r^0\right) + y(t, \varepsilon), \tag{2.21}$$

we arrive at a problem of finding sufficient conditions for the existence of solutions of the problem

$$\dot{y}(t) = A\left(S_{h_0} y\right)(t) + \varepsilon Z\left(S_h\left(z_0 + y\right)(t), t, \varepsilon\right), \quad \ell y = \theta_m, \; t \in [0, b], \tag{2.22}$$

and such that

$$\begin{gathered} y(\cdot, \varepsilon) : [0, b] \longrightarrow \mathbb{R}^n, \\ y(\cdot, \varepsilon) \in D_p[0, b], \\ \dot{y}(\cdot, \varepsilon) \in L_p[0, b], \\ y(t, \cdot) \in C[0, \varepsilon_0], \\ y(t, 0) = \theta. \end{gathered} \tag{2.23}$$

Since the vector function $Z((S_h z)(t), t, \varepsilon)$ is continuously differentiable with respect to z and continuous in ε in the neighborhood of the point

$$(z, \varepsilon) = \left(z_0\left(t, c_r^0\right), 0\right), \tag{2.24}$$

we can separate its linear term as a function depending on y and terms of order zero with respect to ε

$$Z\Big(S_h\Big(z_0\Big(t,c_r^0\Big)+y\Big),t,\varepsilon\Big)=f_0\Big(t,c_r^0\Big)+A_1(t)(S_hy)(t)+R((S_hy)(t),t,\varepsilon), \tag{2.25}$$

where

$$\begin{gathered}
f_0\Big(t,c_r^0\Big):=Z\Big((S_hz_0)\Big(t,c_r^0\Big),t,0\Big),\quad f_0\Big(\cdot,c_r^0\Big)\in L_p[0,b],\\
A_1(t)=A_1\Big(t,c_r^0\Big)=\left.\frac{\partial Z(S_hx,t,0)}{\partial S_hx}\right|_{x=z_0(t,c_r^0)},\quad A_1(\cdot)\in L_p[0,b],\\
R(\theta,t,0)=\theta,\quad \frac{\partial R(\theta,t,0)}{\partial y}=\Theta,\quad R(y,\cdot,\varepsilon)\in L_p[0,b].
\end{gathered} \tag{2.26}$$

We now consider the vector function $Z((S_h(z_0+y))(t),t,\varepsilon)$ in (2.22) as an inhomogeneity and we apply Theorem 1.2 to this system. As the result, we obtain the following representation for the solution of (2.22):

$$y(t,\varepsilon)=X_r(t)c+y^{(1)}(t,\varepsilon). \tag{2.27}$$

In this expression, the unknown vector of constants $c=c(\varepsilon)\in C[0,\varepsilon_0]$ is determined from a condition similar to condition (1.37) for the existence of solution of problem (2.22):

$$B_0c=\int_0^b H(s)\Big[A_1(s)\Big(S_hy^{(1)}\Big)(s,\varepsilon)+R((S_hy)(s,\varepsilon),s,\varepsilon)\Big]ds, \tag{2.28}$$

where

$$B_0=\int_0^b H(s)A_1(s)(S_hX_r)(s)ds \tag{2.29}$$

is a $d\times r$ matrix, and

$$H(s):=P_{Q_d^*}\ell K(\cdot,s)=P_{Q_d^*}\ell e_\tau^{A(\cdot-\tau-s)}. \tag{2.30}$$

The unknown vector function $y^{(1)}(t,\varepsilon)$ is determined by using the generalized Green operator as follows:

$$y^{(1)}(t,\varepsilon)=\varepsilon\Big(G\Big[Z\Big(S_h\Big(z_0\Big(s,c_r^0\Big)+y\Big),s,\varepsilon\Big)\Big]\Big)(t). \tag{2.31}$$

Let $P_{N(B_0)}$ be an $r \times r$ matrix orthoprojector $\mathbb{R}^r \rightarrow N(B_0)$, and let $P_{N(B_0^*)}$ be a $d \times d$ matrix-orthoprojector $\mathbb{R}^d \rightarrow N(B_0^*)$. Equation (2.28) is solvable with respect to $c \in \mathbb{R}^r$ if and only if

$$P_{N(B_0^*)} \int_0^b H(s)\Big[A_1(s)\Big(S_h y^{(1)}\Big)(s,\varepsilon) + R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)\Big] ds = \theta_d. \tag{2.32}$$

For

$$P_{N(B_0^*)} = \Theta_d, \tag{2.33}$$

the last condition is always satisfied and (2.28) is solvable with respect to $c \in \mathbb{R}^r$ up to an arbitrary vector constant $P_{N(B_0)}c \in \mathbb{R}^r$ from the null space of the matrix B_0

$$c = B_0^+ \int_0^b H(s)\Big[A_1(s)\Big(S_h y^{(1)}\Big)(s,\varepsilon) + R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)\Big] ds + P_{N(B_0)}c. \tag{2.34}$$

To find a solution $y = y(t,\varepsilon)$ of (2.28) such that

$$\begin{gathered} y(\cdot,\varepsilon) : [0,b] \longrightarrow R^n, \\ y(\cdot,\varepsilon) \in D_p[0,b], \\ \dot{y}(\cdot,\varepsilon) \in L_p[0,b], \\ y(t,\cdot) \in C[0,\varepsilon_0], \\ y(t,0) = \theta, \end{gathered} \tag{2.35}$$

it is necessary to solve the following operator system:

$$\begin{gathered} y(t,\varepsilon) = X_r(t)c + y^{(1)}(t,\varepsilon), \\ c = B_0^+ \int_0^b H(s)\Big[A_1(s)\Big(S_h y^{(1)}\Big)(s,\varepsilon) + R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)\Big] ds, \\ y^{(1)}(t,\varepsilon) = \varepsilon G\Big[Z\Big(S_h\Big(z_0\Big(s, c_r^0\Big) + y\Big), s, \varepsilon\Big)\Big](t). \end{gathered} \tag{2.36}$$

The operator system (2.36) belongs to the class of systems solvable by the method of simple iterations, convergent for sufficiently small $\varepsilon \in [0,\varepsilon_0]$ (see [10, page 188]). Indeed, system (2.36) can be rewritten in the form

$$u = L^{(1)}u + Fu, \tag{2.37}$$

where $u = \text{col}\,(y(t,\varepsilon), c(\varepsilon), y^{(1)}(t,\varepsilon))$ is a $(2n+r)$-dimensional column vector, $L^{(1)}$ is a linear operator

$$L^{(1)} := \begin{pmatrix} \Theta & X_r(t) & I \\ \Theta_{r,n} & \Theta_{r,r} & L_1 \\ \Theta & \Theta_{n,r} & \Theta \end{pmatrix}, \tag{2.38}$$

where

$$L_1(*) = B_0^+ \int_0^b H(s) A_1(s)(*) ds, \tag{2.39}$$

and F is a nonlinear operator

$$Fu := \begin{pmatrix} \theta \\ B_0^+ \int_0^b H(s) R\big((S_h y)(s,\varepsilon), s, \varepsilon\big) ds \\ \varepsilon\big(G\big[Z\big((S_h z_0)(s, c_r^0), s, 0\big) + A_1(s)\big(S_h y\big)(s,\varepsilon) + R\big(\big(S_h y\big)(s,\varepsilon), s, \varepsilon\big)\big]\big)(t) \end{pmatrix}. \tag{2.40}$$

In view of the structure of the operator $L^{(1)}$ containing zero blocks on and below the main diagonal, the inverse operator

$$\left(I_{2n+r} - L^{(1)}\right)^{-1} \tag{2.41}$$

exists. System (2.37) can be transformed into

$$u = Su, \tag{2.42}$$

where

$$S := \left(I_{2n+r} - L^{(1)}\right)^{-1} F \tag{2.43}$$

is a contraction operator in a sufficiently small neighborhood of the point

$$(z,\varepsilon) = \left(z_0\left(t, c_r^0\right), 0\right). \tag{2.44}$$

Thus, the solvability of the last operator system can be established by using one of the existing versions of the fixed-point principles [12] applicable to the system for sufficiently small $\varepsilon \in [0, \varepsilon_0]$. It is easy to prove that the sufficient condition $P_{N(B_0^*)} = \Theta_d$ for the existence of solutions of the boundary value problem (2.13), (2.14) means that the constant $c_r^0 \in \mathbb{R}^r$ of the equation for generating constant (2.17) is a simple root of equation (2.17) [2]. By using the method of simple iterations, we can find the solution of the operator system and hence the solution of the original boundary value problem (2.13), (2.14). Now, we arrive at the following theorem.

Theorem 2.3 (sufficient condition). *Assume that the boundary value problem* (2.13), (2.14) *satisfies the conditions listed above and the corresponding linear boundary value problem* (1.25) *has an r-dimensional family of linearly independent solutions of the form* (1.38). *Then, for any simple root $c_r = c_r^0 \in \mathbb{R}^r$ of the equation for generating the constants* (2.17), *there exist at least one solution of the boundary value problem* (2.13), (2.14). *The indicated solution $z(t, \varepsilon)$ is such that*

$$\begin{aligned} &z(\cdot, \varepsilon) \in D_p[0, b], \\ &\dot{z}(\cdot, \varepsilon) \in L_p[0, b], \\ &z(t, \cdot) \in C[0, \varepsilon_0], \end{aligned} \tag{2.45}$$

and, for $\varepsilon = 0$, turns into one of the generating solutions (1.38) *with a constant $c_r^0 \in \mathbb{R}^r$; that is, $z(t, 0) = z_0(t, c_r^0)$. This solution can be found by the method of simple iterations, which is convergent for a sufficiently small $\varepsilon \in [0, \varepsilon_0]$.*

Corollary 2.4. *If the number n of unknown variables is equal to the number m of boundary conditions (and hence $r = d$), the boundary value problem* (2.13), (2.14) *has a unique solution. In such a case, the problems considered for functional-differential equations are of Fredholm's type with a zero index. By using the procedure proposed in [2] with some simplifying assumptions, we can generalize the proposed method to the case of multiple roots of equation* (2.17) *to determine sufficient conditions for the existence of solutions of the boundary-value problem* (2.13), (2.14).

3. Example

We will illustrate the above proved theorems on the example of a weakly perturbed linear boundary value problem. Consider the following simplest boundary value problem-a periodic problem for the delayed differential equation:

$$\begin{gathered} \dot{z}(t) = z(t - \tau) + \varepsilon \sum_{i=1}^{k} B_i(t) z(h_i(t)) + g(t), \quad t \in (0, T], \\ z(s) = \psi(s), \quad \text{if } s < 0, \\ z(0) = z(T), \end{gathered} \tag{3.1}$$

where $0 < \tau, T = \text{const}$, B_i are $n \times n$ matrices, $B_i, g \in L_p[0, T]$, $\psi : \mathbb{R}^1 \setminus (0, T] \to \mathbb{R}^n$, $h_i(t) \le t$ are measurable functions. Using the symbols S_{h_i} and ψ^{h_i} (see (1.3), (1.4), (2.9)), we arrive at the following operator system:

$$\begin{gathered} \dot{z}(t) = z(t - \tau) + \varepsilon B(t)(S_h z)(t) + \varphi(t, \varepsilon), \\ \ell z := z(0) - z(T) = \theta_n, \end{gathered} \tag{3.2}$$

where $B(t) := (B_1(t), \ldots, B_k(t))$ is an $n \times N$ matrix $(N = nk)$, and

$$\varphi(t,\varepsilon) := g(t) + \psi^{h_0}(t) + \varepsilon \sum_{i=1}^{k} B_i(t)\psi^{h_i}(t) \in L_p[0,T]. \tag{3.3}$$

We will consider the simplest case with $T \leq \tau$. Utilizing the delayed matrix exponential, it can be easily verified that in this case, the matrix

$$X(t) = e_{\tau}^{I(t-\tau)} = I \tag{3.4}$$

is a normal fundamental matrix for the homogeneous generating system

$$\dot{z}(t) = z(t-\tau). \tag{3.5}$$

Then,

$$\begin{gathered}
Q := \ell X(\cdot) = e_{\tau}^{-I\tau} - e_{\tau}^{I(T-\tau)} = \theta_n, \\
P_Q = P_{Q^*} = I, \quad (r = n, d = m = n), \\
K(t,s) = \begin{cases} e_{\tau}^{I(t-\tau-s)} = I, & 0 \leq s \leq t \leq T, \\ \Theta, & s > t, \end{cases} \\
\ell K(\cdot, s) = K(0,s) - K(T,s) = -I, \\
H(\tau) = P_{Q^*}\ell K(\cdot, s) = -I, \\
(S_{h_i} I)(t) = \chi_{h_i}(t,0) \cdot I = I \cdot \begin{cases} 1, & \text{if } 0 \leq h_i(t) \leq T, \\ 0, & \text{if } h_i(t) \ < 0. \end{cases}
\end{gathered} \tag{3.6}$$

To illustrate the theorems proved above, we will find the conditions for which the boundary value problem (3.1) has a solution $z(t,\varepsilon)$ that, for $\varepsilon = 0$, turns into one of solutions (1.38) $z_0(t,c_r)$ of the generating problem. In contrast to the previous works [7, 9], we consider the case when the unperturbed boundary-value problem

$$\begin{gathered}
\dot{z}(t) = z(t-\tau) + \varphi(t,0), \\
z(0) = z(T)
\end{gathered} \tag{3.7}$$

has an n-parametric family of linear-independent solutions of the form(1.38)

$$z := z_0(t,c_n) = c_n + (G\varphi)(t), \quad \forall c_n \in \mathbb{R}^n. \tag{3.8}$$

For this purpose, it is necessary and sufficient for the vector function

$$\varphi(t) = g(t) + \psi^{h_0}(t) \tag{3.9}$$

to satisfy the condition of type (1.37)

$$\int_0^T H(s)\varphi(s)\ ds = -\int_0^T \varphi(s)\ ds = \theta_n. \tag{3.10}$$

Then, according to the Theorem 2.1, the constant $c_n = c_n^0 \in \mathbb{R}^n$ must satisfy (2.17), that is, the equation

$$F\left(c_n^0\right) := \int_0^T H(s)Z\left((S_h z_0)\left(s, c_n^0\right), s, 0\right)ds = \theta_n, \tag{3.11}$$

which in our case is a linear algebraic system

$$B_0 c_n^0 = -\int_0^T B(s)(S_h(G\varphi))(s)ds, \tag{3.12}$$

with the $n \times n$ matrix B_0 in the form

$$\begin{aligned} B_0 &= \int_0^T H(s)B(s)(S_h I)(s)ds \\ &= -\int_0^T \sum_{i=1}^k B_i(s)(S_{h_i} I)(s)ds = -\sum_{i=1}^k \int_0^T B_i(s)\chi_{h_i}(s,0)ds. \end{aligned} \tag{3.13}$$

According to Corollary 2.4, if $\det B_0 \neq 0$, the problem (3.1) for the case $T \leq \tau$ has a unique solution $z(t,\varepsilon)$ with the properties

$$\begin{gathered} z(\cdot,\varepsilon) \in D_p^n[0,T], \\ \dot{z}(\cdot,\varepsilon) \in L_p^n[0,T], \\ z(t,\cdot) \in C[0,\varepsilon_0], \\ z(t,0) = z_0\left(t, c_n^0\right), \end{gathered} \tag{3.14}$$

for $g \in L_p[0,T]$, $\psi(t) \in L_p[0,T]$, and for measurable delays h_i that which satisfy the criterion (3.10) of the existence of a generating solution where

$$c_n^0 = -B_0^+ \int_0^T B(s)(S_h(G\varphi))(s)ds. \tag{3.15}$$

A solution $z(t,\varepsilon)$ of the boundary value problem (3.1) can be found by the convergent method of simple iterations (see Theorem 2.3).

If, for example, $h_i(t) = t - \Delta_i$, where $0 < \Delta_i = \text{const} < T$, $i = 1, \ldots, k$, then

$$\chi_{h_i}(t,0) = \begin{cases} 1 & \text{if } 0 \leq h_i(t) = t - \Delta_i \leq T, \\ 0 & \text{if } h_i(t) = t - \Delta_i < 0, \end{cases} = \begin{cases} 1 & \text{if } \Delta_i \leq t \leq T + \Delta_i, \\ 0, & \text{if } t < \Delta_i. \end{cases} \tag{3.16}$$

The $n \times n$ matrix B_0 can be rewritten in the form

$$\begin{aligned} B_0 &= \int_0^T H(s) \sum_{i=1}^k B_i(s) \chi_{h_i}(s,0) d\tau \\ &= -\sum_{i=1}^k \int_0^T B_i(s) \chi_{h_i}(s,0) ds = -\sum_{i=1}^k \int_{\Delta_i}^T B_i(s) ds, \end{aligned} \tag{3.17}$$

and the unique solvability condition of the boundary value problem (3.1) takes the form

$$\det \left[\sum_{i=1}^k \int_{\Delta_i}^T B_i(s) ds \right] \neq 0. \tag{3.18}$$

It is easy to see that if the vector function $Z(z(h_i(t)), t, \varepsilon)$ is nonlinear in z, for example as a square, then (3.11) generating the constants will be a square-algebraic system and, in this case, the boundary value problem (3.1) can have two solutions branching from the point $\varepsilon = 0$.

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References

[1] N. V. Azbelev and V. P. Maksimov, "Equations with delayed arguments," *Differential Equations*, vol. 18, pp. 1419–1441, 1983, translated in *Differentsial'nye Uravneniya*, Vol. 18, no. 12, pp. 2027-2050.

[2] A. A. Boichuk and A. M. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary Value Problems*, VSP, Boston, Mass, USA, 2004.

[3] J. Hale, *Theory of Functional Differential Equations*, Applied Mathematical Sciences, Vol. 3, Springer, New York, NY, USA, 2nd edition, 1977.

[4] J. Mallet-Paret, "The fredholm alternative for functional-differential equations of mixed type," *Journal of Dynamics and Differential Equations*, vol. 11, no. 1, pp. 1–47, 1999.

[5] D. Ya. Khusainov and G. V. Shuklin, "Relative controllability in systems with pure delay," *International Applied Mechanics*, vol. 41, no. 2, pp. 210–221, 2005, translated in *Prikladnaya Mekhanika*, Vol. 41, no. 2, pp. 118-130.

[6] A. A. Boichuk, J. Diblík, D. Khusainov, and M. Růžičková, "Fredholm's boundary-value problems for differential systems with a single delay," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 72, no. 5, pp. 2251–2258, 2010.
[7] A. A. Boichuk, J. Diblík, D. Y. A. Khusainov, and M. Růžičková, "Boundary value problems for delay differential systems," *Advances in Difference Equations*, vol. 2010, Article ID 593834, 10 pages, 2010.
[8] M. Medvěd, M. Pospíšil, and L. Škripková, "Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 12, pp. 3903–3911, 2011.
[9] A. A. Boichuk and M. K. Grammatikopoulos, "Perturbed fredholm boundary value problems for delay differential systems," *Abstract and Applied Analysis*, no. 15, pp. 843–864, 2003.
[10] A. A. Boichuk, V. F. Zhuravlev, and A. M. Samoilenko, "Generalized inverse operators and noether boundary-value problems," in *Proceedings of the Institute of Mathematics of the National Academy of Sciences of the Ukraine*, vol. 13, p. 320, Kyiv, Ukrania, 1995.
[11] I. G. Malkin, *Some Problems in the Theory of Nonlinear Oscillations*, Gostekhizdat, Moscow, Russia, 1956.
[12] M. A. Krasnosel'skii, G. M. Vainikko, P. P. Zabreiko, Y. A. B. Rutitskii, and V. Ya. Stetsenko, *Approximate Solution of Operator Equations*, Nauka, Moscow, Russia, 1968, Translation: Noordhoff, Groningen, 1972.