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Research Article

Optimal Inequalities between Harmonic, Geometric, Logarithmic, and Arithmetic-Geometric Means

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We find the least values p, q, and s in $(0, 1/2)$ such that the inequalities $H(pa + (1 - p)b, pb+(1 - p)a) > \mathrm{AG}(a,b)$, $G(qa+(1-q)b, qb+(1-q)a) > \mathrm{AG}(a,b)$, and $L(sa+(1-s)b, sb+(1-s)a) > \mathrm{AG}(a,b)$ hold for all $a, b > 0$ with $a \neq b$, respectively. Here $\mathrm{AG}(a,b)$, $H(a,b)$, $G(a,b)$, and $L(a,b)$ denote the arithmetic-geometric, harmonic, geometric, and logarithmic means of two positive numbers a and b, respectively.

1. Introduction

The classical arithmetic-geometric mean $\mathrm{AG}(a,b)$ of two positive real numbers a and b is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$, which are given by

$$\begin{aligned} a_0 &= a, \qquad b_0 = b, \\ a_{n+1} &= \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}. \end{aligned} \tag{1.1}$$

Let $H(a,b) = 2ab/(a + b)$, $G(a,b) = \sqrt{ab}$, $L(a,b) = (a - b)/(\log a - \log b)$, $I(a,b) = (1/e)(b^b/a^a)^{1/(b-a)}$, $A(a,b) = (a + b)/2$, and $M_p(a,b) = [(a^p + b^p)/2]^{1/p} (p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ be the harmonic, geometric, logarithmic, identric, arithmetic, and p-th power means of two positive numbers a and b with $a \neq b$, respectively. Then it is well known that

$$\begin{aligned} \min\{a,b\} &< H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ &< I(a,b) < A(a,b) = M_{-1}(a,b) < \max\{a,b\} \end{aligned} \tag{1.2}$$

for all $a, b > 0$ with $a \neq b$.

Recently, the inequalities for means have been the subject of intensive research. In particular, many remarkable inequalities for arithmetic-geometric mean can be found in the literature [1–9].

Carlson and Vuorinen [2], and Bracken [9] proved that

$$L(a,b) < \mathrm{AG}(a,b) \tag{1.3}$$

for all $a, b > 0$ with $a \neq b$.

In [3], Vamanamurthy and Vuorinen established the following inequalities:

$$\begin{gathered} \mathrm{AG}(a,b) < I(a,b) < A(a,b), \\ \mathrm{AG}(a,b) < M_{1/2}(a,b), \\ \mathrm{AG}(a,b) < \frac{\pi}{2} L(a,b), \\ M_1(a,b) < \frac{\mathrm{AG}(a^2,b^2)}{\mathrm{AG}(a,b)} < M_2(a,b) \end{gathered} \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

We recall the Gauss identity [6, 7]

$$\mathrm{AG}(1,r')\mathcal{K}(r) = \frac{\pi}{2} \tag{1.5}$$

for $r \in [0,1)$ and $r' = \sqrt{1-r^2}$. As usual, $\mathcal{K}$ and $\mathcal{E}$ denote the complete elliptic integrals [8] given by

$$\begin{aligned} \mathcal{K}(r) &= \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{-1/2} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2,n)^2}{(n!)^2} r^{2n}, \quad \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{E}(r) &= \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{1/2} d\theta = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1/2,n)(1/2,n)}{(n!)^2} r^{2n}, \quad \mathcal{E}'(r) = \mathcal{E}(r'), \end{aligned} \tag{1.6}$$

where $(a,0) = 1$ for $a \neq 0$, and $(a,n) = \prod_{k=0}^{n-1}(a+k)$.

For fixed $a, b > 0$ with $a \neq b$ and $x \in [0, 1/2]$, let

$$f_1(x) = H(xa + (1-x)b, xb + (1-x)a), \tag{1.7}$$

$$f_2(x) = G(xa + (1-x)b, xb + (1-x)a), \tag{1.8}$$

$$f_3(x) = L(xa + (1-x)b, xb + (1-x)a). \tag{1.9}$$

Then it is not difficult to verify that $f_1(x)$, $f_2(x)$, and $f_3(x)$ are continuous and strictly increasing in $[0, 1/2]$, respectively. Note that $f_1(0) = H(a,b) < \mathrm{AG}(a,b) < f_1(1/2) = A(a,b)$, $f_2(0) = G(a,b) < \mathrm{AG}(a,b) < f_2(1/2) = A(a,b)$ and $f_3(0) = L(a,b) < \mathrm{AG}(a,b) < f_3(1/2) = A(a,b)$.

Therefore, it is natural to ask what are the least values p, q, and s in $(0, 1/2)$ such that the inequalities $H(pa + (1-p)b, pb + (1-p)a) > \mathrm{AG}(a,b)$, $G(qa + (1-q)b, qb + (1-q)a) > \mathrm{AG}(a,b)$, and $L(sa+(1-s)b, sb+(1-s)a) > \mathrm{AG}(a,b)$ hold for all $a, b > 0$ with $a \neq b$, respectively. The main purpose of this paper is to answer these questions. Our main results are Theorems 1.1–1.3.

Theorem 1.1. *If $p \in (0, 1/2)$, then inequality*

$$H\big(pa + (1-p)b, pb + (1-p)a\big) > AG(a,b) \tag{1.10}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \geq 1/4$.

Theorem 1.2. *If $q \in (0, 1/2)$, then inequality*

$$G\big(qa + (1-q)b, qb + (1-q)a\big) > AG(a,b) \tag{1.11}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $q \geq 1/2 - \sqrt{2}/4$.

Theorem 1.3. *If $s \in (0, 1/2)$, then inequality*

$$L(sa + (1-s)b, sb + (1-s)a) > AG(a,b) \tag{1.12}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $s \geq 1/2 - \sqrt{3}/4$.

2. Lemmas

In order to establish our main results we need several formulas and lemmas, which we present in this section.

For $0 < r < 1$, the following derivative formulas were presented in [6, Appendix E, pp. 474-475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$
$$\frac{d\left(\mathcal{E} - r'^2\mathcal{K}\right)}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2}, \tag{2.1}$$

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r). \tag{2.2}$$

The following Lemma 2.1 can be found in [6, Theorem 3.21(7) and Exercise 3.43(4)].

Lemma 2.1. (1) $(1 + r'^2)\mathcal{E}(r) - 2r'^2\mathcal{K}(r)$ *is strictly increasing from* $(0, 1)$ *onto* $(0, 1)$*;*
(2) $\mathcal{E}(r)/r'^{1/2}$ *is strictly increasing from* $(0, 1)$ *onto* $(\pi/2, +\infty)$.

Lemma 2.2. *Inequality*

$$\frac{2}{\pi}\mathcal{K}(r)\sqrt{1 - \frac{1}{2}r^2} > 1 \tag{2.3}$$

holds for all $r \in (0, 1)$.

Proof. Let

$$f(r) = \log\left[\frac{2}{\pi}\mathcal{K}(r)\sqrt{1 - \frac{1}{2}r^2}\right]. \tag{2.4}$$

Then simple computations lead to

$$f(0) = 0, \tag{2.5}$$

$$f'(r) = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2\mathcal{K}(r)} - \frac{r}{2 - r^2} = \frac{\left(1 + r'^2\right)\mathcal{E}(r) - 2r'^2\mathcal{K}(r)}{rr'^2(2 - r^2)\mathcal{K}(r)}. \tag{2.6}$$

It follows from Lemma 2.1 (1) and (2.6) that $f'(r) > 0$ for $r \in (0, 1)$, which implies that $f(r)$ is strictly increasing in $(0, 1)$.

Therefore, inequality (2.3) follows from (2.4) and (2.5) together with the monotonicity of $f(r)$. □

Lemma 2.3. *Inequality*

$$\frac{2\sqrt{3}}{\pi}r\mathcal{K}(r) > \log\left(\frac{2 + \sqrt{3}r}{2 - \sqrt{3}r}\right) \tag{2.7}$$

holds for all $r \in (0, 1)$.

Proof. Let

$$g(r) = \frac{2\sqrt{3}}{\pi} r\mathcal{K}(r) - \log\left(\frac{2+\sqrt{3}r}{2-\sqrt{3}r}\right). \tag{2.8}$$

Then simple computations lead to

$$g(0) = 0, \tag{2.9}$$

$$g'(r) = \frac{2\sqrt{3}}{\pi}\left(\mathcal{K}(r) + r\frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}\right) - \frac{4\sqrt{3}}{4-3r^2} = \frac{2\sqrt{3}}{\pi\left(1+3r'^2\right)}\left(\frac{1+3r'^2}{r'^{3/2}}\frac{\mathcal{E}(r)}{r'^{1/2}} - 2\pi\right). \tag{2.10}$$

Clearly the function $r \to (1+3r^2)/r^{3/2}$ is strictly decreasing from $(0,1)$ onto $(4,+\infty)$. Then (2.10) and Lemma 2.1 (2) lead to the conclusion that $g'(r) > 0$ for $r \in (0,1)$. Thus, $g(r)$ is strictly increasing in $(0,1)$.

Therefore, inequality (2.7) follows from (2.8) and (2.9) together with the monotonicity of $g(r)$. □

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. Let $\lambda = 1/4$, then from the monotonicity of the function $f_1(x) = H(xa + (1-x)b, xb + (1-x)a)$ in $[0,1/2]$ we know that to prove inequality (1.10) we only need to prove that

$$\mathrm{AG}(a,b) < H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) \tag{3.1}$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.7) we clearly see that both $\mathrm{AG}(a,b)$ and $H(\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0,1)$ and $r = (1-t)/(1+t)$, then from (1.5) we have

$$H(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a) - \mathrm{AG}(a,b) = \frac{(t+3)(3t+1)}{8(1+t)} - \frac{\pi}{2\mathcal{K}(t')}. \tag{3.2}$$

Let

$$F(t) = \frac{(t+3)(3t+1)}{8(1+t)} - \frac{\pi}{2\mathcal{K}(t')}. \tag{3.3}$$

Then making use of (2.2) we get

$$F(t) = \frac{(2+r)(2-r)}{4(1+r)} - \frac{\pi}{2(1+r)\mathcal{K}(r)} = \frac{\pi}{8(1+r)\mathcal{K}(r)}F_1(r), \tag{3.4}$$

where $F_1(r) = (2/\pi)(4 - r^2)\mathcal{K}(r) - 4$. Note that

$$\begin{aligned} F_1(r) &= \sum_{n=0}^{\infty} \frac{(1/2,n)^2}{(n!)^2} r^{2n}\left(4 - r^2\right) - 4 \\ &= 4r^2 \sum_{n=0}^{\infty} \frac{(1/2,n+1)^2}{[(n+1)!]^2} r^{2n} - r^2 \sum_{n=0}^{\infty} \frac{(1/2,n)^2}{(n!)^2} r^{2n} \\ &= \sum_{n=0}^{\infty} \frac{(1/2,n)^2}{[(n+1)!]^2}\left(3n^2 + 2n\right) r^{2(n+1)} > 0. \end{aligned} \tag{3.5}$$

Therefore, inequality (3.1) follows from (3.2)–(3.5). □

Next, we prove that the parameter $p = \lambda = 1/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.10) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < p < 1/2$ and small $x > 0$,

$$\mathrm{AG}(1, 1-x) = \frac{\pi}{2\mathcal{K}\left(\sqrt{2x - x^2}\right)} = 1 - \frac{1}{2}x - \frac{1}{16}x^2 + o\left(x^3\right), \tag{3.6}$$

$$H(p(1-x) + 1 - p, (1-p)(1-x) + p) = 1 - \frac{1}{2}x + \left(-p^2 + p - \frac{1}{4}\right)x^2 + o\left(x^3\right). \tag{3.7}$$

It follows from (3.6) and (3.7) that inequality $\mathrm{AG}(1, 1-x) \leq H(p(1-x) + 1 - p, (1-p)(1-x) + p)$ holds for small x only $p \geq 1/4$.

Remark 3.1. For $0 < p < 1/2$ and $x > 0$, one has

$$\lim_{x \to 0} \frac{H(px + 1 - p, (1-p)x + p)}{\mathrm{AG}(1, x)} = \lim_{x \to 0} \frac{4[px + 1 - p][(1-p)x + p]}{(1+x)\pi}\mathcal{K}(x') = +\infty. \tag{3.8}$$

Equation (3.8) implies that there does not exist $p \in (0, 1/2)$ such that $\mathrm{AG}(1, x) > H(px + 1 - p, (1-p)x + p)$ for all $x \in (0, 1)$.

Proof of Theorem 1.2. Let $\mu = 1/2 - \sqrt{2}/4$, then from the monotonicity of the function $f_2(x) = G(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.11) we only need to prove that

$$\mathrm{AG}(a, b) < G(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \tag{3.9}$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.8) we clearly see that both $\mathrm{AG}(a,b)$ and $G(\mu a+(1-\mu)b, \mu b+(1-\mu)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a = 1 > b$. Let $t = b \in (0,1)$ and $r = (1-t)/(1+t)$, then from (1.5) we have

$$G(\mu a + (1-\mu)b, \mu b + (1-\mu)a) - \mathrm{AG}(a,b) = \sqrt{[\mu + (1-\mu)t][\mu t + (1-\mu)]} - \frac{\pi}{2\mathcal{K}(t')}. \tag{3.10}$$

Let

$$G(t) = \sqrt{[\mu + (1-\mu)t][\mu t + (1-\mu)]} - \frac{\pi}{2\mathcal{K}(t')}. \tag{3.11}$$

Then making use of (2.2) we have

$$G(t) = \frac{\pi}{2(1+r)\mathcal{K}(r)}\left[\frac{2}{\pi}\mathcal{K}(r)\sqrt{1-\frac{1}{2}r^2} - 1\right]. \tag{3.12}$$

Therefore, inequality (3.9) follows from (3.10)–(3.12) together with Lemma 2.2. □

Next, we prove that the parameter $q = \mu = 1/2 - \sqrt{2}/4$ is the best possible parameter in $(0, 1/2)$ such that inequality (1.11) holds for all $a, b > 0$ with $a \neq b$.

Since for $0 < q < 1/2$ and small $x > 0$,

$$G\big(q(1-x) + 1 - q, (1-q)(1-x) + q\big) = 1 - \frac{1}{2}x + \frac{1}{8}\left(-4q^2 + 4q - 1\right)x^2 + o\left(x^3\right). \tag{3.13}$$

It follows from (3.6) and (3.13) that inequality $\mathrm{AG}(1, 1-x) \leq G(q(1-x) + 1 - q, (1-q)(1-x) + q)$ holds for small x only $q \geq 1/2 - \sqrt{2}/4$.

Remark 3.2. For $0 < q < 1/2$ and $x > 0$, one has

$$\lim_{x\to 0}\frac{G(qx + 1 - q, (1-q)x + q)}{\mathrm{AG}(1,x)} = \lim_{x\to 0}\frac{2}{\pi}\sqrt{[qx + 1 - q][(1-q)x + q]}\,\mathcal{K}(x') = +\infty. \tag{3.14}$$

Equation (3.14) implies that there does not exist $q \in (0, 1/2)$ such that $\mathrm{AG}(1,x) > G(qx + 1 - q, (1-q)x + q)$ for all $x \in (0,1)$.

Proof of Theorem 1.3. Let $\beta = 1/2 - \sqrt{3}/4$, then from the monotonicity of $f_3(x) = L(xa + (1-x)b, xb + (1-x)a)$ in $[0, 1/2]$ we know that to prove inequality (1.12) we only need to prove that

$$\mathrm{AG}(a,b) < L\big(\beta a + (1-\beta)b, \beta b + (1-\beta)a\big) \tag{3.15}$$

for all $a, b > 0$ with $a \neq b$.

From (1.1) and (1.9) we clearly see that both $\mathrm{AG}(a,b)$ and $L(\beta a+(1-\beta)b, \beta b+(1-\beta)a)$ are symmetric and homogeneous of degree 1. Without loss of generality, we can assume that $a=1>b$. Let $t=b\in(0,1)$ and $r=(1-t)/(1+t)$, then from (1.5) one has

$$L(\beta a+(1-\beta)b, \beta b+(1-\beta)a)-\mathrm{AG}(a,b) = \frac{\sqrt{3}(1-t)}{2\log\left[\left(\left(2-\sqrt{3}\right)t+2+\sqrt{3}\right)/\left(\left(2+\sqrt{3}\right)t+2-\sqrt{3}\right)\right]}-\frac{\pi}{2\mathcal{K}(t')}. \tag{3.16}$$

Let

$$J(t)=\frac{\sqrt{3}(1-t)}{2\log\left[\left(\left(2-\sqrt{3}\right)t+2+\sqrt{3}\right)/\left(\left(2+\sqrt{3}\right)t+2-\sqrt{3}\right)\right]}-\frac{\pi}{2\mathcal{K}(t')}. \tag{3.17}$$

Then from (2.2) we get

$$J(t)=\frac{\pi}{2(1+r)\mathcal{K}(r)\log\left(\left(2+\sqrt{3}r\right)/\left(2-\sqrt{3}r\right)\right)}g(r), \tag{3.18}$$

where $g(r)$ is defined as in Lemma 2.3.

Therefore, inequality (3.15) follows from (3.16)–(3.18) together with Lemma 2.3. □

Next, we prove that the parameter $s=\beta=1/2-\sqrt{3}/4$ is the best possible parameter in $(0,1/2)$ such that inequality (1.12) holds for all $a,b>0$ with $a\neq b$.

Since for $0<s<1/2$ and small $x>0$,

$$L(s(1-x)+1-s,(1-s)(1-x)+s)=1-\frac{1}{2}x+\frac{1}{12}\left(-4s^2+4s-1\right)x^2+o\left(x^3\right). \tag{3.19}$$

It follows from (3.6) and (3.19) that inequality $\mathrm{AG}(1,1-x)\leq L(s(1-x)+1-s,(1-s)(1-x)+s)$ holds for small x only $s\geq 1/2-\sqrt{3}/4$.

Remark 3.3. For $0<s<1/2$ and $x>0$, one has

$$\lim_{x\to 0}\frac{L(sx+1-s,(1-s)x+s)}{\mathrm{AG}(1,x)}=\lim_{x\to 0}\frac{2}{\pi}\mathcal{K}(x')\frac{(1-2s)(1-x)}{\log[(sx+1-s)/((1-s)x+s)]}=+\infty. \tag{3.20}$$

Equation (3.20) implies that there exist no values $s\in(0,1/2)$ such that $\mathrm{AG}(1,x)>L(sx+1-s,(1-s)x+s)$ for all $x\in(0,1)$.

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