Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 608329, 10 pages
doi:10.1155/2011/608329

Research Article

Principal Functions of Non-Selfadjoint Difference Operator with Spectral Parameter in Boundary Conditions

Murat Olgun, Turhan Koprubasi, and Yelda Aygar

Department of Mathematics, Ankara University, 06100 Ankara, Turkey

Correspondence should be addressed to Murat Olgun, molgun@science.ankara.edu.tr

Received 21 January 2011; Accepted 6 April 2011

Academic Editor: Svatoslav Staněk

We investigate the principal functions corresponding to the eigenvalues and the spectral singularities of the boundary value problem (BVP) $a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n$, $n \in \mathbb{N}$ and $(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0$, where (a_n) and (b_n) are complex sequences, λ is an eigenparameter, and $\gamma_i, \beta_i \in \mathbb{C}$ for $i = 0, 1$.

1. Introduction

Let us consider the (BVP)

$$\begin{gathered} -y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty, \\ y'(0) - hy(0) = 0 \end{gathered} \tag{1.1}$$

in $L^2(\mathbb{R}_+)$, where q is a complex-valued function and $\lambda \in \mathbb{C}$ is a spectral parameter and $h \in \mathbb{C}$. The spectral theory of the above BVP with continuous and point spectrum was investigated by Naĭmark [1]. He showed that the existence of the spectral singularities in the continuous spectrum of the BVP. He noted that the spectral singularities that belong to the continuous spectrum are the poles of the resolvents kernel but they are not the eigenvalues of the BVP. Also he showed that eigenfunctions and the associated functions (principal functions) corresponding to the spectral singularities are not the element of $L^2(\mathbb{R}_+)$. The spectral singularities in the spectral expansion of the BVP in terms of principal functions have been investigated in [2]. The spectral analysis of the quadratic pencil of Schrödinger, Dirac, and Klein-Gordon operators with spectral singularities was studied in [3–8]. The spectral analysis of a non-selfadjoint difference equation with spectral parameter has been studied in [9]. In this paper, it is proved that the BVP

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N}, \tag{1.2}$$

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0 \tag{1.3}$$

has a finite number of eigenvalues and spectral singularities with a finite multiplicities if

$$\sup_{n\in\mathbb{N}}\left[\exp\left(\varepsilon n^{\delta}\right)(|1-a_n| + |b_n|)\right] < \infty \tag{1.4}$$

for some $\varepsilon > 0$ and $1/2 \leq \delta \leq 1$.

Let L denote difference operator of second order generated in $\ell_2(\mathbb{N})$ by

$$(\ell y)_n = a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1}, \quad n \in \mathbb{N} \tag{1.5}$$

and with boundary condition

$$(\gamma_0 + \gamma_1\lambda)y_1 + (\beta_0 + \beta_1\lambda)y_0 = 0, \quad \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \quad \gamma_1 \neq a_0^{-1}\beta_0, \tag{1.6}$$

where $\{a_n\}_{n\in\mathbb{N}}$, $\{b_n\}_{n\in\mathbb{N}}$ are complex sequences and $a_n \neq 0$ for all $n \in \mathbb{N}\cup\{0\}$ and $\gamma_i, \beta_i \in \mathbb{C}$ for $i = 0, 1$.

In this paper, which is extension of [9], we aim to investigate the properties of the principal functions corresponding to the eigenvalues and spectral singularities of the BVP (1.2)-(1.3).

2. Discrete Spectrum of (1.2)-(1.3)

Let

$$\sup_{n\in\mathbb{N}}\left[\exp\left(\varepsilon n^{\delta}\right)(|1-a_n| + |b_n|)\right] < \infty \tag{2.1}$$

for some $\varepsilon > 0$ and $1/2 \leq \delta \leq 1$. The following result is obtained in [10, 11]: under the condition (2.1), equation (1.2) has the solution

$$e_n(z) = \alpha_n e^{inz}\left(1 + \sum_{m=1}^{\infty} A_{nm}e^{imz}\right), \quad n \in \mathbb{N}\cup\{0\} \tag{2.2}$$

for $\lambda = 2\cos z$, where $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z \geq 0\}$ and α_n, A_{nm} are expressed in terms of (a_n) and (b_n) as

$$\begin{aligned}
\alpha_n &= \left(\prod_{k=n}^{\infty} a_k\right)^{-1}, \\
A_{n,1} &= -\sum_{k=n+1}^{\infty} b_k, \\
A_{n,2} &= -\sum_{k=n+1}^{\infty}\left(1 - a_k^2\right) + \sum_{k=n+1}^{\infty} b_k \sum_{p=k+1}^{\infty} b_p, \\
A_{n,m+2} &= \sum_{k=n+1}^{\infty}\left(1 - a_k^2\right) A_{k+1,m} \sum_{k=n+1}^{\infty} b_k A_{k,m+1} + A_{n+1,m}.
\end{aligned} \tag{2.3}$$

Moreover, A_{nm} satisfies

$$|A_{nm}| \leq C \sum_{k=n+[m/2]}^{\infty} \left(|1 - a_k| + |b_k|\right), \tag{2.4}$$

where $[m/2]$ is the integer part of $m/2$ and $C > 0$ is a constant. So $e(z) = \{e_n(z)\}$ is continuous in $\operatorname{Im} z = 0$ and analytic in $\mathbb{C}_+ := \{z : z \in \mathbb{C},\ \operatorname{Im} z > 0\}$ with respect to z.

Let us define $f(z)$ using (2.2) and the boundary condition (1.3) as

$$f(z) = \left(\gamma_0 + 2\gamma_1 \cos z\right) e_1(z) + \left(\beta_0 + 2\beta_1 \cos z\right) e_0(z). \tag{2.5}$$

The function f is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $f(z) = f(z + 2\pi)$.

We denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From the definition of the eigenvalues and spectral singularities, we have [12]

$$\begin{aligned}
\sigma_d(L) &= \{\lambda : \lambda = 2\cos z,\ z \in P_0,\ F(z) = 0\}, \\
\sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\cos z,\ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ F(z) = 0\right\} \setminus \{0\}.
\end{aligned} \tag{2.6}$$

From (2.2) and (2.5), we get

$$
\begin{aligned}
f(z) &= \left[\gamma_0 + \gamma_1\left(e^{iz} + e^{-iz}\right)\right]\left[\alpha_1 e^{iz}\left(1 + \sum_{m=1}^{\infty} A_{1m} e^{imz}\right)\right] \\
&\quad + \left[\beta_0 + \beta_1\left(e^{iz} + e^{-iz}\right)\right]\left[\alpha_0\left(1 + \sum_{m=1}^{\infty} A_{0m} e^{imz}\right)\right] \\
&= \alpha_0\beta_1 e^{-iz} + \gamma_1\alpha_1 + \alpha_0\beta_0 + \left(\gamma_0\alpha_1 + \alpha_0\beta_1\right)e^{iz} + \gamma_1\alpha_1 e^{i2z} \\
&\quad + \sum_{m=1}^{\infty}\alpha_0\beta_1 A_{0m} e^{i(m-1)z} + \sum_{m=1}^{\infty}\left(\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}\right)e^{imz} \\
&\quad + \sum_{m=1}^{\infty}\left(\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}\right)e^{i(m+1)z} + \sum_{m=1}^{\infty}\gamma_1\alpha_1 A_{1m} e^{i(m+2)z}.
\end{aligned} \tag{2.7}
$$

Let

$$
\begin{aligned}
F(z) = f(z)e^{iz} &= \alpha_0\beta_1 + \left(\gamma_1\alpha_1 + \alpha_0\beta_0\right)e^{iz} + \left(\gamma_0\alpha_1 + \alpha_0\beta_1\right)e^{2iz} + \gamma_1\alpha_1 e^{3iz} \\
&\quad + \sum_{m=1}^{\infty}\alpha_0\beta_1 A_{0m} e^{imz} + \sum_{m=1}^{\infty}\left(\gamma_1\alpha_1 A_{1m} + \alpha_0\beta_0 A_{0m}\right)e^{i(m+1)z} \\
&\quad + \sum_{m=1}^{\infty}\left(\gamma_0\alpha_1 A_{1m} + \alpha_0\beta_1 A_{0m}\right)e^{i(m+2)z} + \sum_{m=1}^{\infty}\gamma_1\alpha_1 A_{1m} e^{i(m+3)z};
\end{aligned} \tag{2.8}
$$

then the function F is analytic in $\mathbb{C}_+$, continuous in $\overline{\mathbb{C}}_+$, and $F(z) = F(z + 2\pi)$. It follows from (2.6) and (2.8) that

$$
\begin{aligned}
\sigma_d(L) &= \{\lambda : \lambda = 2\cos z,\ z \in P_0,\ F(z) = 0\}, \\
\sigma_{ss}(L) &= \left\{\lambda : \lambda = 2\cos z,\ z \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ F(z) = 0\right\} \setminus \{0\}.
\end{aligned} \tag{2.9}
$$

Definition 2.1. The multiplicity of a zero of F in P is called the multiplicity of the corresponding eigenvalue or spectral singularity of the BVP (1.2) and (1.3).

3. Principal Functions

Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ and $\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_q$ denote the zeros of F in $P_0 := \{z : z \in \mathbb{C}, z = x + iy,\ -\pi/2 \le x \le 3\pi/2,\ y > 0\}$ and $[-\pi/2,\ 3\pi/2]$ with multiplicities $m_1, m_2, \ldots, m_p$ and $m_{p+1}, m_{p+2}, \ldots, m_q$, respectively.

Definition 3.1. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the vectors $y^{(0)}, y^{(1)}, \ldots, y^{(s)}$; $y^{(k)} = \{y_n^{(k)}\}_{n\in\mathbb{N}}$ $k = 0, 1, \ldots, s$ satisfy the equations

$$\left(ly^{(0)}\right)_n - \lambda_0 y_n^{(0)} = 0,$$
$$\left(ly^{(k)}\right)_n - \lambda_0 y_n^{(k)} - y_n^{(k-1)} = 0, \quad k = 1, 2, \ldots, s;\ n \in \mathbb{N}, \tag{3.1}$$

then vector $y^{(0)}$ is called the eigenvector corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The vectors $y^{(1)}, \ldots, y^{(s)}$ are called the associated vectors corresponding to $\lambda = \lambda_0$. The eigenvector and the associated vectors corresponding to $\lambda = \lambda_0$ are called the principal vectors of the eigenvalue $\lambda = \lambda_0$.

The principal vectors of the spectral singularities of L are defined similarly.

We define the vectors

$$V_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n(\lambda)\right\}\Bigg|_{\lambda=\lambda_j}, \quad k = 0, 1, \ldots, m_j - 1;\ j = 1, 2, \ldots, p,$$
$$V_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\frac{d^k}{d\lambda^k}E_n(\lambda)\right\}\Bigg|_{\lambda=\lambda_j}, \quad k = 0, 1, \ldots, m_j - 1;\ j = p+1, p+2, \ldots, q, \tag{3.2}$$

where $\lambda = 2\cos z$, $z \in P_0$, and

$$\{E_n(\lambda)\} := \left\{e_n\left(\arccos\frac{\lambda}{2}\right)\right\}, \quad n \in \mathbb{N}. \tag{3.3}$$

Moreover, if $y(\lambda) = \{y_n(\lambda)\}_{n\in\mathbb{N}}$ is a solution of (1.2), then $(d^k/d\lambda^k)y(\lambda) = \{(d^k/d\lambda^k)y_n(\lambda)\}_{n\in\mathbb{N}}$ satisfies

$$a_{n-1}\frac{d^k}{d\lambda^k}y_{n-1}(\lambda) + b_n\frac{d^k}{d\lambda^k}y_n(\lambda) + a_n\frac{d^k}{d\lambda^k}y_{n+1}(\lambda) = \lambda\frac{d^k}{d\lambda^k}y_n(\lambda) + k\frac{d^{k-1}}{d\lambda^{k-1}}y_n(\lambda). \tag{3.4}$$

From (3.2) and (3.4), we get that

$$\left(\ell V^{(0)}(\lambda_j)\right)_n - \lambda_j V_n^{(0)}(\lambda_j) = 0,$$
$$\left(\ell V^{(k)}(\lambda_j)\right)_n - \lambda_j V_n^{(k)}(\lambda_j) - V_n^{(k-1)}(\lambda_j) = 0, \quad k = 1, 2, \ldots, m_j - 1;\ j = 1, 2, \ldots, q. \tag{3.5}$$

Consequently, the vectors $V_n^{(k)}(\lambda_j)$; $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$ and $V_n^{(k)}(\lambda_j)$; $k = 0, 1, \ldots, m_j - 1$, $j = p+1, p+2, \ldots, q$ are the principal vectors of eigenvalues and spectral singularities of L, respectively.

Theorem 3.2.

$$
\begin{aligned}
&V_n^{(k)}(\lambda_j) \in \ell_2(\mathbb{N}); \quad k = 0,1,\dots,m_j - 1,\ j = 1,2,\dots,p,\\
&V_n^{(k)}(\lambda_j) \notin \ell_2(\mathbb{N}); \quad k = 0,1,\dots,m_j - 1,\ j = p+1,\dots,q.
\end{aligned} \tag{3.6}
$$

Proof. Using $E_n(\lambda) = e_n(\arccos(\lambda/2))$, we obtain that

$$
\left\{\frac{d^k}{d\lambda^k} E_n(\lambda)\right\}\bigg|_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} C_\nu \left\{\frac{d^\nu}{d\lambda^\nu} e_n(z)\right\}_{z=z_j}, \quad n \in \mathbb{N}, \tag{3.7}
$$

where $\lambda_j = 2\cos z_j$; $z_j \in P = P_0 \cup [-\pi/2, 3\pi/2]$, $j = 1,2,\dots,q$; C_ν is a constant depending on λ_j.

From (2.2), we find that

$$
\begin{aligned}
\left\{\frac{d^\nu}{dz^\nu} e_n(z)\right\}_{z=z_j} &= \alpha_n e^{inz_j}\left\{(in)^\nu + \sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm} e^{imz_j}\right\}\\
&= \alpha_n e^{inz_j}(in)^\nu + \alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm} e^{imz_j}.
\end{aligned} \tag{3.8}
$$

For the principal vectors $V_n^{(k)}(\lambda_j) = \{V^{(k)}(\lambda_j)\}_{n\in\mathbb{N}}$, $k = 0,1,\dots,m_j - 1$, $j = 1,2,\dots,p$, corresponding to the eigenvalues $\lambda_j = 2\cos z_j$, $j = 1,2,\dots,p$, of L, we get

$$
\left\{\frac{d^k}{d\lambda^k} E_n(\lambda)\right\}\bigg|_{\lambda=\lambda_j} = \sum_{\nu=0}^{k} C_\nu \left\{\alpha_n e^{inz_j}(in)^\nu + \alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm} e^{imz_j}\right\}; \tag{3.9}
$$

then

$$
V_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\sum_{\nu=0}^{k} C_\nu \left[\alpha_n e^{inz_j}(in)^\nu + \alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm} e^{imz_j}\right]\right\} \tag{3.10}
$$

for $k = 0,1,\dots,m_j - 1$, $j = 1,2,\dots,p$.

Since $\operatorname{Im}\lambda_j > 0$, $j = 1, 2, \ldots, p$ from (3.10) we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{1}{k!}\sum_{\nu=0}^{k}C_\nu \alpha_n e^{inz_j}(in)^\nu\right|^2 &\leq \frac{1}{(k!)^2}\left[\sum_{n=1}^{\infty}\sum_{\nu=0}^{k}|C_\nu||\alpha_n|e^{-n\operatorname{Im} z_j}|n^\nu|\right]^2\\ &\leq \frac{A}{(k!)^2}\left[\sum_{n=1}^{\infty}e^{-n\operatorname{Im} z_j}\left(1+n+n^2+\cdots+n^k\right)\right]^2\\ &\leq \frac{A}{(k!)^2}(k+1)^2\left(\sum_{n=1}^{\infty}e^{-n\operatorname{Im} z_j}n^k\right)^2\\ &< \infty,\end{aligned} \tag{3.11}$$

where A is a constant. Now we define the function

$$g_n(z) = \frac{1}{k!}\sum_{\nu=0}^{k}\alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm}e^{imz_j}, \quad j = 1, 2, \ldots, p. \tag{3.12}$$

From (2.4), we obtain that

$$\begin{aligned}\left|g_n(z)\right| &\leq \sum_{\nu=0}^{k}|\alpha_n|e^{-n\operatorname{Im} z_j}\sum_{m=1}^{\infty}|n+m|^\nu|A_{nm}|e^{-m\operatorname{Im} z_j}\\ &\leq |\alpha_n|e^{-n\operatorname{Im} z_j}\left[\sum_{m=1}^{\infty}|A_{nm}|e^{-m\operatorname{Im} z_j}+\sum_{m=1}^{\infty}(n+m)|A_{nm}|e^{-m\operatorname{Im} z_j}\right.\\ &\qquad\left.+\cdots+\sum_{m=1}^{\infty}(n+m)^k|A_{nm}|e^{-m\operatorname{Im} z_j}\right]\\ &< Be^{-n\operatorname{Im} z_j},\end{aligned} \tag{3.13}$$

where $B = |\alpha_n|\sum_{m=1}^{\infty}\sum_{\nu=0}^{k}|A_{nm}|e^{-m\operatorname{Im} z_j}(n+m)^\nu$. Therefore, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left|g_n(z)\right|^2 &\leq \sum_{n=1}^{\infty}B^2e^{-2n\operatorname{Im} z_j}, \quad j = 1, 2, \ldots, p\\ &< \infty.\end{aligned} \tag{3.14}$$

It follows from (3.11) and (3.14) that $V_n^{(k)}(\lambda_j) \in \ell_2(\mathbb{N})$, $k = 0, 1, \ldots, m_j - 1$, $j = 1, 2, \ldots, p$.

If we consider (3.10) for the principal vectors corresponding to the spectral singularities $\lambda_j = 2\cos z_j$, $j = p+1, p+2, \ldots, q$, of L and consider that $\operatorname{Im} z_j = 0$ for the spectral singularities, then we have

$$V_n^{(k)}(\lambda_j) = \frac{1}{k!}\left\{\sum_{\nu=0}^{k} C_\nu \alpha_n e^{inz_j}(in)^\nu + \alpha_n e^{inz_j}\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm} e^{imz_j}\right\} \tag{3.15}$$

for $k = 0, 1, \ldots, m_j - 1$, $j = p+1, p+2, \ldots, q$.

Since $\operatorname{Im}\lambda_j = 0$, $j = p+1, \ldots, q$ from (3.15) we find that

$$\frac{1}{k!}\sum_{n=1}^{\infty}\left|\sum_{\nu=0}^{k} C_\nu \alpha_n e^{inz_j}(in)^\nu\right|^2 = \infty. \tag{3.16}$$

Now we define $t_n(z) = \sum_{\nu=0}^{k}\sum_{m=1}^{\infty}[i(n+m)]^\nu A_{nm} e^{imz_j}$, and using (2.4) we get

$$\begin{aligned}
|t_n(z)| &\le \sum_{\nu=0}^{k}\sum_{m=1}^{\infty}\left|(n+m)^\nu\right||A_{nm}| \\
&\le \sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu C \sum_{k=n+[m/2]}^{\infty}(|1-a_k|+|b_k|) \\
&\le C\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu \sum_{k=n+[m/2]}^{\infty}\exp(-\varepsilon k)\exp(\varepsilon k)(|1-a_k|+|b_k|) \\
&\le C\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu \exp\left[\frac{-\varepsilon}{4}(n+m)\right]\sum_{k=n+[m/2]}^{\infty}\exp(\varepsilon k)(|1-a_k|+|b_k|) \\
&\le C_1\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}(n+m)^\nu \exp\left[\frac{-\varepsilon}{4}(n+m)\right] \\
&= C_1 e^{(-\varepsilon/4)n}\sum_{m=1}^{\infty}\sum_{\nu=0}^{k}(n+m)^\nu \exp\left(\frac{-\varepsilon}{4}m\right) \\
&= A e^{(-\varepsilon/4)n},
\end{aligned} \tag{3.17}$$

where

$$A = C_1\sum_{m=1}^{\infty}\sum_{\nu=0}^{k}(n+m)^\nu \exp\left(\frac{-\varepsilon}{4}m\right). \tag{3.18}$$

If we use (3.17), we obtain that

$$\frac{1}{k!}\sum_{n=1}^{\infty}\left|\alpha_n e^{inz_j}\sum_{\nu=0}^{k}\sum_{m=1}^{\infty}[i(n+m)]^{\nu}A_{nm}e^{imz_j}\right|^2 \le \frac{1}{k!}\sum_{n=1}^{\infty}\alpha_n^2 A^2 e^{-\varepsilon n/2} < \infty. \tag{3.19}$$

So $V_n^{(k)} \notin \ell_2(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1, p+2,\ldots q$. □

Let us introduce Hilbert spaces

$$\begin{gathered} H_k(\mathbb{N}) = \left\{ y = \{y_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}}(1+|n|)^{2k}|y_n|^2 < \infty \right\}, \\ H_{-k}(\mathbb{N}) = \left\{ u = \{u_n\}_{n\in\mathbb{N}} : \sum_{n\in\mathbb{N}}(1+|n|)^{-2k}|u_n|^2 < \infty \right\}, \quad k = 0,1,2,\ldots, \end{gathered} \tag{3.20}$$

with $\|y\|_k^2 = \sum_{n\in\mathbb{N}}(1+|n|)^{2k}|y_n|^2$, $\|u\|_{-k}^2 = \sum_{n\in\mathbb{N}}(1+|n|)^{-2k}|u_n|^2$, respectively. It is obvious that $H_0(\mathbb{N}) = \ell_2(\mathbb{N})$ and

$$H_{k+1}(\mathbb{N}) \subsetneqq H_k(\mathbb{N}) \subsetneqq l_2(\mathbb{N}) \subsetneqq H_{-k}(\mathbb{N}) \subsetneqq H_{-(k+1)}(\mathbb{N}), \quad k = 1,2,\ldots \tag{3.21}$$

Theorem 3.3. $V_n^{(k)}(\lambda_j) \in H_{-(k+1)}(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,\ldots,q$.

Proof. From (3.15), we have

$$\begin{gathered} \sum_{n=1}^{\infty}(1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\sum_{\nu=0}^{k}C_\nu \alpha_n e^{inz_j}(in)^{\nu}\right|^2 < \infty, \\ \sum_{n=1}^{\infty}(1+|n|)^{-2(k+1)}\left|\frac{1}{k!}\sum_{\nu=0}^{k}\alpha_n e^{inz_j}\sum_{m=1}^{\infty}[i(n+m)]^{\nu}A_{nm}e^{imz_j}\right|^2 < \infty \end{gathered} \tag{3.22}$$

for $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots,q$. Therefore, we obtain that $V_n^{(k)}(\lambda_j) \in H_{-(k+1)}(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots,q$. □

Let us choose $m_0 = \max\{m_{p+1}, m_{p+2},\ldots,m_q\}$. By Theorem 3.2 and (3.21), we get the following.

Theorem 3.4. $V_n^{(k)}(\lambda_j) \in H_{-m_0}(\mathbb{N})$, $k = 0,1,\ldots,m_j-1$, $j = p+1,p+2,\ldots,q$.

Proof. The proof of theorem is trivial. □

References

[1] M. A. Naĭmark, “Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint differential operator of the second order on a semi-axis,” *American Mathematical Society Translations*, vol. 16, pp. 103–193, 1960.

[2] V. E. Lyance, “A differential operator with spectral singularities, I, II,” *AMS Translations*, vol. 2, no. 60, pp. 227–283, 1967.

[3] E. Bairamov, Ö. Çakar, and A. O. Çelebi, “Quadratic pencil of Schrödinger operators with spectral singularities: discrete spectrum and principal functions,” *Journal of Mathematical Analysis and Applications*, vol. 216, no. 1, pp. 303–320, 1997.

[4] A. M. Krall, E. Bairamov, and Ö. Çakar, “Spectrum and spectral singularities of a quadratic pencil of a Schrödinger operator with a general boundary condition,” *Journal of Differential Equations*, vol. 151, no. 2, pp. 252–267, 1999.

[5] E. Bairamov, Ö. Çakar, and A. M. Krall, “An eigenfunction expansion for a quadratic pencil of a Schrödinger operator with spectral singularities,” *Journal of Differential Equations*, vol. 151, no. 2, pp. 268–289, 1999.

[6] E. Bairamov and A. O. Çelebi, “Spectrum and spectral expansion for the non-selfadjoint discrete Dirac operators,” *The Quarterly Journal of Mathematics. Oxford. Second Series*, vol. 50, no. 200, pp. 371–384, 1999.

[7] E. Bairamov and A. O. Çelebi, “Spectral properties of the Klein-Gordon s-wave equation with complex potential,” *Indian Journal of Pure and Applied Mathematics*, vol. 28, no. 6, pp. 813–824, 1997.

[8] G. B. Tunca and E. Bairamov, “Discrete spectrum and principal functions of non-selfadjoint differential operator,” *Czechoslovak Mathematical Journal*, vol. 49, no. 4, pp. 689–700, 1999.

[9] E. Bairamov, Y. Aygar, and T. Koprubasi, “The spectrum of eigenparameter-dependent discrete Sturm-Liouville equations,” *Journal of Computational and Applied Mathematics*, vol. 235, pp. 4519–4523, 2011.

[10] G. Š. Guseĭnov, “Determination of an infinite Jacobi matrix from scattering data,” *Doklady Akademii Nauk SSSR*, vol. 227, no. 6, pp. 1289–1292, 1976.

[11] G. S. Guseinov, “The inverse problem of scattering theory for a second order difference equation on the whole axis,” *Doklady Akademii Nauk SSSR*, vol. 17, pp. 1684–1688, 1976.

[12] M. A. Naĭmark, *Linear Differential Operators, II*, Ungar, New York, NY, USA, 1968.