Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 596971, 9 pages
doi:10.1155/2011/596971

Research Article

Approximate Best Proximity Pairs in Metric Space

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Received 8 January 2011; Accepted 12 February 2011

Academic Editor: Norimichi Hirano

Let A and B be nonempty subsets of a metric space X and also $T : A \cup B \to A \cup B$ and $T(A) \subseteq B$, $T(B) \subseteq A$. We are going to consider element $x \in A$ such that $d(x, Tx) \le d(A, B) + \epsilon$ for some $\epsilon > 0$. We call pair (A, B) an approximate best proximity pair. In this paper, definitions of approximate best proximity pair for a map and two maps, their diameters, T-minimizing a sequence are given in a metric space.

1. Introduction

Let X be a metric space and A and B nonempty subsets of X, and $d(A, B)$ is distance of A and B. If $d(x_0, y_0) = d(A, B)$, then the pair (x_0, y_0) is called a best proximity pair for A and B and put

$$\operatorname{prox}(A, B) := \{(x, y) \in A \times B : d(x, y) = d(A, B)\} \tag{1.1}$$

as the set of all best proximity pair (A, B). Best proximity pair evolves as a generalization of the concept of best approximation. That reader can find some important result of it in [1–4].

Now, as in [5] (see also [4, 6–11]), we can find the best proximity points of the sets A and B, by considering a map $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Best proximity pair also evolves as a generalization of the concept of fixed point of mappings. Because if $A \cap B \neq \emptyset$, every best proximity point is a fixed point of T.

We say that the point $x \in A \cup B$ is an approximate best proximity point of the pair (A, B), if $d(x, Tx) \le d(A, B) + \epsilon$, for some $\epsilon > 0$.

In the following, we introduce a concept of approximate proximity pair that is stronger than proximity pair.

Definition 1.1. Let A and B be nonempty subsets of a metric space X and $T : A \cup B \to A \cup B$ a map such that $T(A) \subseteq B$, $T(B) \subseteq A$. put

$$P_T^a(A,B) = \{x \in A \cup B : d(x,Tx) \le d(A,B) + \epsilon \text{ for some } \epsilon > 0\}. \tag{1.2}$$

We say that the pair (A,B) is an approximate best proximity pair if $P_T^a(A,B) \ne \emptyset$.

Example 1.2. Suppose that $X = \mathbf{R}^2$, $A = \{(x,y) \in X : (x-y)^2 + y^2 \le 1\}$, and $B = \{(x,y) \in X : (x+y)^2 + y^2 \le 1\}$ with $T(x,y) = (-x,y)$ for $(x,y) \in X$. Then $d((x,y),T(x,y)) \le d(A,B) + \epsilon$ for some $\epsilon > 0$. Hence $P_T^a(A,B) \ne \emptyset$.

2. Approximate Best Proximity

In this section, we will consider the existence of approximate best proximity points for the map $T : A \cup B \to A \cup B$, such that $T(A) \subseteq B$, $T(B) \subseteq A$, and its diameter.

Theorem 2.1. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$, and*

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1} x\right) = d(A,B) \quad \text{for some } x \in A \cup B. \tag{2.1}$$

Then the pair (A,B) is an approximate best proximity pair.

Proof. Let $\epsilon > 0$ be given and $x \in A \cup B$ such that $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = d(A,B)$; then there exists $N_0 > 0$ such that

$$\forall n \ge N_0 : d\left(T^n x, T^{n+1} x\right) < d(A,B) + \epsilon. \tag{2.2}$$

If $n = N_0$, then $d(T^{N_0}(x), T(T^{N_0}(x))) < d(A,B) + \epsilon$, and $T^{N_0}(x) \in P_T^a(A,B)$ and $P_T^a(A,B) \ne \emptyset$. □

Theorem 2.2. *Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and*

$$d(Tx,Ty) \le \alpha d(x,y) + \beta\left[d(x,Tx) + d(y,Ty)\right] + \gamma d(A,B) \tag{2.3}$$

for all $x,y \in A \cup B$, where $\alpha,\beta,\gamma \ge 0$ and $\alpha + 2\beta + \gamma < 1$. Then the pair (A,B) is an approximate best proximity pair.

Proof. If $x \in A \cup B$, then

$$d\left(Tx, T^2 x\right) \le \alpha d(x,Tx) + \beta\left[d(x,Tx) + d\left(Tx, T^2 x\right)\right] + \gamma d(A,B). \tag{2.4}$$

Therefore,

$$d\left(Tx, T^2x\right) \leq \frac{\alpha+\beta}{1-\beta} d(x, Tx) + \frac{\gamma}{1-\beta} d(A, B). \tag{2.5}$$

Now if $k = (\alpha+\beta)/(1-\beta)$, then

$$d\left(Tx, T^2x\right) \leq k d(x, Tx) + (1-k) d(A, B) \tag{2.6}$$

also

$$d\left(T^2x, T^3x\right) \leq k^2 d(x, Tx) + \left(1-k^2\right) d(A, B). \tag{2.7}$$

Therefore,

$$d\left(T^n x, T^{n+1}x\right) \leq k^n d(x, Tx) + (1-k^n) d(A, B), \tag{2.8}$$

and so

$$d\left(T^n x, T^{n+1}x\right) \longrightarrow d(A, B), \quad \text{as } n \longrightarrow \infty. \tag{2.9}$$

Therefore, by Theorem 2.1, $P_T^a(A,B) \neq \emptyset$; then pair (A,B) is an approximate best proximity pair. □

Definition 2.3. Let A and B be nonempty subsets of a metric space X. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$. We say that the sequence $\{z_n\} \subseteq A \cup B$ is T-minimizing if

$$\lim_{n\to\infty} d(z_n, Tz_n) = d(A, B). \tag{2.10}$$

Theorem 2.4. *Let A and B be nonempty subsets of a metric space X, suppose that the mapping T : $A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$. If $\{T^n x\}$ is a T-minimizing for some $x \in A \cup B$, then (A,B) is an approximate best pair proximity.*

Proof. Since

$$\lim_{n\to\infty} d\left(T^n x, T^{n+1}x\right) = d(A, B) \quad \text{for some } x \in A \cup B, \tag{2.11}$$

therefore, by Theorem 2.1, $P_T^a(A,B) \neq \emptyset$; then pair (A,B) is an approximate best proximity pair. □

Theorem 2.5. *Let A and B be nonempty subsets of a normed space X such that $A \cup B$ is compact. Suppose that the mapping $T : A \cup B \to A \cup B$ is satisfying $T(A) \subseteq B$, $T(B) \subseteq A$, T is continuous and*

$$\|Tx - Ty\| \leq \|x - y\|, \tag{2.12}$$

where $(x, y) \in A \times B$. Then $P_T^a(A, B)$ is nonempty and compact.

Proof. Since $A \cup B$ compact, there exists a $z_0 \in A \cup B$ such that

$$\|z_0 - Tz_0\| = \inf_{z \in A \cup B} \|z - Tz\| \cdot \tag{$*$}$$

If $\|z_0 - Tz_0\| > d(A, B)$, then $\|Tz_0 - T^2 z_0\| < \|z_0 - Tz_0\|$ which contradict to the definition of z_0, ($Tz_0 \in A \cup B$ and by $(*)$ $\|Tz_0 - T(Tz_0)\| \geq \|z_0 - Tz_0\|$). Therefore, $\|z_0 - Tz_0\| = d(A, B) \leq d(A, B) + \epsilon$ for some $\epsilon > 0$ and $z_0 \in P_T^a(A, B)$. Therefore, $P_T^a(A, B)$ is nonempty.

Also, if $\{z_n\} \subseteq P_T^\epsilon(A, B)$, then $\|z_n - Tz_n\| < d(A, B) + \epsilon$, for some $\epsilon > 0$, and by compactness of $A \cup B$, there exists a subsequence z_{n_k} and a $z_0 \in A \cup B$ such that $z_{n_k} \to z_0$ and so

$$\|z_0 - Tz_0\| = \lim_{k \to \infty} \|z_{n_k} - Tz_{n_k}\| < d(A, B) + \epsilon \tag{2.13}$$

for some $\epsilon > 0$, hence $P_T^a(A, B)$ is compact. □

Example 2.6. If $A = [-3, -1]$, $B = [1, 3]$, and $T : A \cup B \to A \cup B$ such that

$$T(x) = \begin{cases} \dfrac{1 - x}{2}, & x \in A, \\[2mm] \dfrac{-1 - x}{2}, & x \in B, \end{cases} \tag{2.14}$$

then $P_T^a(A, B)$ is compact, and we have

$$\begin{aligned} P_T^a(A, B) &= \{x \in A \cup B : \ d(x, Tx) < d(A, B) + \epsilon \text{ for some } \epsilon > 0\} \\ &= \{x \in A \cup B : \ d(x, Tx) < 2 + \epsilon \text{ for some } \epsilon > 0\} \\ &= \{1, -1\}. \end{aligned} \tag{2.15}$$

That is compact.

In the following, by $\operatorname{diam}(P_T^a(A, B))$ for a set $P_T^a(A, B) \neq \emptyset$, we will understand the diameter of the set $P_T^a(A, B)$.

Definition 2.7. Let $T : A \cup B \to A \cup B$ be a continuous map such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $\epsilon > 0$. We define diameter $P_T^a(A, B)$ by

$$\operatorname{diam}\left(P_T^a(A, B)\right) = \sup\left\{d(x, y) : x, y \in P_T^a(A, B)\right\}. \tag{2.16}$$

Theorem 2.8. *Let $T : A \cup B \to A \cup B$, such that $T(A) \subseteq B$, $T(B) \subseteq A$ and $\epsilon > 0$. If there exists an $\alpha \in [0, 1]$ such that for all $(x, y) \in A \times B$*

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{2.17}$$

then

$$\operatorname{diam}\left(P_T^a(A, B)\right) \leq \frac{2\epsilon}{1 - \alpha} + \frac{2d(A, B)}{1 - \alpha}. \tag{2.18}$$

Proof. If $x, y \in P_T^a(A, B)$, then

$$\begin{aligned} d(x, y) &\leq d(x, Tx) + d(Tx, Ty) + d(Ty, y) \\ &\leq \epsilon_1 + \alpha d(x, y) + 2d(A, B) + \epsilon_2. \end{aligned} \tag{2.19}$$

Put $\epsilon = \operatorname{Max}\{\epsilon_1, \epsilon_2\}$, therefore, $d(x, y) \leq 2\epsilon/(1-\alpha) + (2d(A, B))/(1-\alpha)$. Hence $\operatorname{diam}(P_T^a(A, B)) \leq 2\epsilon/(1 - \alpha) + (2d(A, B))/(1 - \alpha)$. □

3. Approximate Best Proximity for Two Maps

In this section, we will consider the existence of approximate best proximity points for two maps $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$, and its diameter.

Definition 3.1. Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup BS : A \cup B \to A \cup B$ two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. A point (x, y) in $A \times B$ is said to be an approximate-pair fixed point for (T, S) in X if there exists $\epsilon > 0$

$$d(Tx, Sy) \leq d(A, B) + \epsilon. \tag{3.1}$$

We say that the pair (T, S) has the approximate-pair fixed property in X if $P_{(T,S)}^a(A, B) \neq \emptyset$, where

$$P_{(T,S)}^a(A, B) = \left\{(x, y) \in A \times B : d(Tx, Sy) \leq d(A, B) + \epsilon \text{ for some } \epsilon > 0\right\}. \tag{3.2}$$

Theorem 3.2. *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If, for every $(x, y) \in A \times B$,*

$$d\left(T^n(x), S^n(y)\right) \longrightarrow d(A, B), \tag{3.3}$$

then (T, S) has the approximate-pair fixed property.

Proof. For $\epsilon > 0$, Suppose $(x, y) \in A \times B$. Since

$$\begin{gathered} d\left(T^n(x), S^n(y)\right) \longrightarrow d(A, B), \\ \exists n_0 > 0 \quad \text{s.t. } \forall n \geq n_0 : d\left(T^n(x), S^n(y)\right) < d(A, B) + \epsilon, \end{gathered} \tag{3.4}$$

then $d(T(T^{n-1}(x), S(S^{n-1}(y)) < d(A, B) + \epsilon$ for every $n \geq n_0$. Put $x_0 = T^{n_0-1}(x)$ and $y_0 = S^{n_0-1}(y))$. Hence $d(T(x_0), S(y_0)) \leq d(A, B) + \epsilon$ and $P^a_{(T,S)}(A, B) \neq \emptyset$. □

Theorem 3.3. *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \rightarrow A \cup B$ and $S : A \cup B \rightarrow A \cup B$ be two maps such that $T(A) \subseteq B$, $S(B) \subseteq A$ and, for every $(x, y) \in A \times B$,*

$$d(Tx, Sy) \leq \alpha d(x, y) + \beta\left[d(x, Tx) + d(y, Sy)\right] + \gamma d(A, B), \tag{3.5}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha+2\beta+\gamma < 1$. Then if x is an approximate fixed point for T, or y is an approximate fixed point for S, then $P^a_{(T,S)}(A, B) \neq \emptyset$.

Proof. If $(x, y) \in A \times B$, then

$$d(Tx, S(Tx)) \leq \alpha d(x, Tx) + \beta[d(x, Tx) + d(Tx, S(Tx))] + \gamma d(A, B). \tag{3.6}$$

Therefore,

$$d(Tx, S(Tx)) \leq \frac{\alpha + \beta}{1 - \beta} d(x, Tx) + \frac{\gamma}{1 - \beta} d(A, B). \tag{3.7}$$

Now if $k = (\alpha + \beta)/(1 - \beta)$, then

$$d(Tx, S(Tx)) \leq kd(x, Tx) + (1 - k)d(A, B) \tag{$*$}$$

also

$$d(Sy, T(Sy)) \leq kd(y, Sy) + (1 - k)d(A, B). \tag{$**$}$$

If x is an approximate fixed point for T, then there exists a $\epsilon > 0$ and by $(*)$

$$\begin{aligned} d(Tx, S(Tx)) &\leq kd(x, Tx) + (1 - k)d(A, B) \\ &\leq k(d(A, B) + \epsilon) + (1 - k)d(A, B) \\ &= d(A, B) + k\epsilon \\ &< d(A, B) + \epsilon. \end{aligned} \tag{3.8}$$

And $(x, Tx) \in P^a_{(T,S)}(A, B)$; also if y is an approximate fixed point for S, then there exists a $\epsilon > 0$ and by $(**)$

$$\begin{aligned} d(Sy, T(Sy)) &\leq kd(y, Sy) + (1-k)d(A, B) \\ &\leq k(d(A, B) + \epsilon) + (1-k)d(A, B) \\ &= d(A, B) + k\epsilon \\ &< d(A, B) + \epsilon. \end{aligned} \tag{3.9}$$

And $(y, Sy) \in P^a_{(T,S)}(A, B)$. Therefore, $P^a_{(T,S)}(A, B) \neq \emptyset$. □

Theorem 3.4. *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be two continuous maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If, for every $(x, y) \in A \times B$,*

$$d(Tx, Sy) \leq \alpha d(x, y) + \gamma d(A, B), \tag{3.10}$$

where $\alpha, \gamma \geq 0$ and $\alpha + \gamma = 1$, also let $\{x_n\}$ and $\{y_n\}$ be as follows:

$$x_{n+1} = Sy_n, \quad y_{n+1} = Tx_n \quad \text{for some } (x_1, y_1) \in A \times B, \ n \in N. \tag{3.11}$$

If $\{x_n\}$ has a convergent subsequence in A, then there exists a $x_0 \in A$ such that $d(x_0, Tx_0) = d(A, B)$.

Proof. We have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(Tx_n, Sy_n) \\ &\leq \alpha d(x_n, y_n) + \gamma(d(A, B) \\ &\leq \cdots \\ &\leq \alpha^{n+1} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^n)\gamma d(A, B). \end{aligned} \tag{3.12}$$

If $\{x_{n_k}\}_{k\geq 1}$ converges to $x_1 \in A$, that is, $x_{n_k} \to x_1$, then

$$d(x_{n_{K+1}}, y_{n_{k+1}}) \leq \alpha^{n_{k+1}} d(x_0, y_0) + (1 + \alpha + \cdots + \alpha^n_k)\gamma d(A, B). \tag{3.13}$$

Since T is continuous, then

$$d(x_{n_{k+1}}, Tx_{n_k}) \longrightarrow \frac{\gamma}{1-\alpha} d(A, B) = d(A, B). \tag{3.14}$$

Therefore, $d(x_1, Tx_1) = d(A, B)$. □

Definition 3.5. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be continues maps such that $T(A) \subseteq B$ and $S(B) \subseteq A$. We define diameter $P^a_{(T,S)}(A,B)$ by*

$$\operatorname{diam}\left(P^a_{(T,S)}(A,B)\right) = \sup\{d(x,y) : d(Tx,Ty) \le \epsilon + d(A,B) \text{ for some } \epsilon > 0\}. \tag{3.15}$$

Example 3.6. Suppose $A = \{(x,0) : 0 \le x \le 1\}$, $B = \{(x,1) : 0 \le x \le 1\}$, $T(x,0) = T(x,1) = (1/2,1)$, and $S(x,1) = S(x,0) = (1/2,0)$. Then $d(T(x,0),S(y,1)) = 1$ and $\operatorname{diam}(P^a_{(T,S)}(A,B)) = \operatorname{diam}(A \times B) = \sqrt{2}$.

Theorem 3.7. *Let $T : A \cup B \to A \cup B$ and $S : A \cup B \to A \cup B$ be continuous maps such that $T(A) \subseteq B$, $S(B) \subseteq A$. If there exists a $k \in [0,1]$,*

$$d(x,Tx) + d(Sy,y) \le kd(x,y), \tag{3.16}$$

then

$$\operatorname{diam}\left(P^a_{(T,S)}(A,B)\right) \le \frac{\epsilon}{1-k} + \frac{d(A,B)}{1-k} \quad \textit{for some } \epsilon > 0. \tag{3.17}$$

Proof. If $(x,y) \in P^a_{(T,S)}(A,B)$, then

$$\begin{aligned} d(x,y) &\le d(x,Tx) + d(Tx,Sy) + d(Sy,y) \\ &\le \epsilon + kd(x,y) + d(A,B). \end{aligned} \tag{3.18}$$

Therefore, $d(x,y) \le \epsilon/(1-k) + (d(A,B))/(1-k)$. Then $\operatorname{diam}(P^a_{(T,S)}(A,B)) \le \epsilon/(1-k) + (d(A,B))/(1-k)$. □

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