

Research Article

Discrete Mittag-Leffler Functions in Linear Fractional Difference Equations

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This paper investigates some initial value problems in discrete fractional calculus. We introduce a linear difference equation of fractional order along with suitable initial conditions of fractional type and prove the existence and uniqueness of the solution. Then the structure of the solutions space is discussed, and, in a particular case, an explicit form of the general solution involving discrete analogues of Mittag-Leffler functions is presented. All our observations are performed on a special time scale which unifies and generalizes ordinary difference calculus and q -difference calculus. Some of our results are new also in these particular discrete settings.

1. Introduction

The fractional calculus is a research field of mathematical analysis which may be taken for an old as well as a modern topic. It is an old topic because of its long history starting from some notes and ideas of G. W. Leibniz and L. Euler. On the other hand, it is a modern topic due to its enormous development during the last two decades. The present interest of many scientists and engineers in the theory of fractional calculus has been initiated by applications of this theory as well as by new mathematical challenges.

The theory of discrete fractional calculus belongs among these challenges. Foundations of this theory were formulated in pioneering works by Agarwal [1] and Diaz and Osler [2], where basic approaches, definitions, and properties of the theory of fractional sums and differences were reported (see also [3, 4]). The cited papers discussed these notions on discrete sets formed by arithmetic or geometric sequences (giving rise to fractional difference calculus or q -difference calculus). Recently, a series of papers continuing this research has appeared (see, e.g., [5, 6]).

The extension of basic notions of fractional calculus to other discrete settings was performed in [7], where fractional sums and differences have been introduced and studied in the framework of (q, h) -calculus, which can be reduced to ordinary difference calculus

and q -difference calculus via the choice $q = h = 1$ and $h = 0$, respectively. This extension follows recent trends in continuous and discrete analysis, characterized by a unification and generalization, and resulting into the origin and progressive development of the time scales theory (see [8, 9]). Discussing problems of fractional calculus, a question concerning the introduction of (Hilger) fractional derivative or integral on arbitrary time scale turns out to be a difficult matter. Although first attempts have been already performed (see, e.g., [10]), results obtained in this direction seem to be unsatisfactory.

The aim of this paper is to introduce some linear nabla (q, h) -fractional difference equations (i.e., equations involving difference operators of noninteger orders) and investigate their basic properties. Some particular results concerning this topic are already known, either for ordinary difference equations or q -difference equations of fractional order (some relevant references will be mentioned in Section 4). We wish to unify them and also present results which are new even also in these particular discrete settings.

The structure of the paper is the following: Section 2 presents a necessary mathematical background related to discrete fractional calculus. In particular, we are going to make some general remarks concerning fractional calculus on arbitrary time scales. In Section 3, we consider a linear nabla (q, h) -difference equation of noninteger order and discuss the question of the existence and uniqueness of the solution for the corresponding initial value problem, as well as the question of a general solution of this equation. In Section 4, we consider a particular case of the studied equation and describe the base of its solutions space by the use of eigenfunctions of the corresponding difference operator. We show that these eigenfunctions can be taken for discrete analogues of the Mittag-Leffler functions.

2. Preliminaries

The basic definitions of fractional calculus on continuous or discrete settings usually originate from the Cauchy formula for repeated integration or summation, respectively. We state here its general form valid for arbitrary time scale \mathbb{T} . Before doing this, we recall the notion of Taylor monomials introduced in [9]. These monomials $\widehat{h}_n : \mathbb{T}^2 \rightarrow \mathbb{R}$, $n \in \mathbb{N}_0$ are defined recursively as follows:

$$\widehat{h}_0(t, s) = 1 \quad \forall s, t \in \mathbb{T} \quad (2.1)$$

and, given \widehat{h}_n for $n \in \mathbb{N}_0$, we have

$$\widehat{h}_{n+1}(t, s) = \int_s^t \widehat{h}_n(\tau, s) \nabla \tau \quad \forall s, t \in \mathbb{T}. \quad (2.2)$$

Now let $f : \mathbb{T} \rightarrow \mathbb{R}$ be ∇ -integrable on $[a, b] \cap \mathbb{T}$, $a, b \in \mathbb{T}$. We put

$${}_a \nabla^{-1} f(t) = \int_a^t f(\tau) \nabla \tau \quad \forall t \in \mathbb{T}, a \leq t \leq b \quad (2.3)$$

and define recursively

$${}_a\nabla^{-n}f(t) = \int_a^t {}_a\nabla^{-n+1}f(\tau)\nabla\tau \tag{2.4}$$

for $n = 2, 3, \dots$. Then we have the following.

Proposition 2.1 (Nabla Cauchy formula). *Let $n \in \mathbb{Z}^+$, $a, b \in \mathbb{T}$ and let $f : \mathbb{T} \rightarrow \mathbb{R}$ be ∇ -integrable on $[a, b] \cap \mathbb{T}$. If $t \in \mathbb{T}$, $a \leq t \leq b$, then*

$${}_a\nabla^{-n}f(t) = \int_a^t \widehat{h}_{n-1}(t, \rho(\tau))f(\tau)\nabla\tau. \tag{2.5}$$

Proof. This assertion can be proved by induction. If $n = 1$, then (2.5) obviously holds. Let $n \geq 2$ and assume that (2.5) holds with n replaced with $n - 1$, that is,

$${}_a\nabla^{-n+1}f(t) = \int_a^t \widehat{h}_{n-2}(t, \rho(\tau))f(\tau)\nabla\tau. \tag{2.6}$$

By the definition, the left-hand side of (2.5) is an antiderivative of ${}_a\nabla^{-n+1}f(t)$. We show that the right-hand side of (2.5) is an antiderivative of $\int_a^t \widehat{h}_{n-2}(t, \rho(\tau))f(\tau)\nabla\tau$. Indeed, it holds

$$\nabla \int_a^t \widehat{h}_{n-1}(t, \rho(\tau))f(\tau)\nabla\tau = \int_a^t \nabla \widehat{h}_{n-1}(t, \rho(\tau))f(\tau)\nabla\tau = \int_a^t \widehat{h}_{n-2}(t, \rho(\tau))f(\tau)\nabla\tau, \tag{2.7}$$

where we have employed the property

$$\nabla \int_a^t g(t, \tau)\nabla\tau = \int_a^t \nabla g(t, \tau)\nabla\tau + g(\rho(t), t) \tag{2.8}$$

(see [9, page 139]). Consequently, the relation (2.5) holds up to a possible additive constant. Substituting $t = a$, we can find this additive constant zero. \square

The formula (2.5) is a corner stone in the introduction of the nabla fractional integral ${}_a\nabla^{-\alpha}f(t)$ for positive reals α . However, it requires a reasonable and natural extension of a discrete system of monomials $(\widehat{h}_n, n \in \mathbb{N}_0)$ to a continuous system $(\widehat{h}_\alpha, \alpha \in \mathbb{R}^+)$. This matter is closely related to a problem of an explicit form of \widehat{h}_n . Of course, it holds $\widehat{h}_1(t, s) = t - s$ for all $t, s \in \mathbb{T}$. However, the calculation of \widehat{h}_n for $n > 1$ is a difficult task which seems to be answerable only in some particular cases. It is well known that for $\mathbb{T} = \mathbb{R}$, it holds

$$\widehat{h}_n(t, s) = \frac{(t - s)^n}{n!}, \tag{2.9}$$

while for discrete time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k, k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$, we have

$$\widehat{h}_n(t, s) = \frac{\prod_{j=0}^{n-1} (t - s + j)}{n!}, \quad \widehat{h}_n(t, s) = \prod_{j=0}^{n-1} \frac{q^j t - s}{\sum_{r=0}^j q^r}, \quad (2.10)$$

respectively. In this connection, we recall a conventional notation used in ordinary difference calculus and q -calculus, namely,

$$(t - s)^{(n)} = \prod_{j=0}^{n-1} (t - s + j), \quad (t - s)_{\tilde{q}}^{(n)} = t^n \prod_{j=0}^{n-1} \left(1 - \frac{\tilde{q}^j s}{t}\right) \quad (0 < \tilde{q} < 1) \quad (2.11)$$

and $[j]_q = \sum_{r=0}^{j-1} q^r$ ($q > 0$), $[n]_q! = \prod_{j=1}^n [j]_q$. To extend the meaning of these symbols also for noninteger values (as it is required in the discrete fractional calculus), we recall some other necessary background of q -calculus. For any $x \in \mathbb{R}$ and $0 < q \neq 1$, we set $[x]_q = (q^x - 1)/(q - 1)$. By the continuity, we put $[x]_1 = x$. Further, the q -Gamma function is defined for $0 < \tilde{q} < 1$ as

$$\Gamma_{\tilde{q}}(x) = \frac{(\tilde{q}, \tilde{q})_{\infty} (1 - \tilde{q})^{1-x}}{(\tilde{q}^x, \tilde{q})_{\infty}}, \quad (2.12)$$

where $(p, \tilde{q})_{\infty} = \prod_{j=0}^{\infty} (1 - p\tilde{q}^j)$, $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$. Note that this function satisfies the functional relation $\Gamma_{\tilde{q}}(x+1) = [x]_{\tilde{q}} \Gamma_{\tilde{q}}(x)$ and the condition $\Gamma_{\tilde{q}}(1) = 1$. Using this, the q -binomial coefficient can be introduced as

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\tilde{q}} = \frac{\Gamma_{\tilde{q}}(x+1)}{\Gamma_{\tilde{q}}(k+1) \Gamma_{\tilde{q}}(x-k+1)}, \quad x \in \mathbb{R}, \quad k \in \mathbb{Z}. \quad (2.13)$$

Note that although the q -Gamma function is not defined at nonpositive integers, the formula

$$\frac{\Gamma_{\tilde{q}}(x+m)}{\Gamma_{\tilde{q}}(x)} = (-1)^m \tilde{q}^{xm + \binom{m}{2}} \frac{\Gamma_{\tilde{q}}(1-x)}{\Gamma_{\tilde{q}}(1-x-m)}, \quad x \in \mathbb{R}, \quad m \in \mathbb{Z}^+ \quad (2.14)$$

permits to calculate this ratio also at such the points. It is well known that if $\tilde{q} \rightarrow 1^-$ then $\Gamma_{\tilde{q}}(x)$ becomes the Euler Gamma function $\Gamma(x)$ (and analogously for the q -binomial coefficient). Among many interesting properties of the q -Gamma function and q -binomial coefficients, we mention q -Pascal rules

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\tilde{q}} = \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_{\tilde{q}} + \tilde{q}^k \begin{bmatrix} x-1 \\ k \end{bmatrix}_{\tilde{q}}, \quad x \in \mathbb{R}, \quad k \in \mathbb{Z}, \quad (2.15)$$

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\tilde{q}} = \tilde{q}^{x-k} \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_{\tilde{q}} + \begin{bmatrix} x-1 \\ k \end{bmatrix}_{\tilde{q}}, \quad x \in \mathbb{R}, \quad k \in \mathbb{Z} \quad (2.16)$$

and the q -Vandermonde identity

$$\sum_{j=0}^m \begin{bmatrix} x \\ m-j \end{bmatrix}_{\tilde{q}} \begin{bmatrix} y \\ j \end{bmatrix}_{\tilde{q}} \tilde{q}^{j^2-mj+xj} = \begin{bmatrix} x+y \\ m \end{bmatrix}_{\tilde{q}}, \quad x, y \in \mathbb{R}, m \in \mathbb{N}_0 \quad (2.17)$$

(see [11]) that turn out to be very useful in our further investigations.

The computation of an explicit form of $\hat{h}_n(t, s)$ can be performed also in a more general case. We consider here the time scale

$$\mathbb{T}_{(q,h)}^{t_0} = \left\{ t_0 q^k + [k]_q h, k \in \mathbb{Z} \right\} \cup \left\{ \frac{h}{1-q} \right\}, \quad t_0 > 0, q \geq 1, h \geq 0, q+h > 1 \quad (2.18)$$

(see also [7]). Note that if $q = 1$ then the cluster point $h/(1-q) = -\infty$ is not involved in $\mathbb{T}_{(q,h)}^{t_0}$. The forward and backward jump operator is the linear function $\sigma(t) = qt + h$ and $\rho(t) = q^{-1}(t - h)$, respectively. Similarly, the forward and backward graininess is given by $\mu(t) = (q - 1)t + h$ and $\nu(t) = q^{-1}\mu(t)$, respectively. In particular, if $t_0 = q = h = 1$, then $\mathbb{T}_{(q,h)}^{t_0}$ becomes \mathbb{Z} , and if $t_0 = 1, q > 1, h = 0$, then $\mathbb{T}_{(q,h)}^{t_0}$ is reduced to $\overline{q^{\mathbb{Z}}}$.

Let $a \in \mathbb{T}_{(q,h)}^{t_0}, a > h/(1-q)$ be fixed. Then we introduce restrictions of the time scale $\mathbb{T}_{(q,h)}^{t_0}$ by the relation

$$\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)} = \left\{ t \in \mathbb{T}_{(q,h)}^{t_0}, t \geq \sigma^i(a) \right\}, \quad i = 0, 1, \dots, \quad (2.19)$$

where the symbol σ^i stands for the i th iterate of σ (analogously, we use the symbol ρ^i). To simplify the notation, we put $\tilde{q} = 1/q$ whenever considering the time scale $\mathbb{T}_{(q,h)}^{t_0}$ or $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^i(a)}$.

Using the induction principle, we can verify that Taylor monomials on $\mathbb{T}_{(q,h)}^{t_0}$ have the form

$$\hat{h}_n(t, s) = \frac{\prod_{j=0}^{n-1} (\sigma^j(t) - s)}{[n]_q!} = \frac{\prod_{j=0}^{n-1} (t - \rho^j(s))}{[n]_{\tilde{q}}!}. \quad (2.20)$$

Note that this result generalizes previous forms (2.10) and, moreover, enables its unified notation. In particular, if we introduce the symbolic (q, h) -power

$$(t - s)_{(\tilde{q},h)}^{(n)} = \prod_{j=0}^{n-1} (t - \rho^j(s)) \quad (2.21)$$

unifying (2.11), then the Cauchy formula (2.5) can be rewritten for $\mathbb{T} = \mathbb{T}_{(q,h)}^{t_0}$ as

$${}_a \nabla^{-n} f(t) = \int_a^t \frac{(t - \rho(\tau))_{(\tilde{q},h)}^{(n-1)}}{[n-1]_{\tilde{q}}!} f(\tau) \nabla \tau. \quad (2.22)$$

Discussing a reasonable generalization of (q, h) -power (2.21) to real values α instead of integers n , we recall broadly accepted extensions of its particular cases (2.11) in the form

$$(t-s)^{(\alpha)} = \frac{\Gamma(t-s+\alpha)}{\Gamma(t-s)}, \quad (t-s)_{\tilde{q}}^{(\alpha)} = t^\alpha \frac{(s/t, \tilde{q})_\infty}{(\tilde{q}^\alpha s/t, \tilde{q})_\infty}, \quad t \neq 0. \quad (2.23)$$

Now, we assume $s, t \in \mathbb{T}_{(q,h)}^{t_0}$, $t \geq s > h/(1-q)$. First, consider (q, h) -power (2.21) corresponding to the time scale $\mathbb{T}_{(q,h)}^{t_0}$, where $q > 1$. Then we can rewrite (2.21) as

$$(t-s)_{(\tilde{q},h)}^{(n)} = \left(t + \frac{h\tilde{q}}{1-\tilde{q}}\right)^n \prod_{j=0}^{n-1} \left(1 - \tilde{q}^j \frac{s + h\tilde{q}/(1-\tilde{q})}{t + h\tilde{q}/(1-\tilde{q})}\right) = ([t] - [s])_{\tilde{q}}^{(n)}, \quad (2.24)$$

where $[t] = t + h\tilde{q}/(1-\tilde{q})$ and $[s] = s + h\tilde{q}/(1-\tilde{q})$. A required extension of (q, h) -power (2.21) is then provided by the formula

$$(t-s)_{(\tilde{q},h)}^{(\alpha)} = ([t] - [s])_{\tilde{q}}^{(\alpha)}. \quad (2.25)$$

Now consider (q, h) -power (2.21) corresponding to the time scale $\mathbb{T}_{(q,h)}^{t_0}$, where $q = 1$. Then

$$(t-s)_{(1,h)}^{(n)} = \prod_{j=0}^{n-1} (t-s+jh) = h^n \prod_{j=0}^{n-1} \left(\frac{t-s}{h} + j\right) = h^n \frac{((t-s)/h + n - 1)!}{((t-s)/h - 1)!} \quad (2.26)$$

and the formula (2.21) can be extended by

$$(t-s)_{(1,h)}^{(\alpha)} = \frac{h^\alpha \Gamma((t-s)/h + \alpha)}{\Gamma((t-s)/h)}. \quad (2.27)$$

These definitions are consistent, since it can be shown that

$$\lim_{\tilde{q} \rightarrow 1^-} ([t] - [s])_{\tilde{q}}^{(\alpha)} = (t-s)_{(1,h)}^{(\alpha)}. \quad (2.28)$$

Now the required extension of the monomial $\widehat{h}_n(t, s)$ corresponding to $\mathbb{T}_{(q,h)}^{t_0}$ takes the form

$$\widehat{h}_\alpha(t, s) = \frac{(t-s)_{(\tilde{q},h)}^{(\alpha)}}{\Gamma_{\tilde{q}}(\alpha+1)}. \quad (2.29)$$

Another (equivalent) expression of $\widehat{h}_\alpha(t, s)$ is provided by the following assertion.

Proposition 2.2. Let $\alpha \in \mathbb{R}$, $s, t \in \mathbb{T}_{(q,h)}^{t_0}$ and $n \in \mathbb{N}_0$ be such that $t = \sigma^n(s)$. Then

$$\widehat{h}_\alpha(t, s) = (\nu(t))^\alpha \begin{bmatrix} \alpha + n - 1 \\ n - 1 \end{bmatrix}_{\tilde{q}} = (\nu(t))^\alpha \begin{bmatrix} -\alpha - 1 \\ n - 1 \end{bmatrix}_{\tilde{q}} (-1)^{n-1} \tilde{q}^{\alpha(n-1) + \binom{n}{2}}. \quad (2.30)$$

Proof. Let $q > 1$. Using the relations

$$[t] = \frac{\nu(t)}{(1 - \tilde{q})}, \quad \frac{[s]}{[t]} = \tilde{q}^n, \quad (2.31)$$

we can derive that

$$\begin{aligned} \widehat{h}_\alpha(t, s) &= \frac{[t]^\alpha ([s]/[t], \tilde{q})_\infty}{\Gamma_{\tilde{q}}(\alpha + 1) (\tilde{q}^\alpha [s]/[t], \tilde{q})_\infty} = \frac{(1 - \tilde{q})^{-\alpha} \nu(t)^\alpha (\tilde{q}^n, \tilde{q})_\infty}{\Gamma_{\tilde{q}}(\alpha + 1) (\tilde{q}^{\alpha+n}, \tilde{q})_\infty} \\ &= (\nu(t))^\alpha \frac{\Gamma_{\tilde{q}}(\alpha + n)}{\Gamma_{\tilde{q}}(\alpha + 1) \Gamma_{\tilde{q}}(n)} = (\nu(t))^\alpha \begin{bmatrix} \alpha + n - 1 \\ n - 1 \end{bmatrix}_{\tilde{q}}. \end{aligned} \quad (2.32)$$

The second equality in (2.30) follows from the identity (2.14). The case $q = 1$ results from (2.27). \square

The key property of $\widehat{h}_\alpha(t, s)$ follows from its differentiation. The symbol $\nabla_{(q,h)}^m$ used in the following assertion (and also undermentioned) is the m th order nabla (q, h) -derivative on the time scale $\mathbb{T}_{(q,h)}^{t_0}$, defined for $m = 1$ as

$$\nabla_{(q,h)} f(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} = \frac{f(t) - f(\tilde{q}(t - h))}{(1 - \tilde{q})t + \tilde{q}h} \quad (2.33)$$

and iteratively for higher orders.

Lemma 2.3. Let $m \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}$, $s, t \in \mathbb{T}_{(q,h)}^{t_0}$ and $n \in \mathbb{Z}^+$, $n \geq m$ be such that $t = \sigma^n(s)$. Then

$$\nabla_{(q,h)}^m \widehat{h}_\alpha(t, s) = \begin{cases} \widehat{h}_{\alpha-m}(t, s), & \alpha \notin \{0, 1, \dots, m-1\}, \\ 0, & \alpha \in \{0, 1, \dots, m-1\}. \end{cases} \quad (2.34)$$

Proof. First let $m = 1$. For $\alpha = 0$ we get $\widehat{h}_0(t, s) = 1$ and the first nabla (q, h) -derivative is zero. If $\alpha \neq 0$, then by (2.30) and (2.16), we have

$$\begin{aligned} \nabla_{(q,h)} \widehat{h}_\alpha(t, s) &= \frac{\widehat{h}_\alpha(t, s) - \widehat{h}_\alpha(\rho(t), s)}{v(t)} \\ &= \frac{1}{v(t)} \left((v(t))^\alpha \left[\begin{matrix} \alpha + n - 1 \\ n - 1 \end{matrix} \right]_{\tilde{q}} - (v(\rho(t)))^\alpha \left[\begin{matrix} \alpha + n - 2 \\ n - 2 \end{matrix} \right]_{\tilde{q}} \right) \\ &= (v(t))^{\alpha-1} \left(\left[\begin{matrix} \alpha + n - 1 \\ n - 1 \end{matrix} \right]_{\tilde{q}} - \tilde{q}^\alpha \left[\begin{matrix} \alpha + n - 2 \\ n - 2 \end{matrix} \right]_{\tilde{q}} \right) = \widehat{h}_{\alpha-1}(t, s). \end{aligned} \quad (2.35)$$

The case $m \geq 2$ can be verified by the induction principle. \square

We note that an extension of this property for derivatives of noninteger orders will be performed in Section 4.

Now we can continue with the introduction of (q, h) -fractional integral and derivative of a function $f : \widetilde{\mathbb{T}}_{(q,h)}^a \rightarrow \mathbb{R}$. Let $t \in \widetilde{\mathbb{T}}_{(q,h)}^a$. Our previous considerations (in particular, the Cauchy formula (2.5) along with the relations (2.22) and (2.29)) warrant us to introduce the nabla (q, h) -fractional integral of order $\alpha \in \mathbb{R}^+$ over the time scale interval $[a, t] \cap \widetilde{\mathbb{T}}_{(q,h)}^a$ as

$${}_a \nabla_{(q,h)}^{-\alpha} f(t) = \int_a^t \widehat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla \tau \quad (2.36)$$

(see also [7]). The nabla (q, h) -fractional derivative of order $\alpha \in \mathbb{R}^+$ is then defined by

$${}_a \nabla_{(q,h)}^\alpha f(t) = \nabla_{(q,h)}^m {}_a \nabla_{(q,h)}^{-(m-\alpha)} f(t), \quad (2.37)$$

where $m \in \mathbb{Z}^+$ is given by $m - 1 < \alpha \leq m$. For the sake of completeness, we put

$${}_a \nabla_{(q,h)}^0 f(t) = f(t). \quad (2.38)$$

As we noted earlier, a reasonable introduction of fractional integrals and fractional derivatives on arbitrary time scales remains an open problem. In the previous part, we have consistently used (and in the sequel, we shall consistently use) the time scale notation of main procedures and operations to outline a possible way out to further generalizations.

3. A Linear Initial Value Problem

In this section, we are going to discuss the linear initial value problem

$$\sum_{j=1}^m p_{m-j+1}(t) {}_a \nabla_{(q,h)}^{\alpha-j+1} y(t) + p_0(t) y(t) = 0, \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}, \quad (3.1)$$

$${}_a \nabla_{(q,h)}^{\alpha-j} y(t) \Big|_{t=\sigma^m(a)} = y_{\alpha-j}, \quad j = 1, 2, \dots, m, \tag{3.2}$$

where $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ are such that $m - 1 < \alpha \leq m$. Further, we assume that $p_j(t)$ are arbitrary real-valued functions on $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}$ ($j = 1, \dots, m - 1$), $p_m(t) = 1$ on $\tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}$ and $y_{\alpha-j}$ ($j = 1, \dots, m$) are arbitrary real scalars.

If α is a positive integer, then (3.1)-(3.2) becomes the standard discrete initial value problem. If α is not an integer, then applying the definition of nabla (q, h) -fractional derivatives, we can observe that (3.1) is of the general form

$$\sum_{i=0}^{n-1} a_i(t) y(\rho^i(t)) = 0, \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}, \quad n \text{ being such that } t = \sigma^n(a), \tag{3.3}$$

which is usually referred to as the equation of Volterra type. If such an equation has two different solutions, then their values differ at least at one of the points $\sigma(a), \sigma^2(a), \dots, \sigma^m(a)$. In particular, if $a_0(t) \neq 0$ for all $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}$, then arbitrary values of $y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a))$ determine uniquely the solution $y(t)$ for all $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}$. We show that the values $y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-m}$, introduced by (3.2), keep the same properties.

Proposition 3.1. *Let $y : \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)} \rightarrow \mathbb{R}$ be a function. Then (3.2) represents a one-to-one mapping between the vectors $(y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a)))$ and $(y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-m})$.*

Proof. The case $\alpha \in \mathbb{Z}^+$ is well known from the literature. Let $\alpha \notin \mathbb{Z}^+$. We wish to show that the values of $y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a))$ determine uniquely the values of

$${}_a \nabla_{(q,h)}^{\alpha-1} y(t) \Big|_{t=\sigma^m(a)}, \quad {}_a \nabla_{(q,h)}^{\alpha-2} y(t) \Big|_{t=\sigma^m(a)}, \dots, \quad {}_a \nabla_{(q,h)}^{\alpha-m} y(t) \Big|_{t=\sigma^m(a)} \tag{3.4}$$

and vice versa. Utilizing the relation

$${}_a \nabla_{(q,h)}^{\alpha-j} y(t) \Big|_{t=\sigma^m(a)} = \sum_{k=1}^m v(\sigma^{m-k+1}(a)) \hat{h}_{j-1-\alpha}(\sigma^m(a), \sigma^{m-k}(a)) y(\sigma^{m-k+1}(a)) \tag{3.5}$$

(see [7, Propositions 1 and 3] with respect to (2.30)), we can rewrite (3.2) as the linear mapping

$$\sum_{k=1}^m r_{jk} y(\sigma^{m-k+1}(a)) = y_{\alpha-j}, \quad j = 1, \dots, m, \tag{3.6}$$

where

$$r_{jk} = v(\sigma^{m-k+1}(a)) \hat{h}_{j-1-\alpha}(\sigma^m(a), \sigma^{m-k}(a)), \quad j, k = 1, \dots, m \tag{3.7}$$

are elements of the transformation matrix R_m . We show that R_m is regular. Obviously,

$$\det R_m = \left(\prod_{k=1}^m \nu(\sigma^k(a)) \right) \det H_m, \quad (3.8)$$

where

$$H_m = \begin{pmatrix} \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \hat{h}_{-\alpha}(\sigma^m(a), a) \\ \hat{h}_{1-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \hat{h}_{1-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \hat{h}_{1-\alpha}(\sigma^m(a), a) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{h}_{m-1-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \hat{h}_{m-1-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \hat{h}_{m-1-\alpha}(\sigma^m(a), a) \end{pmatrix}. \quad (3.9)$$

To calculate $\det H_m$, we employ some elementary operations preserving the value of $\det H_m$. Using the properties

$$\begin{aligned} \hat{h}_{i-\alpha}(\sigma^m(a), \sigma^\ell(a)) - \nu(\sigma^m(a)) \hat{h}_{i-\alpha-1}(\sigma^m(a), \sigma^\ell(a)) &= \hat{h}_{i-\alpha}(\sigma^{m-1}(a), \sigma^\ell(a)) \\ (i = 1, 2, \dots, m-1, \ell = 0, 1, \dots, m-2), \end{aligned} \quad (3.10)$$

$$\hat{h}_{i-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) - \nu(\sigma^m(a)) \hat{h}_{i-\alpha-1}(\sigma^m(a), \sigma^{m-1}(a)) = 0,$$

which follow from Lemma 2.3, we multiply the i th row ($i = 1, 2, \dots, m-1$) of H_m by $-\nu(\sigma^m(a))$ and add it to the successive one. We arrive at the form

$$\left(\begin{array}{c|ccc} \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \hat{h}_{-\alpha}(\sigma^m(a), a) \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{array} \middle| \begin{array}{c} \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) \cdots \hat{h}_{-\alpha}(\sigma^m(a), a) \\ \hline H_{m-1} \end{array} \right). \quad (3.11)$$

Then we apply repeatedly this procedure to obtain the triangular matrix

$$\left(\begin{array}{cccc} \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \hat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \hat{h}_{-\alpha}(\sigma^m(a), a) \\ 0 & \hat{h}_{1-\alpha}(\sigma^{m-1}(a), \sigma^{m-2}(a)) & \cdots & \hat{h}_{1-\alpha}(\sigma^{m-1}(a), a) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{h}_{m-1-\alpha}(\sigma(a), a) \end{array} \right). \quad (3.12)$$

Since $\hat{h}_{i-\alpha}(\sigma^k(a), \sigma^{k-1}(a)) = (\nu(\sigma^k(a)))^{i-\alpha}$ ($i = 0, 1, \dots, m-1$), we get

$$\det H_m = \prod_{k=1}^m \left(\nu \left(\sigma^k(a) \right) \right)^{m-k-\alpha}, \text{ that is, } \det R_m = \prod_{k=1}^m \left(\nu \left(\sigma^k(a) \right) \right)^{m-k-\alpha+1} \neq 0. \quad (3.13)$$

Thus the matrix R_m is regular, hence the corresponding mapping (3.6) is one to one. \square

Now we approach a problem of the existence and uniqueness of (3.1)-(3.2). First we recall the general notion of ν -regressivity of a matrix function and a corresponding linear nabla dynamic system (see [9]).

Definition 3.2. An $n \times n$ -matrix-valued function $A(t)$ on a time scale \mathbb{T} is called ν -regressive provided

$$\det(I - \nu(t)A(t)) \neq 0 \quad \forall t \in \mathbb{T}_\kappa, \quad (3.14)$$

where I is the identity matrix. Further, we say that the linear dynamic system

$$\nabla z(t) = A(t)z(t) \quad (3.15)$$

is ν -regressive provided that $A(t)$ is ν -regressive.

Considering a higher order linear difference equation, the notion of ν -regressivity for such an equation can be introduced by means of its transformation to the corresponding first order linear dynamic system. We are going to follow this approach and generalize the notion of ν -regressivity for the linear fractional difference equation (3.1).

Definition 3.3. Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m - 1 < \alpha \leq m$. Then (3.1) is called ν -regressive provided the matrix

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{p_0(t)}{\nu^{m-\alpha}(t)} & -p_1(t) & \cdots & -p_{m-2}(t) & -p_{m-1}(t) \end{pmatrix} \quad (3.16)$$

is ν -regressive.

Remark 3.4. The explicit expression of the ν -regressivity property for (3.1) can be read as

$$1 + \sum_{j=1}^{m-1} p_{m-j}(t)(\nu(t))^j + p_0(t)(\nu(t))^\alpha \neq 0 \quad \forall t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}. \quad (3.17)$$

If α is a positive integer, then both these introductions agree with the definition of ν -regressivity of a higher order linear difference equation presented in [9].

Theorem 3.5. Let (3.1) be ν -regressive. Then the problem (3.1)-(3.2) has a unique solution defined for all $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$.

Proof. The conditions (3.2) enable us to determine the values of $y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a))$ by the use of (3.6). To calculate the values of $y(\sigma^{m+1}(a)), y(\sigma^{m+2}(a)), \dots$, we perform the transformation

$$z_j(t) = {}_a\nabla_{(q,h)}^{\alpha-m+j-1} y(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^j(a)}, \quad j = 1, 2, \dots, m \quad (3.18)$$

which allows us to rewrite (3.1) into a matrix form. Before doing this, we need to express $y(t)$ in terms of $z_1(t), z_1(\rho(t)), \dots, z_1(\sigma(a))$. Applying the relation ${}_a\nabla_{(q,h)}^{m-\alpha} {}_a\nabla_{(q,h)}^{-(m-\alpha)} y(t) = y(t)$ (see [7]) and expanding the fractional derivative, we arrive at

$$y(t) = {}_a\nabla_{(q,h)}^{m-\alpha} z_1(t) = \frac{z_1(t)}{\nu^{m-\alpha}(t)} + \int_a^{\rho(t)} \hat{h}_{\alpha-m-1}(t, \rho(\tau)) z_1(\tau) \nabla \tau. \quad (3.19)$$

Therefore, the problem (3.1)-(3.2) can be rewritten to the vector form

$$\begin{aligned} {}_a\nabla_{(q,h)} z(t) &= A(t)z(t) + b(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}, \\ z(\sigma^m(a)) &= (y_{\alpha-m}, \dots, y_{\alpha-1})^T, \end{aligned} \quad (3.20)$$

where

$$z(t) = (z_1(t), \dots, z_m(t))^T, \quad b(t) = \left(0, \dots, 0, -p_0(t) \int_a^{\rho(t)} \hat{h}_{\alpha-m-1}(t, \rho(\tau)) z_1(\tau) \nabla \tau \right)^T \quad (3.21)$$

and $A(t)$ is given by (3.16). The ν -regressivity of the matrix $A(t)$ enables us to write

$$z(t) = (I - \nu(t)A(t))^{-1} (z(\rho(t)) + \nu(t)b(t)), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}, \quad (3.22)$$

hence, using the value of $z(\sigma^m(a))$, we can solve this system by the step method starting from $t = \sigma^{m+1}(a)$. The solution $y(t)$ of the original initial value problem (3.1)-(3.2) is then given by the formula (3.19). \square

Remark 3.6. The previous assertion on the existence and uniqueness of the solution can be easily extended to the initial value problem involving nonhomogeneous linear equations as well as some nonlinear equations.

The final goal of this section is to investigate the structure of the solutions of (3.1). We start with the following notion.

Definition 3.7. Let $\gamma \in \mathbb{R}, 0 \leq \gamma < 1$. For m functions $y_j : \tilde{\mathbb{T}}_{(q,h)}^a \rightarrow \mathbb{R} (j = 1, 2, \dots, m)$, we define the γ -Wronskian $W_\gamma(y_1, \dots, y_m)(t)$ as determinant of the matrix

$$V_\gamma(y_1, \dots, y_m)(t) = \begin{pmatrix} {}_a\nabla_{(q,h)}^{-\gamma} y_1(t) & {}_a\nabla_{(q,h)}^{-\gamma} y_2(t) & \cdots & {}_a\nabla_{(q,h)}^{-\gamma} y_m(t) \\ {}_a\nabla_{(q,h)}^{1-\gamma} y_1(t) & {}_a\nabla_{(q,h)}^{1-\gamma} y_2(t) & \cdots & {}_a\nabla_{(q,h)}^{1-\gamma} y_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ {}_a\nabla_{(q,h)}^{m-1-\gamma} y_1(t) & {}_a\nabla_{(q,h)}^{m-1-\gamma} y_2(t) & \cdots & {}_a\nabla_{(q,h)}^{m-1-\gamma} y_m(t) \end{pmatrix}, \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^m(a)}. \tag{3.23}$$

Remark 3.8. Note that the first row of this matrix involves fractional order integrals. It is a consequence of the form of initial conditions utilized in our investigations. Of course, this introduction of $W_\gamma(y_1, \dots, y_m)(t)$ coincides for $\gamma = 0$ with the classical definition of the Wronskian (see [8]). Moreover, it holds $W_\gamma(y_1, \dots, y_m)(t) = W_0({}_a\nabla_{(q,h)}^{-\gamma} y_1, \dots, {}_a\nabla_{(q,h)}^{-\gamma} y_m)(t)$.

Theorem 3.9. Let functions $y_1(t), \dots, y_m(t)$ be solutions of the ν -regressive equation (3.1) and let $W_{m-\alpha}(y_1, \dots, y_m)(\sigma^m(a)) \neq 0$. Then any solution $y(t)$ of (3.1) can be written in the form

$$y(t) = \sum_{k=1}^m c_k y_k(t), \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}, \tag{3.24}$$

where c_1, \dots, c_m are real constants.

Proof. Let $y(t)$ be a solution of (3.1). By Proposition 3.1, there exist real scalars $y_{\alpha-1}, \dots, y_{\alpha-m}$ such that $y(t)$ is satisfying (3.2). Now we consider the function $u(t) = \sum_{k=1}^m c_k y_k(t)$, where the m -tuple (c_1, \dots, c_m) is the unique solution of

$$V_{m-\alpha}(y_1, \dots, y_m)(\sigma^m(a)) \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} y_{\alpha-m} \\ y_{\alpha-m+1} \\ \vdots \\ y_{\alpha-1} \end{pmatrix}. \tag{3.25}$$

The linearity of (3.1) implies that $u(t)$ has to be its solution. Moreover, it holds

$${}_a\nabla_{(q,h)}^{\alpha-j} u(t) \Big|_{t=\sigma^m(a)} = y_{\alpha-j}, \quad j = 1, 2, \dots, m, \tag{3.26}$$

hence $u(t)$ is a solution of the initial value problem (3.1)-(3.2). By Theorem 3.5, it must be $y(t) = u(t)$ for all $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ and (3.24) holds. \square

Remark 3.10. The formula (3.24) is essentially an expression of the general solution of (3.1).

4. Two-Term Equation and (q, h) -Mittag-Leffler Function

Our main interest in this section is to find eigenfunctions of the fractional operator ${}_a\nabla_{(q,h)}^\alpha$, $\alpha \in \mathbb{R}^+$. In other words, we wish to solve (3.1) in a special form

$${}_a\nabla_{(q,h)}^\alpha \mathbf{y}(t) = \lambda \mathbf{y}(t), \quad \lambda \in \mathbb{R}, \quad t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}. \quad (4.1)$$

Throughout this section, we assume that ν -regressivity condition for (4.1) is ensured, that is,

$$\lambda(\nu(t))^\alpha \neq 1. \quad (4.2)$$

Discussions on methods of solving fractional difference equations are just at the beginning. Some techniques how to explicitly solve these equations (at least in particular cases) are exhibited, for example, in [12–14], where a discrete analogue of the Laplace transform turns out to be the most developed method. In this section, we describe the technique not utilizing the transform method, but directly originating from the role which is played by the Mittag-Leffler function in the continuous fractional calculus (see, e.g., [15]). In particular, we introduce the notion of a discrete Mittag-Leffler function in a setting formed by the time scale $\tilde{\mathbb{T}}_{(q,h)}^a$ and demonstrate its significance with respect to eigenfunctions of the operator ${}_a\nabla_{(q,h)}^\alpha$. These results generalize and extend those derived in [16, 17].

We start with the power rule stated in Lemma 2.3 and perform its extension to fractional integrals and derivatives.

Proposition 4.1. *Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ and $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$. Then it holds*

$${}_a\nabla_{(q,h)}^{-\alpha} \hat{h}_\beta(t, a) = \hat{h}_{\alpha+\beta}(t, a). \quad (4.3)$$

Proof. Let $t \in \tilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ be such that $t = \sigma^n(a)$ for some $n \in \mathbb{Z}^+$. We have

$$\begin{aligned} {}_a\nabla_{(q,h)}^{-\alpha} \hat{h}_\beta(t, a) &= \sum_{k=0}^{n-1} \hat{h}_{\alpha-1}(t, \rho^{k+1}(t)) \nu(\rho^k(t)) \hat{h}_\beta(\rho^k(t), a) \\ &= \sum_{k=0}^{n-1} (\nu(t))^{\alpha-1} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_{\tilde{q}} (-1)^k \tilde{q}^{(\alpha-1)k + \binom{k+1}{2}} \tilde{q}^k \nu(t) \\ &\quad \times (\nu(\rho^k(t)))^\beta \begin{bmatrix} -\beta-1 \\ n-k-1 \end{bmatrix}_{\tilde{q}} (-1)^{n-k-1} \tilde{q}^{\beta(n-k-1) + \binom{n-k}{2}} \\ &= (\nu(t))^{\alpha+\beta} \sum_{k=0}^{n-1} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_{\tilde{q}} \begin{bmatrix} -\beta-1 \\ n-k-1 \end{bmatrix}_{\tilde{q}} (-1)^{n-1} \tilde{q}^{k^2 - k(n-1) + k\alpha + \binom{n}{2} + \beta(n-1)} \end{aligned}$$

$$\begin{aligned}
 &= (\nu(t))^{\alpha+\beta} \sum_{k=0}^{n-1} \begin{bmatrix} -\alpha \\ n-k-1 \end{bmatrix}_{\tilde{q}} \begin{bmatrix} -\beta-1 \\ k \end{bmatrix}_{\tilde{q}} \\
 &\quad \times (-1)^{n-1} \tilde{q}^{(n-k-1)^2 - (n-k-1)(n-1) + (n-k-1)\alpha + \binom{n}{2} + \beta(n-1)} \\
 &= (\nu(t))^{\alpha+\beta} \sum_{k=0}^{n-1} \begin{bmatrix} -\alpha \\ n-k-1 \end{bmatrix}_{\tilde{q}} \begin{bmatrix} -\beta-1 \\ k \end{bmatrix}_{\tilde{q}} (-1)^{n-1} \tilde{q}^{k^2 - k(n-1) - k\alpha + (\alpha+\beta)(n-1) + \binom{n}{2}} \\
 &= (\nu(t))^{\alpha+\beta} \begin{bmatrix} -\alpha - \beta - 1 \\ n-1 \end{bmatrix}_{\tilde{q}} (-1)^{n-1} \tilde{q}^{(\alpha+\beta)(n-1) + \binom{n}{2}} = \widehat{h}_{\alpha+\beta}(t, a),
 \end{aligned} \tag{4.4}$$

where we have used (2.30) on the second line and (2.17) on the last line. □

Corollary 4.2. *Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}$, where $m \in \mathbb{Z}^+$ is satisfying $m - 1 < \alpha \leq m$. Then*

$${}_a\nabla_{(q,h)}^\alpha \widehat{h}_\beta(t, a) = \begin{cases} \widehat{h}_{\beta-\alpha}(t, a), & \beta - \alpha \notin \{-1, \dots, -m\}, \\ 0, & \beta - \alpha \in \{-1, \dots, -m\}. \end{cases} \tag{4.5}$$

Proof. Proposition 4.1 implies that

$${}_a\nabla_{(q,h)}^\alpha \widehat{h}_\beta(t, a) = \nabla_{(q,h)}^m \left({}_a\nabla_{(q,h)}^{-(m-\alpha)} \widehat{h}_\beta(t, a) \right) = \nabla_{(q,h)}^m \widehat{h}_{m+\beta-\alpha}(t, a). \tag{4.6}$$

Then the statement is an immediate consequence of Lemma 2.3. □

Now we are in a position to introduce a (q, h) -discrete analogue of the Mittag-Leffler function. We recall that this function is essentially a generalized exponential function, and its two-parameter form (more convenient in the fractional calculus) can be introduced for $\mathbb{T} = \mathbb{R}$ by the series expansion

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{R}^+, t \in \mathbb{R}. \tag{4.7}$$

The fractional calculus frequently employs (4.7), because the function

$$t^{\beta-1} E_{\alpha,\beta}(\lambda t^\alpha) = \sum_{k=0}^{\infty} \lambda^k \frac{t^{\alpha k + \beta - 1}}{\Gamma(\alpha k + \beta)} \tag{4.8}$$

(a modified Mittag-Leffler function, see [15]) satisfies under special choices of β a continuous (differential) analogy of (4.1). Some extensions of the definition formula (4.7) and their utilization in special fractional calculus operators can be found in [18, 19].

Considering the discrete calculus, the form (4.8) seems to be much more convenient for discrete extensions than the form (4.7), which requires, among others, the validity of the law

of exponents. The following introduction extends the discrete Mittag-Leffler function defined and studied in [20] for the case $q = h = 1$.

Definition 4.3. Let $\alpha, \beta, \lambda \in \mathbb{R}$. We introduce the (q, h) -Mittag-Leffler function $E_{\alpha, \beta}^{s, \lambda}(t)$ by the series expansion

$$E_{\alpha, \beta}^{s, \lambda}(t) = \sum_{k=0}^{\infty} \lambda^k \widehat{h}_{\alpha k + \beta - 1}(t, s) \left(= \sum_{k=0}^{\infty} \lambda^k \frac{(t-s)_{(\tilde{q}, h)}^{(\alpha k + \beta - 1)}}{\Gamma_{\tilde{q}}(\alpha k + \beta)} \right), \quad s, t \in \widetilde{\mathbb{T}}_{(q, h)}^a, \quad t \geq s. \quad (4.9)$$

It is easy to check that the series on the right-hand side converges (absolutely) if $|\lambda|(\nu(t))^\alpha < 1$. As it might be expected, the particular (q, h) -Mittag-Leffler function

$$E_{1,1}^{a, \lambda}(t) = \prod_{k=0}^{n-1} \frac{1}{1 - \lambda \nu(\rho^k(t))}, \quad (4.10)$$

where $n \in \mathbb{Z}^+$ satisfies $t = \sigma^n(a)$, is a solution of the equation

$$\nabla_{(q, h)} y(t) = \lambda y(t), \quad t \in \widetilde{\mathbb{T}}_{(q, h)}^{\sigma(a)}, \quad (4.11)$$

that is, it is a discrete (q, h) -analogue of the exponential function.

The main properties of the (q, h) -Mittag-Leffler function are described by the following assertion.

Theorem 4.4. (i) Let $\eta \in \mathbb{R}^+$ and $t \in \widetilde{\mathbb{T}}_{(q, h)}^{\sigma(a)}$. Then

$${}_a \nabla_{(q, h)}^{-\eta} E_{\alpha, \beta}^{a, \lambda}(t) = E_{\alpha, \beta + \eta}^{a, \lambda}(t). \quad (4.12)$$

(ii) Let $\eta \in \mathbb{R}^+$, $m \in \mathbb{Z}^+$ be such that $m - 1 < \eta \leq m$ and let $\alpha k + \beta - 1 \notin \{0, -1, \dots, -m + 1\}$ for all $k \in \mathbb{Z}^+$. If $t \in \widetilde{\mathbb{T}}_{(q, h)}^{\sigma^{m+1}(a)}$, then

$${}_a \nabla_{(q, h)}^\eta E_{\alpha, \beta}^{a, \lambda}(t) = \begin{cases} E_{\alpha, \beta - \eta}^{a, \lambda}(t), & \beta - \eta \notin \{0, -1, \dots, -m + 1\}, \\ \lambda E_{\alpha, \beta - \eta + \alpha}^{a, \lambda}(t), & \beta - \eta \in \{0, -1, \dots, -m + 1\}. \end{cases} \quad (4.13)$$

Proof. The part (i) follows immediately from Proposition 4.1. Considering the part (ii), we can write

$${}_a \nabla_{(q, h)}^\eta E_{\alpha, \beta}^{a, \lambda}(t) = {}_a \nabla_{(q, h)}^\eta \sum_{k=0}^{\infty} \lambda^k \widehat{h}_{\alpha k + \beta - 1}(t, a) = \sum_{k=0}^{\infty} \lambda^k {}_a \nabla_{(q, h)}^\eta \widehat{h}_{\alpha k + \beta - 1}(t, a) \quad (4.14)$$

due to the absolute convergence property.

If $k \in \mathbb{Z}^+$, then Corollary 4.2 implies the relation

$${}_a \nabla_{(q,h)}^\eta \widehat{h}_{\alpha k + \beta - 1}(t, a) = \widehat{h}_{\alpha k + \beta - \eta - 1}(t, a) \tag{4.15}$$

due to the assumption $\alpha k + \beta - 1 \notin \{0, -1, \dots, -m + 1\}$. If $k = 0$, then two possibilities may occur. If $\beta - \eta \notin \{0, -1, \dots, -m + 1\}$, we get (4.15) with $k = 0$ which implies the validity of (4.13)₁. If $\beta - \eta \in \{0, -1, \dots, -m + 1\}$, the nabla (q, h) -fractional derivative of this term is zero and by shifting the index k , we obtain (4.13)₂. \square

Corollary 4.5. *Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m - 1 < \alpha \leq m$. Then the functions*

$$E_{\alpha,\beta}^{a,\lambda}(t), \quad \beta = \alpha - m + 1, \dots, \alpha - 1, \alpha \tag{4.16}$$

define eigenfunctions of the operator ${}_a \nabla_{(q,h)}^\alpha$ on each set $[\sigma(a), b] \cap \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, where $b \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ is satisfying $|\lambda|(\nu(b))^\alpha < 1$.

Proof. The assertion follows from Theorem 4.4 by the use of $\eta = \alpha$. \square

Our final aim is to show that any solution of (4.1) can be written as a linear combination of (q, h) -Mittag-Leffler functions (4.16).

Lemma 4.6. *Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m - 1 < \alpha \leq m$. Then*

$$W_{m-\alpha} \left(E_{\alpha,\alpha-m+1}^{a,\lambda}, E_{\alpha,\alpha-m+2}^{a,\lambda}, \dots, E_{\alpha,\alpha}^{a,\lambda} \right) (\sigma^m(a)) = \prod_{k=1}^m \frac{1}{1 - \lambda(\nu(\sigma^k(a)))^\alpha} \neq 0. \tag{4.17}$$

Proof. The case $m = 1$ is trivial. For $m \geq 2$, we can formally write $\lambda E_{\alpha,\alpha-\ell}^{a,\lambda}(t) = E_{\alpha,-\ell}^{a,\lambda}(t)$ for all $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^m(a)}$ ($\ell = 0, \dots, m - 2$). Consequently, applying Theorem 4.4, the Wronskian can be expressed as

$$W_{m-\alpha} \left(E_{\alpha,\alpha-m+1}^{a,\lambda}, E_{\alpha,\alpha-m+2}^{a,\lambda}, \dots, E_{\alpha,\alpha}^{a,\lambda} \right) (\sigma^m(a)) = \det M_m(\sigma^m(a)), \tag{4.18}$$

where

$$M_m(\sigma^m(a)) = \begin{pmatrix} E_{\alpha,1}^{a,\lambda}(\sigma^m(a)) & E_{\alpha,2}^{a,\lambda}(\sigma^m(a)) & \dots & E_{\alpha,m}^{a,\lambda}(\sigma^m(a)) \\ E_{\alpha,0}^{a,\lambda}(\sigma^m(a)) & E_{\alpha,1}^{a,\lambda}(\sigma^m(a)) & \dots & E_{\alpha,m-1}^{a,\lambda}(\sigma^m(a)) \\ \dots & \dots & \ddots & \dots \\ E_{\alpha,2-m}^{a,\lambda}(\sigma^m(a)) & E_{\alpha,3-m}^{a,\lambda}(\sigma^m(a)) & \dots & E_{\alpha,1}^{a,\lambda}(\sigma^m(a)) \end{pmatrix}. \tag{4.19}$$

Using the q -Pascal rule (2.15), we obtain the equality

$$E_{\alpha,i}^{a,\lambda}(\sigma^m(a)) - \nu(\sigma(a)) E_{\alpha,i-1}^{a,\lambda}(\sigma^m(a)) = E_{\alpha,i}^{\sigma(a),\lambda}(\sigma^m(a)), \quad i \in \mathbb{Z}, i \geq 3 - m. \tag{4.20}$$

Starting with the first row, $\binom{m}{2}$ elementary row operations of the type (4.20) transform the matrix $M_m(\sigma^m(a))$ into the matrix

$$\widehat{M}_m(\sigma^m(a)) = \begin{pmatrix} E_{\alpha,1}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) & E_{\alpha,2}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) & \dots & E_{\alpha,m}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) \\ E_{\alpha,0}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) & E_{\alpha,1}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) & \dots & E_{\alpha,m-1}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) \\ \dots & \dots & \ddots & \dots \\ E_{\alpha,2-m}^{a,\lambda}(\sigma^m(a)) & E_{\alpha,3-m}^{a,\lambda}(\sigma^m(a)) & \dots & E_{\alpha,1}^{a,\lambda}(\sigma^m(a)) \end{pmatrix} \quad (4.21)$$

with the property $\det \widehat{M}_m(\sigma^m(a)) = \det M_m(\sigma^m(a))$. By Lemma 2.3, we have

$$\begin{aligned} E_{\alpha,p}^{\sigma^i(a),\lambda}(\sigma^m(a)) - \nu(\sigma^m(a))E_{\alpha,p-1}^{\sigma^i(a),\lambda}(\sigma^m(a)) &= E_{\alpha,p}^{\sigma^i(a),\lambda}(\sigma^{m-1}(a)), \quad i = 0, \dots, m-2, \\ E_{\alpha,p}^{\sigma^i(a),\lambda}(\sigma^m(a)) - \nu(\sigma^m(a))E_{\alpha,p-1}^{\sigma^i(a),\lambda}(\sigma^m(a)) &= 0, \quad i = m-1, \end{aligned} \quad (4.22)$$

where $p \in \mathbb{Z}$, $p \geq 3 - m + i$. Starting with the last column, using $m - 1$ elementary column operations of the type (4.22), we obtain the matrix

$$\left(\begin{array}{c|ccc} E_{\alpha,1}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) & 0 & \dots & 0 \\ E_{\alpha,0}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) & \hline & \widehat{M}_{m-1}(\sigma^{m-1}(a)) & & \\ \vdots & & & \\ E_{\alpha,2-m}^{a,\lambda}(\sigma^m(a)) & & & \end{array} \right) \quad (4.23)$$

preserving the value of $\det \widehat{M}_m(\sigma^m(a))$. Since

$$E_{\alpha,1}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) = \sum_{k=0}^{\infty} \lambda^k (\nu(\sigma^m(a)))^{\alpha k} = \frac{1}{1 - \lambda(\nu(\sigma^m(a)))^\alpha}, \quad (4.24)$$

we can observe the recurrence

$$\det \widehat{M}_m(\sigma^m(a)) = \frac{1}{1 - \lambda(\nu(\sigma^m(a)))^\alpha} \det \widehat{M}_{m-1}(\sigma^{m-1}(a)), \quad (4.25)$$

which implies the assertion. □

Now we summarize the results of Theorem 3.9, Corollary 4.5, and Lemma 4.6 to obtain

Theorem 4.7. *Let $y(t)$ be any solution of (4.1) defined on $[\sigma(a), b] \cap \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, where $b \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ is satisfying $|\lambda|(\nu(b))^\alpha < 1$. Then*

$$y(t) = \sum_{j=1}^m c_j E_{\alpha, \alpha-m+j}^{a, \lambda}(t), \tag{4.26}$$

where c_1, \dots, c_m are real constants.

We conclude this paper by the illustrating example.

Example 4.8. Consider the initial value problem

$$\begin{aligned} {}_a \nabla_{(q,h)}^\alpha y(t) &= \lambda y(t), \quad \sigma^3(a) \leq t \leq \sigma^n(a), \quad 1 < \alpha \leq 2, \\ {}_a \nabla_{(q,h)}^{\alpha-1} y(t) \Big|_{t=\sigma^2(a)} &= y_{\alpha-1}, \\ {}_a \nabla_{(q,h)}^{\alpha-2} y(t) \Big|_{t=\sigma^2(a)} &= y_{\alpha-2}, \end{aligned} \tag{4.27}$$

where n is a positive integer given by the condition $|\lambda| \nu(\sigma^n(a))^\alpha < 1$. By Theorem 4.7, its solution can be expressed as a linear combination

$$y(t) = c_1 E_{\alpha, \alpha-1}^{a, \lambda}(t) + c_2 E_{\alpha, \alpha}^{a, \lambda}(t). \tag{4.28}$$

The constants c_1, c_2 can be determined from the system

$$V_{2-\alpha} \left(E_{\alpha, \alpha-1}^{a, \lambda}, E_{\alpha, \alpha}^{a, \lambda} \right) (\sigma^2(a)) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_{\alpha-2} \\ y_{\alpha-1} \end{pmatrix} \tag{4.29}$$

with the matrix elements

$$\begin{aligned} v_{11} = v_{22} &= \frac{[1]_q + ([\alpha]_q - [1]_q) \lambda \nu(\sigma(a))^\alpha}{(1 - \lambda \nu(\sigma(a))^\alpha) (1 - \lambda \nu(\sigma^2(a))^\alpha)}, \\ v_{12} &= \frac{[2]_q \nu(\sigma(a)) + ([\alpha]_q - [2]_q) \lambda \nu(\sigma(a))^{\alpha+1}}{(1 - \lambda \nu(\sigma(a))^\alpha) (1 - \lambda \nu(\sigma^2(a))^\alpha)}, \\ v_{21} &= \frac{[\alpha]_q \lambda \nu(\sigma(a))^{\alpha-1}}{(1 - \lambda \nu(\sigma(a))^\alpha) (1 - \lambda \nu(\sigma^2(a))^\alpha)}. \end{aligned} \tag{4.30}$$

By Lemma 4.6, the matrix $V_{2-\alpha} (E_{\alpha, \alpha-1}^{a, \lambda}, E_{\alpha, \alpha}^{a, \lambda}) (\sigma^2(a))$ has a nonzero determinant, hence applying the Cramer rule, we get

$$\begin{aligned} c_1 &= \frac{y_{\alpha-2} v_{22} - y_{\alpha-1} v_{12}}{W_{2-\alpha} \left(E_{\alpha, \alpha-1}^{a, \lambda}, E_{\alpha, \alpha}^{a, \lambda} \right) (\sigma^2(a))}, \\ c_2 &= \frac{y_{\alpha-1} v_{11} - y_{\alpha-2} v_{21}}{W_{2-\alpha} \left(E_{\alpha, \alpha-1}^{a, \lambda}, E_{\alpha, \alpha}^{a, \lambda} \right) (\sigma^2(a))}. \end{aligned} \tag{4.31}$$

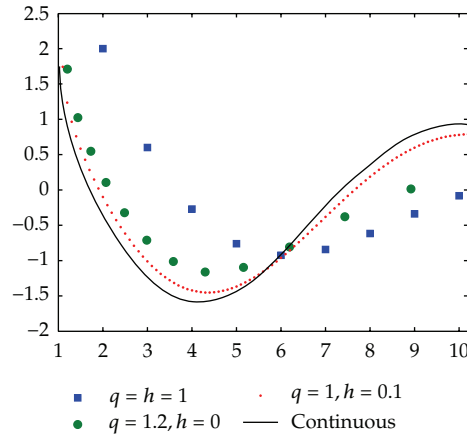


Figure 1: $\alpha = 1.8, a = 1, \lambda = -1/3, y_{\alpha-1} = -1, y_{\alpha-2} = 1$.

Now we make a particular choice of the parameters $\alpha, a, \lambda, y_{\alpha-1}$ and $y_{\alpha-2}$ and consider the initial value problem in the form

$$\begin{aligned}
 {}_1\nabla_{(q,h)}^{1.8} y(t) &= -\frac{1}{3} y(t), \quad \sigma^3(1) \leq t \leq \sigma^n(1), \\
 {}_1\nabla_{(q,h)}^{0.8} y(t) \Big|_{t=\sigma^2(1)} &= -1, \\
 {}_1\nabla_{(q,h)}^{-0.2} y(t) \Big|_{t=\sigma^2(1)} &= 1,
 \end{aligned} \tag{4.32}$$

where n is a positive integer satisfying $\nu(\sigma^n(1)) < 3^{5/9}$. If we take the time scale of integers (the case $q = h = 1$), then the solution $y(t)$ of the corresponding initial value problem takes the form

$$y(t) = \frac{14}{5} \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k \frac{\prod_{j=1}^{t-2} (j + 1.8k - 0.2)}{(t-2)!} - \frac{2}{15} \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k \frac{\prod_{j=1}^{t-2} (j + 1.8k + 0.8)}{(t-2)!}, \quad t = 2, 3, \dots \tag{4.33}$$

Similarly we can determine $y(t)$ for other choices of q and h . For comparative reasons, Figure 1 depicts (in addition to the above case $q = h = 1$) the solution $y(t)$ under particular choices $q = 1.2, h = 0$ (the pure q -calculus), $q = 1, h = 0.1$ (the pure h -calculus) and also the solution of the corresponding continuous (differential) initial value problem.

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