Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 543035, 13 pages
doi:10.1155/2011/543035

Research Article

Uniqueness of Positive Solutions for a Class of Fourth-Order Boundary Value Problems

J. Caballero, J. Harjani, and K. Sadarangani

Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Campus de Tafira Baja, 35017 Las Palmas de Gran Canaria, Spain

Correspondence should be addressed to K. Sadarangani, ksadaran@dma.ulpgc.es

Received 31 December 2010; Accepted 23 March 2011

Academic Editor: Yuri V. Rogovchenko

The purpose of this paper is to investigate the existence and uniqueness of positive solutions for the following fourth-order boundary value problem: $y^{(4)}(t) = f(t, y(t))$, $t \in [0,1]$, $y(0) = y(1) = y'(0) = y'(1) = 0$. Moreover, under certain assumptions, we will prove that the above boundary value problem has a unique symmetric positive solution. Finally, we present some examples and we compare our results with the ones obtained in recent papers. Our analysis relies on a fixed point theorem in partially ordered metric spaces.

1. Introduction

The purpose of this paper is to consider the existence and uniqueness of positive solutions for the following fourth-order two-point boundary value problem:

$$\begin{gathered} y^{(4)}(t) = f(t, y(t)), \quad t \in [0,1], \\ y(0) = y(1) = y'(0) = y'(1) = 0, \end{gathered} \tag{1.1}$$

which describes the bending of an elastic beam clamped at both endpoints.

There have been extensive studies on fourth-order boundary value problems with diverse boundary conditions. Some of the main tools of nonlinear analysis devoted to the study of this type of problems are, among others, lower and upper solutions [1–4], monotone iterative technique [5–7], Krasnoselskii fixed point theorem [8], fixed point index [9–11], Leray-Schauder degree [12, 13], and bifurcation theory [14–16].

2. Background

In this section, we present some basic facts which are necessary for our results.

In our study, we will use a fixed point theorem in partially ordered metric spaces which appears in [17].

Let $\mathcal{M}$ denote the class of those functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \quad \text{implies } t_n \to 0. \tag{2.1}$$

Now, we recall the above mentioned fixed point theorem.

Theorem 2.1 (see 1, Theorem 2.1). *Let $(X,\leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq Tx_0$. Suppose that there exists $\beta \in \mathcal{M}$ such that*

$$d(Tx,Ty) \leq \beta(d(x,y)) \cdot d(x,y), \quad \text{for any } x,y \in X \text{ with } x \geq y. \tag{2.2}$$

Assume that either T is continuous or X is such that

$$\text{if } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \longrightarrow x, \text{ then } x_n \leq x \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Besides, suppose that

$$\text{for each } x,y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{2.4}$$

Then T has a unique fixed point.

In our considerations, we will work with a subset of the classical Banach space $C[0,1]$. This space will be considered with the standard metric

$$d(x,y) = \sup_{0\leq t\leq 1} |x(t) - y(t)|. \tag{2.5}$$

This space can be equipped with a partial order given by

$$x,y \in C[0,1], \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0,1]. \tag{2.6}$$

In [18], it is proved that $(C[0,1],\leq)$ with the above mentioned metric satisfies condition (2.3) of Theorem 2.1. Moreover, for $x,y \in C[0,1]$, as the function $\max(x,y) \in C[0,1]$, $(C[0,1],\leq)$ satisfies condition (2.4).

On the other hand, the boundary value problem (1.1) can be rewritten as the integral equation (see, e.g., [19])

$$y(t) = \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s, \quad \text{for } t \in [0,1], \tag{2.7}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{1}{6}\begin{cases} t^2(1-s)^2[(s-t)+2(1-t)s], & 0 \le t \le s \le 1, \\ s^2(1-t)^2[(t-s)+2(1-s)t], & 0 \le s \le t \le 1. \end{cases} \tag{2.8}$$

Note that $G(t,s)$ satisfies the following properties:

(i) $G(t,s)$ is a continuous function on $[0,1] \times [0,1]$,

(ii) $G(0,s) = G(1,s) = 0$, for $s \in [0,1]$,

(iii) $G(t,s) \ge 0$, for $t,s \in [0,1]$.

3. Main Results

Our starting point in this section is to present the class of functions $\mathcal{A}$ which we use later. By $\mathcal{A}$ we denote the class of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(i) ϕ is nondecreasing,

(ii) for any $x > 0$, $\phi(x) < x$,

(iii) $\beta(x) = \phi(x)/x \in \mathcal{M}$.

Examples of functions in $\mathcal{A}$ are $\phi(x) = \mu x$ with $0 \le \mu < 1$, $\phi(x) = x/(1+x)$ and $\phi(x) = \ln(1+x)$. In the sequel, we formulate our main result.

Theorem 3.1. *Consider problem* (1.1) *assuming the following hypotheses:*

(a) $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous,*

(b) $f(t,y)$ *is nondecreasing with respect to the second variable, for each* $t \in [0,1]$,

(c) *suppose that there exists* $0 < \alpha \le 384$, *such that, for* $x, y \in [0,\infty)$ *with* $y \ge x$,

$$f(t,y) - f(t,x) \le \alpha\phi(y-x), \quad \text{with } \phi \in \mathcal{A}. \tag{3.1}$$

Then, problem (1.1) *has a unique nonnegative solution.*

Proof. Consider the cone

$$P = \{x \in C[0,1] : x \ge 0\}. \tag{3.2}$$

Obviously, (P,d) with $d(x,y) = \sup\{|x(t) - y(t)| : t \in [0,1]\}$ is a complete metric space satisfying condition (2.3) and condition (2.4) of Theorem 2.1.

Consider the operator defined by

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\mathrm{d}s, \quad \text{for } x \in P, \tag{3.3}$$

where $G(t,s)$ is the Green's function defined in Section 2.

It is clear that T applies the cone P into itself since $f(t,x)$ and $G(t,s)$ are nonnegative continuous functions.

Now, we check that assumptions in Theorems 2.1 are satisfied.

Firstly, the operator T is nondecreasing.

Indeed, since f is nondecreasing with respect to the second variable, for $u, v \in P$, $u \geq v$ and $t \in [0,1]$, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \\ &\geq \int_0^1 G(t,s)f(s,v(s))\mathrm{d}s \\ &= (Tv)(t).\end{aligned} \tag{3.4}$$

On the other hand, a straightforward calculation gives us

$$\begin{gathered}\int_0^1 G(t,s)\mathrm{d}s = \int_0^t G(t,s)\mathrm{d}s + \int_t^1 G(t,s)\mathrm{d}s = \frac{t^2}{24} - \frac{t^3}{12} + \frac{t^4}{24}, \\ \max_{0\leq t\leq 1} \int_0^1 G(t,s)\mathrm{d}s = \max_{0\leq t\leq 1}\left(\frac{t^2}{24} - \frac{t^3}{12} + \frac{t^4}{24}\right) = \frac{1}{384}.\end{gathered} \tag{3.5}$$

Taking into account this fact and our hypotheses, for $u, v \in P$ and $u > v$, we can obtain the following estimate:

$$\begin{aligned}d(Tu,Tv) &= \sup_{0\leq t\leq 1}\Big|(Tu)(t) - (Tv)(t)\Big| \\ &= \sup_{0\leq t\leq 1}((Tu)(t) - (Tv)(t)) \\ &= \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\big(f(s,u(s)) - f(s,v(s))\big)\mathrm{d}s \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\alpha\phi(u(s) - v(s))\mathrm{d}s \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\alpha\phi(d(u,v))\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&= \alpha\phi(d(u,v)) \sup_{0\le t\le 1}\int_0^1 G(t,s)\mathrm{d}s \\
&= \alpha\phi(d(u,v)) \cdot \frac{1}{384} \\
&\le \phi(d(u,v)) \\
&= \frac{\phi(d(u,v))}{d(u,v)} \cdot d(u,v).
\end{aligned} \tag{3.6}$$

This gives us, for $u, v \in P$ and $u > v$,

$$d(Tu, Tv) \le \beta(d(u,v)) \cdot d(u,v), \tag{3.7}$$

where $\beta(x) = \phi(x)/x \in \mathcal{M}$.

Obviously, the last inequality is satisfied for $u = v$.

Therefore, the contractive condition appearing in Theorem 2.1 is satisfied for $u \ge v$. Besides, as f and G are nonnegative functions,

$$T0 = \int_0^1 G(t,s) f(s,0)\mathrm{d}s \ge 0. \tag{3.8}$$

Finally, Theorem 2.1 tells us that T has a unique fixed point in P, and this means that problem (1.1) has a unique nonnegative solution.

This finishes the proof. □

Now, we present a sufficient condition for the existence and uniqueness of positive solutions for our problem (1.1) (positive solution means $x(t) > 0$, for $t \in (0,1)$). The proof of the following theorem is similar to the proof of Theorem 3.6 of [8]. We present a proof for completeness.

Theorem 3.2. *Under assumptions of Theorem 3.1 and suppose that $f(t_0, 0) \ne 0$ for certain $t_0 \in [0,1]$, problem* (1.1) *has a unique positive solution.*

Proof. Consider the nonnegative solution $x(t)$ given by Theorem 3.1 of problem (1.1).

Notice that this solution satisfies

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\mathrm{d}s. \tag{3.9}$$

Now, we will prove that x is a positive solution.

In contrary case, suppose that there exists $0 < t^* < 1$ such that $x(t^*) = 0$ and, consequently,

$$x(t^*) = \int_0^1 G(t^*, s) f(s, x(s)) \mathrm{d}s = 0. \tag{3.10}$$

Since $x \geq 0$, f is nondecreasing with respect to the second variable and $G(t, s) \geq 0$, we have

$$0 = x(t^*) = \int_0^1 G(t^*, s) f(s, x(\mathrm{s})) \mathrm{d}s \geq \int_0^1 G(t^*, s) f(s, 0) \mathrm{d}s \geq 0, \tag{3.11}$$

and this gives us

$$\int_0^1 G(t^*, s) f(s, 0) \mathrm{d}s = 0. \tag{3.12}$$

This fact and the nonnegative character of $G(t, s)$ and $f(t, x)$ imply

$$G(t^*, s) \cdot f(s, 0) = 0 \quad \mathrm{a} \cdot \mathrm{e}\ (s). \tag{3.13}$$

As $G(t^*, s) \neq 0 \mathrm{a} \cdot \mathrm{e}\ (s)$, because $G(t^*, s)$ is given by a polynomial, we obtain

$$f(s, 0) = 0 \quad \mathrm{a} \cdot \mathrm{e}\ (s). \tag{3.14}$$

On the other hand, as $f(t_0, 0) \neq 0$ for certain $t_0 \in [0, 1]$ and $f(t_0, x) \geq 0$, we have that $f(t_0, 0) > 0$.

The continuity of f gives us the existence of a set $A \subset [0, 1]$ with $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure, satisfying that $f(t, 0) > 0$ for any $t \in A$. This contradicts (3.14).

Therefore, $x(t) > 0$ for $t \in (0, 1)$.

This finishes the proof. □

Now, we present an example which illustrates our results.

Example 3.3. Consider the nonlinear fourth-order two-point boundary value problem

$$\begin{gathered} y^{(4)}(t) = c + \lambda \arctan\bigl(y(t)\bigr), \quad t \in (0, 1),\ c, \lambda > 0, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered} \tag{3.15}$$

In this case, $f(t, y) = c + \lambda \arctan y$. It is easily seen that $f(t, y)$ satisfies (a) and (b) of Theorem 3.1.

In order to prove that $f(t,y)$ satisfies (c) of Theorem 3.1, previously, we will prove that the function $\phi : [0,\infty) \to [0,\infty)$, defined by $\phi(x) = \arctan x$, satisfies

$$\phi(u) - \phi(v) \leq \phi(u - v) \quad \text{for } u \geq v. \tag{3.16}$$

In fact, put $\phi(u) = \arctan u = \alpha$ and $\phi(v) = \arctan v = \beta$ (notice that, as $u \geq v$ and ϕ is nondecreasing, $\alpha \geq \beta$). Then, from

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}, \tag{3.17}$$

as $\alpha, \beta \in [0, \pi/2)$, then $\tan\alpha, \tan\beta \in [0,\infty)$, we obtain

$$\tan(\alpha - \beta) \leq \tan\alpha - \tan\beta. \tag{3.18}$$

Applying ϕ to this inequality and taking into account the nondecreasing character of ϕ, we have

$$\alpha - \beta \leq \arctan(\tan\alpha - \tan\beta) \tag{3.19}$$

or, equivalently,

$$\phi(u) - \phi(v) = \arctan u - \arctan v \leq \arctan(u - v) = \phi(u - v). \tag{3.20}$$

This proves our claim.

In the sequel, we prove that $f(t,y)$ satisfies assumption (c) of Theorem 3.1.

In fact, for $y \geq x$ and $t \in [0,1]$, we can obtain

$$\begin{aligned} f(t,y) - f(t,x) &= \lambda(\arctan y - \arctan x) \\ &\leq \lambda \arctan(y - x). \end{aligned} \tag{3.21}$$

Now, we will prove that $\phi(x) = \arctan x$ belongs to $\mathcal{A}$. In fact, obviously ϕ takes $[0,\infty)$ into itself and, as $\phi'(x) = 1/(1+x^2)$, ϕ is nondecreasing. Besides, as the derivative of $\varphi(x) = x - \phi(x)$ is $\varphi'(x) = 1 - 1/(1+x^2) > 0$ for $x > 0$, φ is strictly increasing, and, consequently, $\phi(x) < x$ for $x > 0$ (notice that $\varphi(0) = 0$). Notice that if $\beta(x) = \phi(x)/x = \arctan x/x$ and $\beta(t_n) \to 1$, then (t_n) is a bounded sequence because, in contrary case, $t_n \to \infty$ and, thus, $\beta(t_n) \to 0$. Suppose that $t_n \nrightarrow 0$. Then, we can find $\epsilon > 0$ such that, for each $n \in \mathbb{N}$, there exists $p_n \geq n$ with $t_{p_n} \geq \epsilon$. The bounded character of (t_n) gives us the existence of a subsequence (t_{k_n}) of (t_{p_n}) with (t_{k_n}) convergent. Suppose that $t_{k_n} \to a$. From $\beta(t_n) \to 1$, we obtain $\arctan t_{k_n}/t_{k_n} \to \arctan a/a = 1$ and, as the unique solution of $\arctan x = x$ is $x_0 = 0$, we obtain $a = 0$. Thus, $t_{k_n} \to 0$, and this contradicts the fact that $t_{k_n} \geq \epsilon$ for any $n \in \mathbb{N}$. Therefore, $t_n \to 0$. This proves that $f(t,y)$ satisfies assumption (c) of Theorem 3.1. Finally, as $f(t,0) = c > 0$, Problem (3.15) has a unique positive solution for $0 < \lambda \leq 384$ by Theorems 3.1 and 3.2.

Remark 3.4. In Theorem 3.2, the condition $f(t_0, 0) \neq 0$ for certain $t_0 \in [0,1]$ seems to be a strong condition in order to obtain a positive solution for Problem (1.1), but when the solution is unique, we will see that this condition is very adjusted one. More precisely, under assumption that Problem (1.1) has a unique nonnegative solution $x(t)$, then

$$f(t,0) = 0 \quad \text{for } t \in [0,1] \text{ iff } x(t) \equiv 0. \tag{3.22}$$

In fact, if $f(t,0) = 0$ for $t \in [0,1]$, then it is easily seen that the zero function satisfies Problem (1.1) and the uniqueness of solution gives us $x(t) \equiv 0$.

The other implication is obvious since if the zero function is solution of Problem (1.1), then $0 = f(t,0)$ for any $t \in [0,1]$.

Remark 3.5. Notice that assumptions in Theorem 3.1 are invariant by continuous perturbations. More precisely, if $f(t,0) = 0$ for any $t \in [0,1]$ and f satisfies (a), (b), and (c) of Theorem 3.1, then $g(t,x) = a(t) + f(t,x)$, with $a : [0,1] \to [0,\infty)$ continuous and $a \neq 0$, satisfies assumptions of Theorem 3.2, and this means that the following boundary value problem

$$\begin{gathered} y^{(4)}(t) = g(t, y(t)), \quad t \in [0,1], \\ y(0) = y(1) = y'(0) = y'(1) = 0, \end{gathered} \tag{3.23}$$

has a unique positive solution.

4. Some Remarks

In this section, we compare our results with the ones obtained in recent papers. Recently, in [19], the authors present as main result the following theorem.

Theorem 4.1 (Theorem 3.1 of [19]). *Suppose that*

- $(H1)$ $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous,*
- $(H2)$ $f(t,y)$ *is nondecreasing in* y*, for each* $t \in [0,1]$*,*
- $(H3)$ $f(t,y) = f(1-t,y)$ *for each* $(t,y) \in [0,1] \times [0,\infty)$.

Moreover, suppose that there exist positive numbers $a > b$ *such that*

$$\max_{0 \le t \le 1} f(t,a) \le a \cdot A, \qquad \min_{1/4 \le t \le 3/4} f\left(t, \frac{b}{16}\right) \ge b \cdot B, \tag{4.1}$$

where

$$A = \left(\max_{0 \le t \le 1} \int_0^1 G(t,s)\mathrm{d}s \right)^{-1}, \qquad B = \left(\max_{0 \le t \le 1} \int_{1/4}^{3/4} G(t,s)\mathrm{d}s \right)^{-1}, \tag{4.2}$$

with $G(t,s)$ being the Green's function defined in Section 2.Then, Problem (1.1) *has at least one symmetric positive solution $y^* \in C[0,1]$ such that $b \le \|y^*\| \le a$ and, moreover, $y^* = \lim_{k\to\infty} T^k y_0$ in the uniform norm, where T is the operator defined by*

$$(Tx)(t) = \int_0^1 G(t,s) f(s,x(s))\mathrm{d}s, \quad \textit{for } x \in C[0,1] \tag{4.3}$$

and y_0 is the function given by $y_0(t) = b \cdot q(t)$, for $t \in [0,1]$, with $q(t) = \min(t^2,(1-t)^2)$, for $t \in [0,1]$ (symmetric solution means a solution $y(t)$ satisfying $y(t) = y(1-t)$, for $t \in [0,1]$).

In what follows, we present a parallel result to Theorem 3.2 where we obtain uniqueness of a symmetric positive solution of Problem (1.1).

Theorem 4.2. *Adding assumption $(H3)$ of Theorem 4.1 to the hypotheses of Theorem 3.2, one obtains a unique symmetric positive solution of Problem* (1.1)*.*

Proof. As in the proof of Theorem 3.1, instead of P, we consider the following set K

$$K = \{x \in C[0,1] : x \ge 0 \ \text{ and } x \text{ is symmetric}\}. \tag{4.4}$$

It is easily seen that K is a closed subset of $C[0,1]$. Thus, (K,d), where d is the induced metric given by

$$d(x,y) = \sup_{0\le t\le 1} |x(t) - y(t)|, \quad \text{for } x,y \in K, \tag{4.5}$$

is a complete metric space.

Moreover, K with the induced order by $(C[0,1],\le)$ satisfies condition (2.3) of Theorem 2.1, and it is easily proved that the function $\max(x,y) \in K$, for $x,y \in K$ and, consequently, $(K,\le)$, satisfies condition (2.4) of Theorem 2.1.

Now, as in Theorem 2.1, we consider the operator defined by

$$(Tx)(t) = \int_0^1 G(t,s) f(s,x(s))\mathrm{d}s, \quad \text{for } x \in K. \tag{4.6}$$

In the sequel, we prove that, under our assumptions, T applies K into itself.

In fact, suppose that x is symmetric, then for $t \in [0,1]$, we have

$$(Tx)(1-t) = \int_0^1 G(1-t,s) f(s,x(s))\mathrm{d}s. \tag{4.7}$$

Making the change of variables $s = 1 - u$, we obtain

$$\begin{aligned}(Tx)(1-t) &= -\int_1^0 G(1-t,1-u)f(1-u,x(1-u))\mathrm{d}u \\ &= \int_0^1 G(1-t,1-u)f(1-u,x(1-u))\mathrm{d}u.\end{aligned} \tag{4.8}$$

Now, it is easily seen that $G(t,s) = G(1-t,1-s)$ for $t,s \in [0,1]$ and taking into account assumption $(H3)$ of Theorem 4.1 and the symmetric character of x, we have

$$\begin{aligned}(Tx)(1-t) &= \int_0^1 G(t,u)f(u,x(1-u))\mathrm{d}u \\ &= \int_0^1 G(t,u)f(u,x(u))\mathrm{d}u \\ &= (Tx)(t).\end{aligned} \tag{4.9}$$

The rest of the proof follows the lines of Theorems 3.1 and 3.2.

This finishes the proof. □

Now, we present an example which illustrates Theorem 4.2.

Example 4.3. Consider the following problem

$$\begin{gathered} y^{(4)}(t) = c + \lambda \sin(\pi t)\arctan\bigl(y(t)\bigr), \quad t \in (0,1),\ c,\lambda > 0, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered} \tag{4.10}$$

In this case, $f(t,y) = c + \lambda \sin(\pi t) \arctan y$. It is easily checked that $f(t,y)$ satisfies (a) and (b) of Theorem 3.1 and $f(t,y) = f(1-t,y)$, for $(t,y) \in [0,1] \times [0,\infty)$.

On the other hand, taking into account Example 3.3, we can obtain, for $y \geq x$ and $t \in [0,1]$,

$$\begin{aligned} f(t,y) - f(t,x) &= \lambda \sin \pi t\bigl[\arctan y - \arctan x\bigr] \\ &\leq \lambda \sin \pi t\bigl[\arctan(y-x)\bigr] \\ &\leq \lambda \arctan(y-x). \end{aligned} \tag{4.11}$$

Finally, as it is proved in Example 3.3, $\phi(x) = \arctan x$ belongs to $\mathcal{A}$. Therefore, Theorem 4.2 tells us that Problem (4.10) has a unique symmetric positive solution for $0 < \lambda \leq 384$. In what follows, we prove that Problem (4.10) can be treated using Theorem 4.1. In fact, in this case, $f(t,y) = c + \lambda \sin(\pi t) \arctan y$. Moreover, $A = 384$ (see proof of Theorem 3.1);

it can be proved that $B = 531.61$. As we have seen in Example 4.3, $f(t, y)$ satisfies assumptions $(H1)$, $(H2)$, and $(H3)$ of Theorem 4.1. Moreover,

$$\max_{0 \le t \le 1} f(t, a) = f\left(\frac{1}{2}, a\right) = c + \lambda \arctan a,$$
$$\min_{1/4 \le t \le 3/4} f\left(t, \frac{b}{16}\right) = f\left(\frac{1}{4}, \frac{b}{16}\right) = c + \lambda \sin\frac{\pi}{4} \arctan\left(\frac{b}{16}\right) = c + \lambda \frac{\sqrt{2}}{2} \arctan\left(\frac{b}{16}\right). \tag{4.12}$$

Consider the function $\varphi(a) = 384 \cdot a - (c + \lambda \arctan a)$, with $0 < \lambda \le 384$ and $a \in [0, \infty)$. Obviously, $\varphi(0) = -c < 0$ and, as $\lim_{a \to \infty} \varphi(a) = \infty$, we can find $a_0 > 0$ such that $\varphi(a_0) > 0$. This means that

$$c + \lambda \arctan a_0 \le 384 a_0. \tag{4.13}$$

On the other hand, we consider the function $\psi(b) = c + \lambda(\sqrt{2}/2)\arctan(b/16) - 531.61 \cdot b$, with $0 < \lambda \le 384$ and $b \in [0, \infty)$.

Then, as $\psi(0) = c > 0$ and ψ is a continuous function, we can find b_0 such that

$$\min_{1/4 \le t \le 3/4} f\left(t, \frac{b_0}{16}\right) = c + \lambda \frac{\sqrt{2}}{2} \arctan\left(\frac{b_0}{16}\right) \ge b_0 \cdot 531.61. \tag{4.14}$$

Therefore, Problem (4.10) can be treated using Theorem 4.1, and we obtain the existence of a symmetric positive solution.

Our main contribution is the uniqueness of the solution.

In what follows, we present the following example which can be treated by Theorem 4.2 and Theorem 4.1 cannot be used.

Example 4.4. Consider the following problem which is a variant of Example 4.3:

$$y^{(4)}(t) = c(t) + \lambda \sin(\pi t) \arctan(y(t)), \quad t \in (0,1), \ \lambda > 0$$
$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{4.15}$$

where $c(t)$ is a symmetric positive function satisfying $c(1/4) = 0$, for example,

$$c(t) = \begin{cases} 1 - 4t, & 0 \le t \le \dfrac{1}{4}, \\ 0, & \dfrac{1}{4} \le t \le \dfrac{3}{4} \\ 4t - 3, & \dfrac{3}{4} \le t \le 1. \end{cases} \tag{4.16}$$

In this case, $f(t, y) = c(t) + \lambda \sin(\pi t) \arctan(y(t))$. Taking into account Example 4.3, it is easily proved that $f(t, y)$ satisfies assumptions of Theorem 4.2, and, consequently, Problem (4.15) has a unique symmetric positive solution for $0 < \lambda \le 384$.

Now, we prove that $f(t,y)$ does not satisfy assumptions of Theorem 4.1 and, consequently, Problem (4.15) cannot be treated using this theorem. In fact, in this case (notice that $c(1/4) = 0$),

$$\min_{1/4\le t\le 3/4} f\left(t, \frac{b}{16}\right) = f\left(\frac{1}{4}, \frac{b}{16}\right) = \lambda \sin\frac{\pi}{4} \arctan\left(\frac{b}{16}\right) = \lambda\frac{\sqrt{2}}{2} \arctan\left(\frac{b}{16}\right), \tag{4.17}$$

and we cannot find a positive number b_0 such that

$$\lambda\frac{\sqrt{2}}{2} \arctan\left(\frac{b_0}{16}\right) \ge b_0 \cdot 531.61, \quad \text{for } 0 < \lambda \le 384. \tag{4.18}$$

This proves that Problem (4.15) cannot be treated by Theorem 4.1.

Now, we compare our results with the ones obtained in [14]. In [14], the author studies positive solutions of the problem

$$\begin{gathered} u^{(iv)}(x) = \lambda f(u(x)), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{4.19}$$

using theory of bifurcation.

His main result works with functions $f(u)$ satisfying

(i) $f(u) > 0$, for $u \le 0$,

(ii) $\lim_{u\to\infty} f(u)/u = \infty$,

(iii) $f'(0) \ge 0$,

(iv) $f''(u) > 0$, for $u > 0$,

and the author proves that there exists a critical λ_0 such that Problem (4.19) has exactly two, exactly one, or no symmetric positive solution depending on whether $0 < \lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$.

Our Example 3.3 cannot be treated by the results of [14], because, in this case, $f(u) = c + \lambda \arctan u$ and f does not satisfy assumptions (ii) and (iv) above mentioned.

Acknoledgment

This paper is dedicated to Professor Antonio Martin_on on the occasion of his 60th birth-day. This research was partially supported by "Ministerio de Educaci_on y Ciencia", Project MTM 2007=65706

References

[1] A. Cabada, J. A. Cid, and L. Sánchez, "Positivity and lower and upper solutions for fourth order boundary value problems," *Nonlinear Analysis*, vol. 67, no. 5, pp. 1599–1612, 2007.

[2] J. Ehme, P. W. Eloe, and J. Henderson, "Upper and lower solution methods for fully nonlinear boundary value problems," *Journal of Differential Equations*, vol. 180, no. 1, pp. 51–64, 2002.

[3] D. Franco, D. O'Regan, and J. Perán, "Fourth-order problems with nonlinear boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 174, no. 2, pp. 315–327, 2005.

[4] F. Minhós, T. Gyulov, and A. I. Santos, "Existence and location result for a fourth order boundary value problem," *Discrete and Continuous Dynamical System Suplement*, vol. 2005, pp. 662–671, 2005.

[5] Z. Bai, "The method of lower and upper solutions for a bending of an elastic beam equation," *Journal of Mathematical Analysis and Applications*, vol. 248, no. 1, pp. 195–202, 2000.

[6] Y. Guo and Y. Gao, "The method of upper and lower solutions for a Lidstone boundary value problem," *Czechoslovak Mathematical Journal*, vol. 55, no. 3, pp. 639–652, 2005.

[7] P. Habets and L. Sánchez, "A monotone method for fourth order boundary value problems involving a factorizable linear operator," *Portugaliae Mathematica*, vol. 64, no. 3, pp. 255–279, 2007.

[8] J. Caballero, J. Harjani, and K. Sadarangani, "Existence and uniqueness of positive and non-decreasing solutions for a class of singular fractional boundary value problem," *Boundary Value Problems*, vol. 2009, Article ID 421310, 2009.

[9] Z. Bai and H. Wang, "On positive solutions of some nonlinear fourth order beam equations," *Journal of Mathematical Analysis and Applications*, vol. 270, no. 2, pp. 357–368, 2002.

[10] X. L. Liu and W. T. Li, "Existence and multiplicity of solutions for fourth-order boundary value problems with parameters," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 1, pp. 362–375, 2007.

[11] J. R. L. Webb, G. Infante, and D. Franco, "Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions," *Proceedings of the Royal Society of Edinburgh*, vol. 138, no. 2, pp. 427–446, 2008.

[12] C. P. Gupta, "Existence and uniqueness theorems for some fourth order fully quasilinear boundary value problems," *Applicable Analysis*, vol. 36, no. 3-4, pp. 157–169, 1990.

[13] G. E. Hernández and R. Manasevich, "Existence and multiplicity of solutions of a fourth order equation," *Applicable Analysis*, vol. 54, no. 3-4, pp. 237–250, 1994.

[14] P. Korman, "Uniqueness and exact multiplicity of solutions for a class of fourth-order semilinear problems," *Proceedings of the Royal Society of Edinburgh*, vol. 134, no. 1, pp. 179–190, 2004.

[15] R. Ma, "Existence of positive solutions of a fourth-order boundary value problem," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 1219–1231, 2005.

[16] B. P. Rynne, "Infinitely many solutions of superlinear fourth order boundary value problems," *Topological Methods in Nonlinear Analysis*, vol. 19, no. 2, pp. 303–312, 2002.

[17] A. Amini-Harandi and H. Emami, "A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations," *Nonlinear Analysis*, vol. 72, no. 5, pp. 2238–2242, 2010.

[18] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[19] M. Pei and S. K. Chang, "Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem," *Mathematical and Computer Modelling*, vol. 51, no. 9-10, pp. 1260–1267, 2010.