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Research Article

On a General Contractive Condition for Cyclic Self-Mappings

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This paper is concerned with $p(\geq 2)$-cyclic self-mappings $T : \bigcup_{i\in\overline{p}} A_i \rightarrow \bigcup_{i\in\overline{p}} A_i$ in a metric space (X, d), with $A_i \subset X$, $T(A_i) \subseteq A_{i+1}$ for $i = 1, 2, \ldots, p$, under a general contractive condition which includes as particular cases several of the existing ones in the literature. The existence and uniqueness of fixed points and best proximity points is discussed as well as the convergence to them of the iterates generated by the self-mapping from given initial points.

1. Introduction

There are exhaustive results about fixed point theory concerning the use of general contractive conditions in Banach spaces or in complete metric spaces and in partially ordered metric spaces which include as particular cases previous ones in the background literature. See, for instance, [1, 2] and references therein. On the other hand, important attention is being paid to the study of fixed points and best proximity points of $(p \geq 2)$-cyclic contractive mappings and p-cyclic Meir-Keeler contractive mappings, [3–7]. Generally speaking, cyclic contract self-mappings $T : \bigcup_{i\in\overline{p}} A_i \rightarrow \bigcup_{i\in\overline{p}} A_i$ on the union of p nonempty closed convex subsets A_i of a complete metric space (X, d), subject to $T(A_i) \subseteq A_{i+1}$, have a unique fixed point located in the intersection of such p subsets if such subsets intersect, [3, 4]. If the p-subsets are disjoint convex closed nonempty subsets of a uniformly convex Banach space, then there is a unique best proximity point at each of the p subsets. The above properties also hold for cyclic Meir-Keeler contractions, [5–7]. In this paper, a contractive condition for p-cyclic self-mapping on the union of p subsets of a metric space which includes as particular cases a number of the existing ones in the background literature is proposed, and their basic associate properties are discussed. It is discussed the existence and uniqueness of fixed points if the metric space is complete and the set of subsets involved in the cyclic contractive condition have nonempty intersections and the existence and uniqueness of best proximity points within each of the subsets $A_i \subset X$ if they are convex, closed and disjoint and X is a uniformly convex Banach space. The asymptotic convergence of the iterates from given initial point to best proximity points at each subset or to the unique fixed point if the subsets intersect is also discussed.

2. Main Results for a General Contractive Condition

This section contains the main results of the paper for $p(\geq 2)$-cyclic self-mapping on the union of a set of p nonempty subsets of a metric space (X, d) under a very general contractive condition which contains as particular cases several previous ones being known in the background of the literature for the noncyclic case $(p = 1)$.

Theorem 2.1. *Let (X, d) be a metric space with p nonempty closed subsets A_i of $X, \forall i \in \overline{p} := \{1, 2, \ldots, p\}$ such that $A_{pj+\ell} \equiv A_\ell; \forall \ell \in \overline{p-1}$, and let $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ be a continuous $p(\geq 2)$-cyclic self-mapping subject to $T(A_i) \subseteq A_{i+1}; \forall i \in \overline{p}$ and satisfying the following contractive condition:*

$$\begin{aligned} d(Tx, Ty) \leq{} & \alpha_i \frac{d(x,Tx)d(y,Ty) + d(x,Ty)d(y,Tx)}{d(x,y)} \\ & + \frac{\beta_{1i} d(x,Tx)(d(x,Ty) + d(y,Ty)) + \beta_{2i} d(y,Ty)(d(y,Tx) + d(x,Tx))}{d(x,y) + d(x,Ty) + d(y,Tx)} \\ & + \frac{(\gamma_{1i} d(x,Tx) + \gamma_{2i} d(y,Ty))(d(y,Tx) + d(x,Ty))}{d(x,Tx) + d(y,Tx) + d(y,Ty) + d(x,Ty)} \\ & + \delta_{1i} d(x,Tx) + \delta_{2i} d(y,Ty) + \eta_i (d(x,Ty) + d(y,Tx)) + \mu_i d(x,y) + \omega_i D_i, \end{aligned} \tag{2.1}$$

for all $x \in A_i, \forall y(\neq x) \in A_{i+1}$; $\alpha_i \geq 0$, $\beta_{ji} \geq 0$, $\gamma_{ji} \geq 0$, $\delta_{ji} \geq 0, \mu_i \geq 0$, and $\omega_i > 0$ if $D_i \neq 0$; $j = 1, 2$, fo rall $i \in \overline{p}$.

Then, the following properties hold:

(i)

$$D_i \leq d\left(T^{p(n+1)+1}x, T^{p(n+1)}x\right) \leq \left(\prod_{i=1}^{p} [K_i]\right) d\left(T^{pn+1}x, T^{pn}x\right) + \sum_{\ell=1}^{p} \left(\prod_{j=i+1}^{p} [K_j]\right) \omega_\ell D_\ell, \tag{2.2}$$

for any $x \in A_i$ and any given $i \in \overline{p}$ provided that

$$\left\{ x \in A_i : \left(\prod_{i=1}^{p} [K_i]\right) d\left(T^{pn+1}x, T^{pn}x\right) + \sum_{\ell=1}^{p} \left(\prod_{j=i+1}^{p+i-1} [K_j]\right) \omega_\ell D_\ell \geq D_i \right\} \neq \emptyset, \tag{2.3}$$

where

$$K_i := \frac{\delta_{1i} + \mu_i + \gamma_{1i} + \beta_{1i} + \eta_i}{1 - \alpha_i - \beta_{1i} - \beta_{2i} - \delta_{2i} - \gamma_{2i} - \eta_i}, \quad \forall i \in \overline{p}. \tag{2.4}$$

If, in addition, the distances between any pairs of adjacent subsets $D_i = \operatorname{dist}(A_i, A_{i+1}) = D$ are identical; $\forall i \in \overline{p}$, then $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic self-mapping. If, furthermore,

$$\begin{gathered} \alpha_i + \beta_{2i} + \gamma_{1i} + \gamma_{2i} + \delta_{1i} + \delta_{2i} + \mu_i + 2(\beta_{1i} + \eta_i) \le 1, \quad \forall i \in \overline{p}, \\ \prod_{i\in\overline{p}} \left[\frac{\delta_{1i} + \mu_i + \gamma_{1i} + \beta_{1i} + \eta_i}{1 - \alpha_i - \beta_{1i} - \beta_{2i} - \delta_{2i} - \gamma_{2i} - \eta_i}\right] < 1, \end{gathered} \tag{2.5}$$

then all the iterates $d(T^{n+1}x, T^n x)$ fulfil the following constraints:

$$\begin{gathered} d\left(T^{n+1}x, T^n x\right) \le K^n d(Tx, x) + \frac{(1-K_0)D}{1-K} \le d(Tx, x) + \frac{(1-K_0)D}{1-K}, \quad \forall n \in \mathbf{N}, \\ \limsup_{n\to\infty} d\left(T^{pn+1}x, T^n x\right) \le \frac{(1-K_0)D}{1-K} < \infty, \end{gathered} \tag{2.6}$$

$$\begin{aligned} d\left(T^{np+\ell+1}x, T^{np+\ell}x\right) &\le \left(\prod_{i=1}^{p}[K_i]\right)^{pn} \max_{\ell_1\in\overline{p}}\left(\prod_{i=\ell_1}^{\ell_1+\ell}[K_i]^i\right) d(Tx, x) + \frac{(1-K_0)D}{1-K} \\ &\le M d(Tx, x) + \frac{(1-K_0)D}{1-K}, \quad \forall n \in \mathbf{N}, \end{aligned} \tag{2.7}$$

$$d\left(T^{np+\ell+1}x, T^{np+\ell}x\right) \le \frac{(1-K_0)D}{1-K} < \infty, \tag{2.8}$$

where $\ell \in \mathbf{N}_0, M \in \mathbf{R}_+, D := \max_{i\in\overline{p}} D_i, K := \prod_{i=1}^{p}[K_i] < 1$ and $K_0 = \min_{i\in\overline{p}} K_i < 1; \forall n \in \mathbf{N}$ for any given $x \in \bigcup_{i\in\overline{p}} A_i$, with $\omega_i = 1 - K_i, \forall i \in \overline{p}$.

If (2.4) is replaced by $K_i \in [0, 1)$, that is,

$$\alpha_i + \beta_{2i} + \gamma_{1i} + \gamma_{2i} + \delta_{1i} + \delta_{2i} + \mu_i + 2(\beta_{1i} + \eta_i) < 1, \quad \forall i \in \overline{p}, \tag{2.9}$$

then the inequalities (2.6)–(2.8) trivially hold.

(ii) Assume that the contractive condition (2.1) satisfies (2.5). Then,

$$D_i \le \limsup_{n\to\infty} d\left(T^{pn+1}x, T^{pn}x\right) = \frac{\sum_{\ell=1}^{p+i-1}\left(\prod_{j=i+1}^{p+i-1}[K_j]\right)\omega_\ell D_\ell}{1 - \prod_{i=1}^{p}[K_i]} < \infty, \tag{2.10}$$

$\forall x \in A_i, \forall i \in \overline{p}$ under the necessary condition (2.3).

If, in particular, (2.9) holds, that is, $K_i \in [0,1)$, then $\omega_i := ((1-K_i)/D_i)\overline{D}$ for any $D_i \neq 0$ and some $\overline{D} \geq D, \forall i \in \overline{p}$, then

$$D_i \leq \lim_{n\to\infty} \sup d\left(T^{pn+1}x, T^{pn}x\right) \leq \overline{D}, \quad \forall x \in A_i,\ \forall i \in \overline{p}. \tag{2.11}$$

If (2.9) holds, $D_i = D$ and

$$\omega_i = 1 - K_i = \alpha_i + \beta_{2i} + \gamma_{1i} + \gamma_{2i} + \delta_{1i} + \delta_{2i} + \mu_i + 2\left(\beta_{1i} + \eta_i\right), \quad \forall i \in \overline{p}, \tag{2.12}$$

then the limit below exists:

$$\lim_{n\to\infty} d\left(T^{pn+1}x, T^{pn}x\right) = D; \quad \forall x \in \bigcup_{i\in\overline{p}} A_i, \forall i \in \overline{p} \tag{2.13}$$

which is guaranteed by the condition (2.3).

(iii) If the constants of (2.4) fulfil $K = \prod_{i=1}^{p}[K_i] < 1$, then the p subsets A_i of $X, \forall i \in \overline{p}$ have nonempty intersection (i.e., $D_i = D = 0, \forall i \in \overline{p}$), and, if furthermore, the metric space (X,d) is complete, then $\lim_{n\to\infty} d(T^{pn+1}x, T^{pn}x) = 0, \forall x \in \bigcup_{i\in\overline{p}} A_i$ and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has a unique fixed point in $\bigcap_{i\in\overline{p}} A_i$ to which all the sequences $\{T^n x\}_{n\in\mathbf{N}_0}$, which are then bounded, converge, $\forall x \in \bigcup_{i\in\overline{p}} A_i$.

If A_i are disjoint, closed, and convex, $\forall i \in \overline{p}, X$ is uniformly convex and $\prod_{i=1}^{p}[K_i] < 1$ and (2.12) holds with $D_i = D > 0, \forall i \in \overline{p}$, then all sequences $\{T^{pn}x\}_{n\in\mathbf{N}_0}, \forall x \in A_i$ converge to a best proximity point of $A_i, \forall i \in \overline{p}$.

Proof. Let x_0 be an arbitrary point in $\bigcup_{i\in\overline{p}} A_i \subset X$ and take $x = x(n, x_0) \equiv x_n = T^n x_0 \in A_j = A_j(n,i)$ and $y \equiv x_{n+1}(n+1, x_0) = Tx_n = T^{n+1}x_0 \in A_{j+1}, \forall n \in \mathbf{N}_0(:= \mathbf{N} \cup \{0\})$, where $T^0 = id$, and $j = \widehat{j} := [(i+n)/p]$ if $(i+n)/p \in \mathbf{N}$ and $j = \widehat{j} + 1$, otherwise. Then, from (2.1),

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\leq \alpha_i d(x_{n+1}, x_{n+2}) \\
&\quad + \frac{\beta_{1i} d(x_{n+1}, x_{n+2}) + \beta_{2i} d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1}) + d(x_n, x_{n+2})} d(x_n, x_{n+1}) \\
&\quad + \left(\frac{\beta_{1i} d(x_n, x_{n+1})}{d(x_n, x_{n+1}) + d(x_n, x_{n+2})} + \eta_i\right) d(x_n, x_{n+2}) \\
&\quad + \frac{\gamma_{1i} d(x_n, x_{n+1}) + \gamma_{2i} d(x_{n+1}, x_{n+2})}{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_n, x_{n+2})} d(x_n, x_{n+2}) \\
&\quad + \left(\delta_{1i} + \mu_i\right) d(x_n, x_{n+1}) + \delta_{2i} d(x_{n+1}, x_{n+2}) + \omega_i D_i \\
&\leq \alpha_i d(x_{n+1}, x_{n+2}) + \omega_i D_i \\
&\quad + \left(\left(\frac{\beta_{1i} + \beta_{2i}}{d(x_n, x_{n+1}) + d(x_n, x_{n+2})} + \delta_{1i} + \mu_i\right) d(x_n, x_{n+1}) + \delta_{2i}\right) d(x_{n+1}, x_{n+2})
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{\gamma_{1i}d(x_n,x_{n+1})+\gamma_{2i}d(x_{n+1},x_{n+2})}{d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_n,x_{n+2})}+\frac{\beta_{1i}d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_n,x_{n+2})}+\eta_i\right)\\
&\times d(x_n,x_{n+2})\\
\le\ &\alpha_i d(x_{n+1},x_{n+2})+(\delta_{1i}+\mu_i)d(x_n,x_{n+1})+(\beta_{1i}+\beta_{2i}+\delta_{2i})d(x_{n+1},x_{n+2})+\omega_i D_i\\
&+\eta_i d(x_n,x_{n+2})+\gamma_{1i}d(x_n,x_{n+1})+\gamma_{2i}d(x_{n+1},x_{n+2}),
\end{aligned}
\tag{2.14}
$$

and one gets

$$
\begin{aligned}
&(1-\alpha_i-\beta_{1i}-\beta_{2i}-\delta_{2i}-\gamma_{2i}-\eta_i)d(x_{n+1},x_{n+2})\\
&\quad\le(\delta_{1i}+\mu_i+\gamma_{1i}+\beta_{1i})d(x_n,x_{n+1})+\eta_i d(x_n,x_{n+2})+\omega_i D_i\\
&\quad\le(\delta_{1i}+\mu_i+\gamma_{1i}+\beta_{1i}+\eta_i)d(x_n,x_{n+1})+\omega_i D_i,
\end{aligned}
\tag{2.15}
$$

so that, since

$$
\left(\frac{\beta_{1i}d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_n,x_{n+2})}+\eta_i\right)d(x_n,x_{n+2})\le\beta_{1i}d(x_n,x_{n+1})+\eta_i(d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})).
\tag{2.16}
$$

Thus, it follows proceeding recursively with (2.14) subject to (2.9), (2.16), and $\omega_i=1-K_i$, $\forall i\in\overline{p}$

$$
\begin{aligned}
d\left(T^{pn+j+1}x,T^{pn+j}x\right)&\le\left(\prod_{j=i+1}^{p+i-1}[K_j]\right)^n d(Tx,x)+\sum_{\ell=i}^{p+i-1}\left(\prod_{j=i+1}^{p+i-1}[K_j]\right)\omega_\ell D_\ell\\
&\le K^{pn}d(Tx,x)+\frac{(1-K_0)D}{1-K},
\end{aligned}
\tag{2.17}
$$

where the first inequality holds irrespective of the identities $\omega_i=1-K_i;\forall i\overline{p}$ and it implies directly (2.2) since $\omega_{i+jp}\equiv\omega_i$ and $K_{i+jp}\equiv K_i$ and $D_{i+jp}\equiv D_i;\forall i\in\overline{p},\forall j\in\mathbf{N}_0$, so that $\sum_{\ell=1}^{p}(\prod_{j=i+1}^{p}[K_j])\omega_i D_i=\sum_{\ell=i}^{p+i-1}(\prod_{j=i+1}^{p+i-1}[K_j])\omega_\ell D_\ell$. The second inequality follows in the case that $\omega_i=1-K_i$, if (2.9) holds so that $K^p<K=\prod_{i=1}^{p}[K_i]<1,\forall i\in\overline{p}$ if (2.3) leading directly to (2.5)–(2.7). Property (i) has been proven.

Property (ii) is proven by taking $x\in A_i$ for any $i\in\overline{p}$ and proceeding recursively with the first inequality of (2.17) to obtain directly (2.10) and (2.11) since $\prod_{i=1}^{p}[K_i]<1$ and (2.13) if, in addition, (2.12) holds. To prove Property (iii), note from (2.8), (2.9), (2.12), and (2.13) that $0=\lim_{n\to\infty}d(T^{pn+j}x,T^{pn+j-1}x)\le\lim_{n\to\infty}d(T^{pn+1}x,T^{pn}x)=0,\forall x\in\bigcup_{i\in\overline{p}}A_i$,if $D_i=D=0$ and $\prod_{i=1}^{p}[K_i]<1;\forall i,j\in\overline{p}$. If, furthermore, (X,d) is a complete metric space, then each sequence $\{T^n x\}_{n\in\mathbf{N}_0};\forall x\in\bigcup_{i\in\overline{p}}A_i$ is a Cauchy sequence with a limit $\overline{x}=\lim_{n\to\infty}T^n x=\lim_{n\to\infty}T^{pn}x$ in $\bigcap_{i\in\overline{p}}A_i$ since this set intersection is nonempty and closed since all the intersected sets are nonempty and closed. Since the sequences $\{T^n x\}_{n\in\mathbf{N}_0}$ are convergent to a limit $\overline{x}$, they are bounded. Also, since $T:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ is continuous in $\bigcap_{i\in\overline{p}}A_i$, then

$T\overline{x} = T(\lim_{n\to\infty} T^{pn}x) \le \lim_{n\to\infty} T^{pn+1}x = \overline{x}$ so that $\overline{x} \in \mathrm{Fix}(T) \subset \bigcap_{i\in\overline{p}} A_i$. It is proven by contradiction that there exists a unique fixed point. Assume that there exist $u = \overline{x}$ and v subject to $u = Tu \neq v = Tv$ in $\mathrm{Fix}(T)$, the set of fixed points of T. Then, the subsequent contradiction follows from (2.1) for $D_i = D = 0, \forall i \in \overline{p}$, by using $d(u,Tu) = d(v,Tv) = 0, d(u,Tv) = d(v,Tu) = d(u,v) \neq 0$ and (2.4):

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\le \alpha_i \frac{d(u,Tu)d(v,Tv) + d(u,Tv)d(v,Tu)}{d(u,v)} \\ &+ \frac{\beta_{1i} d(u,Tu)\big(d(u,Tv) + d(y,Ty)\big) + \beta_{2i} d(v,Tv)(d(v,Tu) + d(u,Tu))}{d(u,v) + d(u,Tv) + d(v,Tu)} \\ &+ \frac{\big(\gamma_{1i} d(u,Tu) + \gamma_{2i} d(v,Tv)\big)(d(v,Tu) + d(u,Tv))}{d(u,Tu) + d(v,Tu) + d(v,Tv) + d(u,Tv)} \\ &+ \delta_{1i} d(u,Tu) + \delta_{2i} d(v,Tv) + \eta_i (d(u,Tv) + d(v,Tu)) + \mu_i d(u,v) \\ &\le \max_{i\in\overline{p}} (\alpha_i + 2\eta_i + \mu_i) d(u,v) < d(u,v), \end{aligned} \tag{2.18}$$

so that $u = v$. Property (iii) has been proven.

Note that it cannot be concluded from Theorem 2.1 that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ under the contractive condition (2.1) is either a p-cyclic nonexpansive self-mapping from Theorem 2.1(i), even if all the contractive constants in (2.7) are identical and all the distances between adjacent subsets of X are also identical, or a p-cyclic contraction under Theorem 2.1(ii) since (2.6)–(2.8), or its respective versions with strict inequalities, are only guaranteed for the iterates $\{T^n x\}_{n\in\mathbf{N}_0}$ and $\{T^{pn} x\}_{n\in\mathbf{N}_0}$ for any $x \in \bigcup_{i\in\overline{p}} A_i$. Assume that the norm $\| \ \|$ of the uniformly convex (Banach) space $(X, \| \ \|)$ induces the metric $d : X\times X \to \mathbf{R}_{0+}$ being used. Otherwise, any alternative equivalent metric $d_1 : X \times X \to \mathbf{R}_{0+}$ may be used to conclude the result. If (2.9) and (2.12) hold with distances between adjacent subsets $D_i = D > 0; \forall i \in \overline{p}$, then all sequences $\{T^{pn} x\}_{n\in\mathbf{N}_0}$ are Cauchy sequences, $\forall x \in A_i$ converge to a best proximity point $\overline{x}_i \in A_i$ of A_i for any given $i \in \overline{p}$ from (2.13) and the continuity of $T^p : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ ensured by that of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$. □

Remark 2.2. If $\prod_{i=1}^{q} [K_i]^p \in [0,1]$, with some of the K_i being eventually larger than one, then $d(T^{n+1}x, T^n x), \forall n \in \mathbf{N}$ is bounded provided that $d(Tx,x)$ is finite. If, furthermore, $\prod_{i=1}^{q} [K_i]^p \in [0,1)$ and $D = 0$, then all the composed mappings $\widehat{T}_i : A_i \to \mathrm{Im}\ \widehat{T}_i \subset A_i$ defined by $\widehat{T}_i(A_i) = (T)^p(A_i) = (T \circ T^{p-1})(A_i); \forall i \in \overline{p}$ satisfies $\limsup_{n\to\infty} d(\widehat{T}_i^{n+1}x, \widehat{T}_i^{n}x) = 0; \forall x \in A_i, \forall i \in \overline{p}$, so that $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ satisfies and (2.13) holds with some of the K_i being eventually not less than one. The last part of the proof of Theorem 2.1(iii) leads to the conclusion that if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is not continuous while the composed self-mapping $T^p : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is continuous, then the convergence of the iterates to a best proximity point in each adjacent subset $A_i (i \in \overline{p})$ is still ensured.

The existence of a unique fixed point is guaranteed if the p subsets are closed and intersect even if $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is not continuous and all the constants (2.4) are not less than unity provided that (X, d) is complete and $\prod_{i=1}^{p} [K_i] < 1$ as follows.

Theorem 2.3. *Let (X, d) be a complete metric space with $\underline{p}$ nonempty closed subsets A_i of $X, \forall i \in \overline{p}$, with nonempty intersection satisfying $A_{pj+\ell} \equiv A_\ell; \forall \ell \in \overline{p-1}$, and let $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ be a $p(\geq 2)$-cyclic self-mapping subject to $T(A_i) \subseteq A_{i+1}; \forall i \in \overline{p}$ and satisfying the contractive condition* (2.1) *subject to $\prod_{i=1}^{p}[K_i] < 1$ for constants defined in* (2.8)*. Then, all the sequences $\{T^{pn+j}x\}_{n\in\mathbf{N}_0}$ are bounded and converge to a unique fixed point in $\bigcap_{i\in\overline{p}} A_i; \forall i \in \overline{p}$ of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$.*

Proof. Since $D_i = 0$ (since $\bigcap_{i\in\overline{p}} A_i \neq \emptyset$), $\forall i \in \overline{p}, \prod_{i=1}^{p}[K_i] < 1, T(A_i) \subseteq A_{i+1}, \bigcap_{i\in\overline{p}} A_i$ is nonempty and closed and (X, d) is complete, it follows that

$$0 = \lim_{n\to\infty} d\left(T^{pn+j+1}x, T^{pn+j}x\right) \leq \left(\prod_{j=i+1}^{p}[K_j]\right) \lim_{n\to\infty} d\left(T^{pn+1}x, T^{pn}x\right) = 0, \tag{2.19}$$

for any $x \in A_i$, and then for any $x \in \bigcup_{i\in\overline{p}} A_i$, from Theorem 2.1(iii) irrespective of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ being continuous or not. Thus, for any $\varepsilon \in \mathbf{R}_+, \exists n_0 \in \mathbf{N}_0$ such that from triangle inequality for distances, one has

$$d\left(T^{pn+j}x, T^{pn+j-1}x\right) < \frac{\varepsilon}{p} < \varepsilon; \quad \forall j \in \overline{p}, \qquad d\left(T^{pn+j}x, T^{pn+j}x\right) \leq \sum_{j=1}^{p} d\left(T^{pn+j}x, T^{pn+j-1}x\right) < \varepsilon, \tag{2.20}$$

$\forall x \in \bigcup_{i\in\overline{p}} A_i, \forall n > n_0$ so that each $\{T^{pn+j}x\}_{n\in\mathbf{N}_0}$ is a Cauchy sequence with a limit in the closed nonempty set $\bigcap_{i\in\overline{p}} A_i$ which is also bounded since it is convergent. Hence, it follows the existence of fixed points in $\bigcap_{i\in\overline{p}} A_i$ being each of those limits. The uniqueness of the fixed point follows from its counterpart in the proof of Theorem 2.1(iii) where the continuity of $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has not been used. Hence, the theorem. □

The following result allows to fulfil (2.10) in Theorem 2.1 under some negative scalars ω_i and constants K_i exceeding unity provided that a set of necessary conditions involving distances between the adjacent subsets and such scalars are satisfied.

Corollary 2.4. *Assume that* (2.1) *holds and $\prod_{i=1}^{p}[K_i] < 1$ and $D_i \neq 0, \forall i \in \overline{p}$. Then,* (2.3)*, and thus* (2.2)*, equivalently* (2.17)*, holds if $\alpha_i \geq 0, \beta_{ji} \geq 0, \gamma_{ji} \geq 0, \delta_{ji} \geq 0, \mu_i \geq 0, \forall i \in \overline{p}$ provided that the distances between adjacent subsets and the real scalars ω_i satisfy the joint constraints:*

$$D_i \leq \frac{\sum_{\ell(\neq i)=1}^{p+i-1}\left(\prod_{j=i+1}^{p+i-1}[K_j]\right)\omega_\ell D_\ell}{1 - \prod_{i=1}^{p}[K_i] - \prod_{j=i+1}^{p+i-1}[K_j]\omega_i}, \quad \forall i \in \overline{p}. \tag{2.21}$$

Proof. A necessary condition for (2.2)-(2.3) to hold, with a nonnegative second right-hand side term in (2.2) is that the constraints (2.21) hold. □

It is wellknown that p-cyclic nonexpansive self-maps require that the adjacent subsets have all the same pairwise distances. In the case that the relevant self-mappings are contractive are Meir-Keeler contractions, they have a unique fixed point if all the subsets intersect and the metric space is complete. In the case that the subsets do not intersect, there is a unique convergence best proximity point at each subset, to which the iterates through the self-mapping T converge asymptotically, provided that the subsets are nonempty convex and closed and the vector space X defining the metric space is uniformly convex, then also being reflexive and strictly convex, [8]. It is still required that all the distances between adjacent subsets be identical so that, otherwise, the self-mappings from the union of the subsets to itself cannot be nonexpansive [9, 10]; hence, they cannot be contractive. In the following, the condition of all the distances between the adjacent subsets being identical is not longer being required. The price to be paid is that the convergence of the iterates through the self-mapping do not necessarily converge to best proximity points located in the boundaries of the sets but to best proximity points located at the boundaries of appropriate nonempty closed convex subsets of the original subsets of X. In order to facilitate the formalism for the case of distinct distances between adjacent subsets, the maps of interest are restricting their images as the iterations progress in order to asymptotically reach a new set of adjacent subsets all possessing identical pairwise distances. For such a subsequent analysis, first proceed as follows by first introducing the following hypotheses.

Hypotheses

(H1) Assume that a sequence of nonempty closed sets $\{\widehat{A}_{ij}\}_{j\in\mathbf{N}_0}$exists such that $\widehat{A}_{i0}\equiv A_i$ and $\widehat{A}_{i,j+1}\subseteq\widehat{A}_{ij};\forall i\in\overline{p},\ j\in\mathbf{N}$ and that $\exists D:=\lim_{j\to\infty}D_{ij}=\lim_{j\to\infty}\operatorname{dist}(\widehat{A}_{ij},\widehat{A}_{i+1,j})>0;\forall i\in\overline{p}$ satisfying $D:=\lim_{j\to\infty}\operatorname{dist}(\widehat{A}_{ij},\widehat{A}_{i+1,j})\geq\max_{i\in\overline{p}}D_i,\forall i\in\overline{p}$.

(H2) Assume that any sequence of $p(\geq 2)$-cyclic self-mappings $T_j:\bigcup_{i\in\overline{p}}\widehat{A}_{ij}\to\bigcup_{i\in\overline{p}}\widehat{A}_{ij}$ is subject to $T_j(\widehat{A}_{ij})\subseteq\widehat{A}_{i+1,j};\forall i\in\overline{p},\forall j\in\mathbf{N}_0$ while all its elements satisfy the contractive condition (2.1), satisfying (2.8)-(2.9), with $K_{ij}=K_i$ with $\prod_{i=1}^{p}[K_i]<1$ and $\omega_{ij}\equiv\omega_i:=D/D_i-K_i;\forall i\in\overline{p},\forall j\in\mathbf{N}_0$.

(H3) Define composed self-mapping $T_{\{j_\ell\}}^{pn+k}:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}A_i$ as follows (with a certain abuse of notation):

$$T_{\{j_\ell\}}^{pn}:=\left(T_{j_{n_2}}\overbrace{\circ\circ\cdots\circ\circ}^{p}T_{j_{n_1}}\right)\overbrace{\circ\cdots\circ}^{n}\left(T_{j_2}\overbrace{\circ\circ\cdots\circ\circ}^{p}T_1\right),$$

$$T_{\{j_\ell\}}^{pn+k}:=\left(T_{j_{\overline{n}_2}}\overbrace{\circ\circ\cdots\ \circ\circ}^{i}T_{j_{\overline{n}_1}}\right)\circ\left(T_{j_{n_2}}\overbrace{\circ\circ\cdots\ \circ\circ}^{p}T_{j_{n_1}}\right)\overbrace{\circ\ \cdots\ \circ}^{n}\left(T_{j_2}\overbrace{\circ\circ\cdots\ \circ\circ}^{p}T_1\right),$$
$$\forall k\in\overline{p-1}\cup\{0\},\tag{2.22}$$

where $\{j_\ell\}_{\ell\in\mathbf{N}}$ is a (nonnecessarily monotone) increasing sequence of natural numbers fulfilling $j_1=1,p=j_{n_2}-j_{n_1}+1$ for the given $j_{n_{1,2}}\in\mathbf{N}$, for some $\overline{n}_2\geq\overline{n}_1\geq n_2$ if $k\geq 1$, with the sequence of natural numbers$j_\ell$satisfying $1\leq j_\ell\leq j_{\overline{n}_2},n_{1,2},k\in\overline{p-1}\cup\{0\}$. Note that in fact $T_{\{j_\ell\}}^{pn+k}:\bigcup_{i\in\overline{p}}A_i\to\bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}}\widehat{A}_{ij})$ since its image is always restricted to be contained in $\bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}}\widehat{A}_{ij})$ by construction.

The reason of the abuse of notation when defining the composed mapping is useful since it is explicitly indicated that we are dealing with the composition of n times groups of p compositions of the sequence of self-mappings $T_j : \bigcup_{i\in\overline{p}} \widehat{A}_{ij} \to \bigcup_{i\in\overline{p}} \widehat{A}_{ij}$. The following result holds.

Theorem 2.5. *Any composed mapping $T_{\{j_\ell\}}^{pn+k}$ from $\bigcup_{i\in\overline{p}} A_i$ to $\bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$, defined by* (2.22) *under the Hypothesis (H1)–(H3), has the following properties.*

(i) *If the sequence $\{j_\ell\}_{\ell\in N_\alpha}$, defined for some $N_\alpha \subset \mathbf{N}$, is unbounded then,*

$$\exists \lim_{n\to\infty} d\left(T_{\{j_\ell\}}^{pn+k}x, T_{\{j_\ell\}}^{pn+k+1}x\right) = D; \quad \forall x \in \bigcup_{i\in\overline{p}} A_i,\ \forall k \in \overline{p-1}\cup\{0\}. \tag{2.23}$$

(ii) *Assume that the sequence $T_j : \bigcup_{i\in\overline{p}} \widehat{A}_{ij} \to \bigcup_{i\in\overline{p}} \widehat{A}_{ij}$, with $T_j(\widehat{A}_{ij}) \subseteq \widehat{A}_{i+1,j}$, converges uniformly on some nonempty subset $S \subset \bigcup_{i\in\overline{p}} A_i$ (under proper set inclusion) as $j \to \infty$. Then, any sequence $T_{\{j_\ell\}}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ converges uniformly to a limit self-mapping, dependent on the sequence $\{j_\ell\}_{\ell\in N_\alpha}, \widehat{T}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ as $j_\ell,\ n \to \infty$ provided that $S \supseteq \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$. Also,*

$$\exists \lim_{n\to\infty} d\left(\widehat{T}^{pn+k}x, \widehat{T}^{pn+k+1}x\right) = \lim_{n\to\infty} d\left(T_{\{j_\ell\}}^{pn+k}x, T_{\{j_\ell\}}^{pn+k+1}x\right) = D; \quad \forall x \in \bigcup_{i\in\overline{p}} A_i,\ \forall k \in \overline{p-1}\cup\{0\}. \tag{2.24}$$

(iii) *The limit $\widehat{T}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ of any composed sequence of mappings $T_{\{j_\ell\}}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij}), \forall x \in \bigcup_{i\in\overline{p}} A_i$ generated by some unbounded sequence $\{j_\ell\}_{\ell\in N_\alpha}$ has a best proximity point $\overline{x}_i$ between the adjacent sets$(\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ and $(\bigcap_{j=1}^{\infty} \widehat{A}_{i+1,j})$ at each $\partial(\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ which is also in $A_i, \forall i \in \overline{p}$ to which all the sequences $\{\widehat{T}^{pn}x\}_{n\in\mathbf{N}_0}$, which are then bounded, converge; $\forall x \in A_i, \forall i \in \overline{p}$.*

If $D_i = D > 0, \forall i \in \overline{p}$, then all sequences $\{T^{pn}x\}_{n\in\mathbf{N}_0}; \forall x \in A_i$ converges to a best proximity point of $A_i, \forall i \in \overline{p}$.

If, in addition, the metric space (X, d) is complete and $D_i = D = 0, \forall i \in \overline{p}$ then all sequences $\{T^{pn}x\}_{n\in\mathbf{N}_0}, \forall x \in A_i$ converge to a unique fixed point of $A_i, \forall i \in \overline{p}$.

(iv) *Assume that X is a uniformly convex space and that the subsets A_i are convex and closed. Then, the best proximity points $\overline{x}_i \in (\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ of the adjacent subsets $(\bigcap_{j=1}^{\infty} \widehat{A}_{ij}) \subseteq A_i$ and $(\bigcap_{j=1}^{\infty} \widehat{A}_{i,j+1}) \subseteq A_{i+1}$ are unique for each $i \in \overline{p}$. Furthermore, $\overline{x}_{i+j} = T^j\overline{x}_i$ and $d(\overline{x}_i, \widehat{T}\overline{x}_i) = d(\widehat{T}^j\overline{x}_i, \widehat{T}^{j+1}\overline{x}_i) = d(\overline{x}_{i+j}, \overline{x}_{i+j+1}) = D; \forall i, j \in \overline{p}$ with $\overline{x}_{i+np} = \overline{x}_i; \forall n \in \mathbf{N}_0, \forall i \in \overline{p}$.*

Proof. Note that the necessary condition (2.3) for Property (i) holds by construction and take, with no loss of generality, $n = (m_1 j_{\overline{n}_2} + m_2 - k)/p$ for $k \in \overline{p}$ and $n := m_1 j_{n_2} + m_2/p$ for $k = 0$ for any given finite positive natural numbers $m_{1,2}$, dependent on n and k, so that $j_{\overline{n}_1} := \max_{1\le\ell\le\overline{n}_2} j_\ell \to \infty$, if $k = 0$ in (2.22), and also $j_{\overline{n}_2} := \max_{1\le\ell\le\overline{n}_2} j_\ell \to \infty$, if $k \in \overline{p-1}$, as $n \to \infty$ in (2.22) for $\{j_\ell\}_{\ell\in N_\alpha}$ being some unbounded sequence of natural numbers as $\ell, j_\ell \to \infty$. Since $K_{ij} \equiv K_i$ with $\prod_{i=1}^{p}[K_i] < 1$ and $\omega_{ij} \equiv \omega_i := D/D_i - K_i; \forall i \in \overline{p}, \forall j \in \mathbf{N}_0, D := \lim_{j\to\infty} \operatorname{dist}(\widehat{A}_{ij}, \widehat{A}_{i+1,j}) \ge \max_{i\in\overline{p}} D_i$ and $T_j : \bigcup_{i\in\overline{p}} \widehat{A}_{ij} \to \bigcup_{i\in\overline{p}} \widehat{A}_{ij}$ subject to

$T_j(\widehat{A}_{ij}) \subseteq \widehat{A}_{i+1,j}; \forall i \in \overline{p}, \forall j \in \mathbf{N}_0$, it follows from Cantor's intersection theorem that any arbitrary intersection $\bigcap_{j\in\widehat{\mathrm{N}}_0} \widehat{A}_{ij}$ is nonempty and closed; $\forall i \in \overline{p}$ and $\widehat{N}_0 \subseteq \mathbf{N}_0$ since $\widehat{A}_{i0} \equiv A_i$and $\widehat{A}_{i,j+1} \subseteq \widehat{A}_{ij}; \forall i \in \overline{p}, \forall j \in \mathbf{N}_0$. Then, the composed $\forall k \in \overline{p-1} \cup \{0\}$ mapping $T_{\{j_\ell\}}^{pn}$ from $\bigcup_{i\in\overline{p}} A_i$ to $\bigcup_{i\in\overline{p}} A_i$, whose nonempty image is restricted by construction to the subset $\bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ of $\bigcup_{i\in\overline{p}} A_i$, is well-posed for any arbitrary sequence$\{j_\ell\}$ and for any $x \in \bigcup_{i\in\overline{p}} A_i$. Thus, one gets from Theorem 2.1, (2.10), that for any given $\varepsilon \in \mathbf{R}_+, \exists n_0 = n_0(\varepsilon) \in \mathbf{N}$ such that:

$$D \leq d\left(T_{\{j_\ell\}}^{pn+k} x, T_{\{j_\ell\}}^{pn+k+1} x\right) \leq D + \varepsilon; \quad \forall x \in \bigcup_{i\in\overline{p}} A_i,\ \forall n(\in \mathbf{N}_0) \geq n_0,\ \forall x \in \bigcup_{i\in\overline{p}} A_i,\ \forall k \in \overline{p-1} \cup \{0\}, \tag{2.25}$$

where the sequence of natural numbers $\{j_\ell\}_{\ell\in\mathbf{N}}$ being subject to $n = (m_1 j_{\overline{n}_2} + m_2 - k)/p$ for $k \in \overline{p}$ and $n := m_1 j_{n_2} + m_2/p$ for $k = 0$. Thus, it follows that (2.23) holds from (2.25) since $m_1 j_{\overline{n}_2} + m_2 = np + k \to \infty$ that implies $j_{n_2}, j_{\overline{n}_2} \to \infty$ for $k \in \overline{p}$, and $j_{n_2} \to \infty$ for $k = 0$ so that $\{j_\ell\}_{\ell\in\widehat{N}_0}$diverges to infinity as $\ell \to \infty$ taking values in a subset $\widehat{N}_0$ of $\mathbf{N}$ of infinite cardinal. Hence, Property (i).

To prove Property (ii), note first that the composed self-mapping map $T_{\{j_\ell\}}^{pn+k}$: $\bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ has an image restricted to $\bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ by construction. If $T_j : \bigcup_{i\in\overline{p}} \widehat{A}_{ij} \to \bigcup_{i\in\overline{p}} \widehat{A}_{ij}$, satisfying $T_j(\widehat{A}_{ij}) \subseteq \widehat{A}_{i+1,j}$, converges uniformly as $j \to \infty$ on some subset S of X satisfying $\bigcup_{i\in\overline{p}} A_i \supset S \supseteq \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$, then $T_{\{j_\ell\}}^{pn+k}$ can converge uniformly in $\bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ as $j_\ell \to \infty$. Proceed by contradiction. If $T_{\{j_\ell\}}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ do not converge uniformly, while T^{pn+k} converges does, then, for any given $\varepsilon \in \mathbf{R}_+$, there exist some $\varepsilon_0 \in \mathbf{R}_+$, some $x \in \bigcup_{i\in\overline{p}} A_i$ and some $n_0 = n_0(\varepsilon_0), \overline{n}_0 = \overline{n}_0(\overline{\varepsilon}) \in \mathbf{N}_0$ and some sequences of nonnegative ordered integers $\{n_t\}$ and $\{m_t\}$ of minimal element $\widehat{n}_0 = p \max(n_0, \overline{n}_0)$ such that:

$$\begin{aligned} \varepsilon + g &\geq d\left(T_{\{n_t\}}^{pn+k} x, T^{pm+k+1} x\right) + d\left(T^{pm+k+1} x, T^{pn+k} x\right) + d\left(T^{pn+k} x, T_{\{m_t\}}^{pn+k+1} x\right) \\ &\geq d\left(T_{\{n_t\}}^{pn+k} x, T_{\{m_t\}}^{pn+k+1} x\right) \\ &\geq \varepsilon_0, \end{aligned} \tag{2.26}$$

after using the triangle property for distances, for some $g = g(x) \in \mathbf{R}_{0+}$ and $n, m \in \mathbf{N}$ satisfying

$$n_t \geq pn + k \geq \widehat{n}_0, \qquad m_t \geq pm + k \geq \widehat{n}_0, \quad \forall k \in \overline{p-1} \cup \{0\}. \tag{2.27}$$

The chained inequalities (2.25) contradict a choice of $\varepsilon \in \mathbf{R}_+$ satisfying $\varepsilon < \varepsilon_0$. Since $\varepsilon \in \mathbf{R}_+$ is arbitrary, $T_{\{j_\ell\}}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ converges uniformly in $(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ as $j_{\overline{n}_1} \to \infty$, or trivially, if $j_{\overline{n}_1}$is finite, for all $k \in \overline{p-1} \cup \{0\}$. On the other hand, the triangle inequality yields

$$d\left(\widehat{T}^{pn+k}x, \widehat{T}^{pn+k+1}x\right) \leq d\left(\widehat{T}^{pn+k}x, T_{\{j_\ell\}}^{pn+k}x\right) + d\left(T_{\{j_\ell\}}^{pn+k}x, \widehat{T}^{pn+k+1}x\right) \longrightarrow 0 \quad \text{as } j_\ell, n \longrightarrow \infty, \tag{2.28}$$

$\forall x \in \bigcup_{i\in\overline{p}} A_i, \forall k \in \overline{p-1} \cup \{0\}$, since a uniform convergence of the sequence $T_{\{j_\ell\}}^{pn+k}x \to \widehat{T}^{pn+k}x$ as $j_\ell, n \to \infty$, $\forall x \in \bigcup_{i\in\overline{p}} A_i$ has been proven, so that (2.24) follows. Hence, Property (ii).

To prove Property (iii), remember that form Property (ii), $T_j : \bigcup_{i\in\overline{p}} \widehat{A}_{ij} \to \bigcup_{i\in\overline{p}} \widehat{A}_{ij}$ and the composed $T_{\{j_\ell\}}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}}(\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij})$ fulfil:

$$T_j x \longrightarrow \widehat{T}x \text{ as } j \longrightarrow \infty \Longrightarrow T_{\{j_\ell\}}^{pn+k}x \longrightarrow \widehat{T}^{pn+k}x, \quad \forall k \in \overline{p-1} \cup \{0\} \text{ as } j_\ell,\ n \longrightarrow \infty, \forall x \in \bigcup_{i\in\overline{p}} A_i. \tag{2.29}$$

From (2.24), $\forall \varepsilon \in \mathbf{R}_+ \Rightarrow \exists n_1 = n_1(\varepsilon)$ such that for any integers $n, m \geq n_1$,

$$d\left(\widehat{T}^{pm}x, \widehat{T}^{pn}x\right) \leq \varepsilon; \qquad d\left(\widehat{T}^{pn}x, \widehat{T}^{pn+1}x\right) = d\left(T_{\{j_\ell\}}^{pn}x,\ T_{\{j_\ell\}}^{pn+1}x\right) < D + \varepsilon, \qquad \forall x \in \bigcup_{i\in\overline{p}} A_i. \tag{2.30}$$

Proceed by contradiction to prove the convergence of the iterates to best proximity points of $\bigcap_{j=1}^{\infty} \widehat{A}_{ij}; \forall i \in \overline{p}$. Assume that there is no $\overline{x}_i \in A_i$ such that $\overline{x}_i = \widehat{T}^{np}\overline{x}_i \in \bigcap_{j=1}^{\infty} \widehat{A}_{ij}$,and $D = d(\overline{x}_i, \widehat{T}\overline{x}_i) = d(\overline{x}_{i+1}, \widehat{T}\overline{x}_{i+1}); \forall i \in \overline{p}$ so that there exist some integers $\overline{n}, \overline{m} \geq \max(n_1, n_2), \varepsilon_1 \in \mathbf{R}_+$ and $x \in A_i$ such that

$$D + 2\varepsilon > d\left(\widehat{T}^{p\overline{m}}x, \widehat{T}^{p\overline{n}}x\right) + d\left(\widehat{T}^{p\overline{n}}x, \widehat{T}^{p\overline{n}+1}x\right) \geq d\left(\widehat{T}^{p\overline{m}}x, \widehat{T}^{p\overline{n}+1}x\right) \geq D + \varepsilon_1. \tag{2.31}$$

Since ε is arbitrary, one can choose $\varepsilon < \varepsilon_1/2$ which yields a contradiction in (2.31). Then, $\exists \overline{x}_i = \widehat{T}^{np}\overline{x}_i \in \bigcap_{j=1}^{\infty} \widehat{A}_{ij}$ such that $D = d(\overline{x}_i, \widehat{T}\overline{x}_i) = d(\overline{x}_{i+1}, \widehat{T}\overline{x}_{i+1}), \forall i \in \overline{p}$. Furthermore, $\overline{x}_i \in A_i$ since $T_j(\widehat{A}_{ij}) \subseteq \widehat{A}_{i+1,j} \subseteq \widehat{A}_{i+1,0} \equiv A_{i+1}, \forall i \in \overline{p}, \forall j \in \mathbf{N}_0$. If all the distances are nonzero and identical then the best proximity points are at the boundaries of the closed subsets $A_i, \forall i \in \overline{p}$. If, in addition, the metric space is complete and the subsets intersect (i.e., all the distances between adjacent subsets are zero), then $\omega_i = 1 - K_i$ and the best proximity points are also coincident in a unique fixed point (Theorem 2.1(iii)) in the nonempty intersection of all such subsets. Property (iii) has been proven.

Property (iv) is proven by contradiction. Assume that there are two distinct best proximity points $\overline{x}_i, \overline{y}_i \in (\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ for any given $i \in \overline{p}$. Define the following real sequences $\{x_n\}_{n\in\mathbf{N}}$, $\{y_n\}_{n\in\mathbf{N}}$, and $\{z_n\}_{n\in\mathbf{N}}$ by its general terms as follows:

$$\begin{aligned} x_n &= T_{\{j_\ell\}}^{pn+1} x \longrightarrow \widehat{T}^{pn+1} x \longrightarrow \widehat{T}\overline{x}_i = \widehat{T}^{pn}\overline{x}_i \quad \text{as } n \longrightarrow \infty, \\ y_n &= T_{\{j_\ell\}}^{pn} x \longrightarrow \widehat{T}^{pn} x \longrightarrow \overline{x}_i = \widehat{T}^{pn}\overline{x}_i \quad \text{as } n \longrightarrow \infty, \\ z_n &= T_{\{j_\ell\}}^{pn+1} y \longrightarrow \widehat{T}^{pn}\overline{y}_i = \overline{y}_i, \end{aligned} \tag{2.32}$$

generated by a uniformly convergent mapping $T_{\{j_\ell\}}^{pn+k} : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} (\bigcap_{j=1}^{j_{\overline{n}_2}} \widehat{A}_{ij}), k \in \overline{p-1} \cup \{0\}$ for some $x, y(\neq x) \in A_i$, some $i \in \overline{p}$ as the integer sequences$j_\ell, j_{\overline{n}_2} \to \infty$. The limits above exist from Property (iii). Also, one gets from Property (i), (2.23), and Lemma 3.8 of [4], since X is uniformly convex and $(\bigcap_{j=1}^{\infty} \widehat{A}_{ij})$ are nonempty and closed $\forall i \in \overline{p}$, that

$$\begin{aligned} &\left(d\left(T_{\{j_\ell\}}^{pn+1} x, T_{\{j_\ell\}}^{pn} x \right) \longrightarrow d\left(\widehat{T}^{pn+1} x, \widehat{T}^{pn} x \right) \longrightarrow d\left(\overline{x}_i, \widehat{T}\overline{x}_i \right) = D \text{ as } n \longrightarrow \infty \right) \\ &\wedge \left(d\left(T_{\{j_\ell\}}^{pn+1} y, T_{\{j_\ell\}}^{pn} x \right) \longrightarrow d\left(\widehat{T}^{pn+1} y, \widehat{T}^{pn} x \right) \longrightarrow d\left(\overline{x}_i, \widehat{T}\overline{y}_i \right) = D \text{ as } n \longrightarrow \infty \right) \end{aligned} \tag{2.33}$$

$$\Longrightarrow \left(d\left(T_{\{j_\ell\}}^{pn} x,\ T_{\{j_\ell\}}^{pn} y \right) \longrightarrow d\left(\widehat{T}^{pn} x, \widehat{T}^{pn} y \right) \longrightarrow d(\overline{x}_i, \overline{y}_i) = 0 \right), \tag{2.34}$$

since the uniformly convex space $(X, \|\cdot\|)$ with a norm $\|\cdot\|$ is a Banach space so that we could rewrite the above constraints by replacing the metric $d : X \times X \to \mathbf{R}_{0+}$ by a metric $d_1 : X \times X \to \mathbf{R}_{0+}$, induced by $\|\cdot\|$, which is always equivalent to $d : X \times X \to \mathbf{R}_{0+}$ even in the case that both metric functions are not coincident. Equation (2.34) contradicts $\overline{x}_i \neq \overline{y}_i$, since there is no $n_0 = n_0(\varepsilon)$ for any given $\varepsilon \in \mathbf{R}_+$ such that $d(T_{\{j_\ell\}}^{pn} x, T_{\{j_\ell\}}^{pn} y) \leq \varepsilon, \forall n \geq n_0$ implying that $d(T_{\{j_\ell\}}^{pm} x, T_{\{j_\ell\}}^{pm} y) > \varepsilon$ for some $m \geq n_0$. Thus, (2.33) is false unless $\overline{x}_i = \overline{y}_i; \forall i \in \overline{p}$. It follows also as a result that the p best proximity points $\overline{x}_i \in (\bigcap_{j=1}^{\infty} \widehat{A}_{ij}), \forall i \in \overline{p}$ satisfy $\overline{x}_{i+j} = T^j \overline{x}_i$and $d(\overline{x}_i, \widehat{T}\overline{x}_i) = d(\widehat{T}^j \overline{x}_i, \widehat{T}^{j+1}\overline{x}_i) = d(\overline{x}_{i+j}, \overline{x}_{i+j+1}) = D, \forall i, j \in \overline{p}$ with $\overline{x}_{i+np} = \overline{x}_i, \forall n \in \mathbf{N}_0, \forall i \in \overline{p}$. □

Note that the existence of nonempty subsets $\widehat{A}_i \subseteq A_i, \forall i \in \overline{p}$, in Theorem 2.5 are guaranteed if A_i are bounded (although nonnecessarily closed) and $D_i \leq D \leq D_i + \operatorname{diam} A_i$; $\forall i \in \overline{p}$. Note also that $\prod_{i=1}^{p} [K_i + \omega_i] \geq 1$ is guaranteed since $\omega_i := D/D_i - K_i \geq 1 - K_i, \forall i \in \overline{p}$.

3. Links with General Kannan Mappings and p-Cyclic Kannan Mappings

The next result is concerned with p-cyclic Kannan self-mappings [7, 11, 12] which can eventually satisfy contractive conditions within the class (2.1).

Theorem 3.1. *Assume that a $p(\geq 2)$-cyclic self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ satisfies the contractive condition* (2.1) *with $\alpha_i = 0, D_i = D; \forall i \in \overline{p}$. Then, the following properties hold.*

(i) *If*

$$\gamma_i := \frac{1}{1-2\eta_i-\mu_i}\left[\eta_i+\mu_i+\frac{\beta_{1i}+\beta_{2i}}{2}+\max\left(\beta_{1i}+\gamma_{1i}+\delta_{1i},\beta_{2i}+\gamma_{2i}+\delta_{2i}\right)\right] < \frac{1}{2}, \quad \forall i \in \overline{p}, \tag{3.1}$$

and, furthermore, $\omega_i = (2\eta_i + \mu_i)(1 - K_i)$ with the constants K_i defined in (2.4)*, $\forall i \in \overline{p}$ then, $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic Kannan self-mapping.*

(ii) *If, in addition,* (2.9) *is replaced by the constraints*

$$\beta_{2i}+\gamma_{1i}+\gamma_{2i}+3\left(\delta_{1i}+\delta_{2i}+\mu_i+2\left(\beta_{1i}+\eta_i\right)\right) < 1, \quad \forall i \in \overline{p}, \tag{3.2}$$

and, furthermore,

$$\begin{gathered} 2\eta_i+\mu_i < 1, \quad \forall i \in \overline{p}, \\ \frac{\delta_{1i}+\mu_i+\gamma_{1i}+\beta_{1i}+\eta_i}{1-2\beta_{1i}-\beta_{2i}-\delta_{1i}-\delta_{2i}-\gamma_{2i}-2\eta_i-\mu_i-\gamma_{1i}} \\ \leq \gamma_i := \frac{1}{1-2\eta_i-\mu_i}\left[\eta_i+\mu_i+\frac{\beta_{1i}+\beta_{2i}}{2}+\max\left(\beta_{1i}+\gamma_{1i}+\delta_{1i},\beta_{2i}+\gamma_{2i}+\delta_{2i}\right)\right] < \frac{1}{2}, \quad \forall i \in \overline{p} \end{gathered} \tag{3.3}$$

then

$$\lim_{n\to\infty} d\left(T^{pn+1}x, T^{pn}x\right) = D, \quad \forall x \in A_i, \ \forall i \in \overline{p}, \tag{3.4}$$

which is guaranteed by the condition (2.3)*. If the p subsets A_i of $X, \forall i \in \overline{p}$, have nonempty intersection and if the metric space (X,d) is complete, then $\lim_{n\to\infty} d(T^{pn+1}x, T^{pn}x) = 0; \forall x \in \bigcup_{i\in\overline{p}} A_i$ and $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ have a unique fixed point in $\bigcap_{i\in\overline{p}} A_i$ to which all the sequences $\{T^n x\}_{n\in\mathbf{N}_0}$, which are then bounded, converge; $\forall x \in \bigcup_{i\in\overline{p}} A_i$. If the above p subsets of X are disjoint, closed, and convex, $\forall i \in \overline{p}$, X is uniformly convex and $D > 0$ then all sequences $\{T^{pn}x\}_{n\in\mathbf{N}_0}, \forall x \in A_i$ converge to a best proximity point $\overline{x}_i$ of A_i and $\overline{x}_{i+1} = T\overline{x}_i = T^p\overline{x}_i, \forall i \in \overline{p}$.*

Proof. If $\alpha_i = 0, \forall i \in \overline{p}$, then the contractive condition (2.1) may be upper-bounded as follows for all $\forall x, y(\neq x) \in A_i$; any $i \in \overline{p}$ by using the triangle inequality where necessary and upper-bounding the necessary fractions by unity when the denominator is not less than the numerator in the right-hand side of (2.1):

$$\begin{aligned} d(Tx,Ty) \leq{}& (2\eta_i+\mu_i)d(Tx,Ty)+\left(\beta_{1i}+\frac{\beta_{1i}+\beta_{2i}}{2}+\gamma_{1i}+\eta_i+\mu_i\right)d(x,Tx) \\ &+\left(\beta_{2i}+\frac{\beta_{1i}+\beta_{2i}}{2}+\gamma_{2i}+\delta_{2i}+\eta_i+\mu_i\right)d(y,Ty)+\omega_i D, \end{aligned} \tag{3.5}$$

and, since $2\eta_i + \mu_i < 1$, one gets

$$d(Tx, Ty) \le \gamma_i(d(x, Tx) + d(y, Ty)) + \omega_i D_i, \quad \forall x, y(\ne x) \in A_i, \ \forall i \in \overline{p}, \tag{3.6}$$

with $\gamma_i \in [0, 1/2), \forall i \in \overline{p}$. Thus, $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ is a p-cyclic Kannan self-mapping. Hence, Property (i).

Furthermore, since (2.9) is replaced by (3.1) for $\alpha_i = 0$ so that if $K_i \in [0, 1/3)$ in (2.4) and $\omega_i = (2\eta_i + \mu_i)(1 - K_i), \forall i \in \overline{p}$ in (2.1), it follows from (3.5) that (3.1)–(3.3) guarantee that $K_i/(1 - K_i) \le \gamma_i < 1/2$ under the necessary condition $K_i \in [0, 1/3), \forall i \in \overline{p}$, [11, 12]. Then, the self-mapping $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ satisfies the following contractive condition for $y = Tx(\ne x) \in A_i$ and any $i \in \overline{p}$:

$$\begin{aligned} &d(Tx, Ty) \le \gamma_i(d(x, Tx) + d(y, Ty)) \\ &d(Tx, Ty) \le K_i d(x, y) \le K_i d(Tx, Ty) + K_i(d(x, Tx) + d(y, Ty)) \\ &\qquad\qquad \le \gamma_i(d(x, Tx) + d(y, Ty)); \quad \forall x, y(\ne x) \in A_i; \ \forall i \in \overline{p} \\ \Longrightarrow\ &d(Tx, Ty) \le \frac{K_i}{1 - K_i}(d(x, Tx) + d(y, Ty)). \end{aligned} \tag{3.7}$$

Hence, Property (ii) follows directly from Theorem 2.1. □

Remark 3.2. Note that there are several contractive conditions discussed in the literature for the noncyclic ($p = 1$) case and eventually for the cyclic ($p \ge 2$) case which are particular cases of (2.1), under (2.9), as follows.

(1) If $\mu_1 \in [0, 1)$ while all the remaining constants are zero, the contractive condition reduces to that of Banach contraction principle, [13]. A similar extended condition for $i \in \overline{p}$, $p \ge 2$ relies on Banach contraction principle for p-cyclic self-mappings.

(2) If $\alpha_1 \ne 0$ and $\mu_1 \in [0, 1)$, then the contractive condition (2.1) is of the same type as that of [2] and generalizes it.

(3) If $\delta = \delta_{i1} \ne 0$ $(i = 1, 2)$, $\mu_1 \ne 0$ with appropriate constraints and the remaining constants being zero then the contractive condition is that proposed by Chaterjee [14].

(4) If $\eta_1 \ne 0, \mu_1 \ne 0$, with appropriate constraints and the remaining constants being zero, then the contractive condition is that proposed by Fisher [15].

(5) If $1/2 > \delta = \delta_{i1} \ne 0$ $(i = 1, 2)$ with the remaining constants being zero, then the contractive condition is that proposed by Kannan, [16, 17], recently generalized in [12]. See also [7, 11]. A similar extended condition for $i \in \overline{p}, p \ge 2$ relies on Banach contraction principle for p-cyclic Kannan self-mappings which can be simultaneously contractive [7].

(6) $\delta_1 \ne 0, \eta_1 \ne 0, \mu_1 \ne 0$ with appropriate constraints and the remaining constants being zero, then the contractive condition is that proposed by Reich [18].

(7) $\gamma_i = \beta_2 = 0$ for $i = 1, 2$ and the remaining constants are either zero or nonzero then the contractive condition is that of Bhardwaj et al. [1].

(8) The contractive condition also can include a generalization of Kannan's fixed point theorem due to Kikkawa and Syuzuki [19], reported in Enjouji et al. [12] if an

implication is established to get Kannan's condition supported by the use of a non-increasing real function $\varphi : [0,1/2) \to (1/2,1]$ defined as:

$$\varphi(\delta) := \begin{cases} 1, & \text{if } 0 \le \delta < \sqrt{2}-1, \\ 1-\alpha, & \text{if } \sqrt{2}-1 < \delta \le \dfrac{1}{2}, \end{cases} \tag{3.8}$$

so that

$$\varphi(\delta) d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le \delta\big(d(x,Tx) + d(y,Ty)\big), \quad \forall x,y \in X. \tag{3.9}$$

This can be addressed with (2.1) as follows. Take $\alpha \neq 0$ and $1/2 > \delta = \delta_{i1} \neq 0 (i = 1,2)$ and all the remaining constants in (2.1) being zero by defining

$$\varphi(\delta) = \frac{\delta}{1-\alpha(\varepsilon)-\delta} = \begin{cases} 1, & \text{if } 0 \le \delta < \delta_0, \\ 1-\delta+\varepsilon, & \text{if } \delta_0 < \delta \le \dfrac{1}{2}, \end{cases} \tag{3.10}$$

where $\delta_0 = \sqrt{2}-1$ subject to the constraints $\delta \in (0,1/2)$, and

$$\alpha = 1-2\delta \quad \text{if } 0 \le \delta < \sqrt{2}-1, \tag{3.11}$$

$$\alpha = \alpha(\varepsilon) = 1-\delta_0 + \frac{\delta_0+\varepsilon-1}{\delta_0} \quad \text{if } \delta_0 < \delta \le \frac{1}{2} \quad \text{with } \varepsilon \ge 1-\delta_0(2-\delta_0). \tag{3.12}$$

(9) If $p \ge 2, \mu_i \in [0,1)$ and $\omega_i = 1-\mu_i, \forall i \in \overline{p}$, then the contractive condition is a p-cyclic contractive condition, [3–7].

Remark 3.3. An important property of Kannan mappings on a metric space (X,d) is that it is complete if and only if every Kannan mapping on X has a fixed point, [12, 19, 20].

The first property of the following result follows directly from the contractive condition (2.1) and Remark 3.2.7 by using the fixed point result of [19]. The second part follows directly from Theorem 3.1 by taking the particular condition for Kannan $p(\ge 2)p$-cyclic mappings which are simultaneously contractive (see Remarks 3.2.4 and 3.2.8).

Theorem 3.4. *The following properties hold.*

(i) *Let T be a self-mapping on a complete metric space (X,d) and that* (3.9)–(3.12) *hold for all $x,y \in X$ with $\alpha > 0$ and $1/2 > \delta > 0 (i = 1,2)$. Then, T has a fixed point $\overline{x} \in X$ satisfying $\overline{x} = \lim_{n\to\infty} T^n x, \forall x \in X$.*

(ii) *Let $T : \bigcup_{i\in\overline{p}} A_i \to \bigcup_{i\in\overline{p}} A_i$ be a $p(\ge 2)$-cyclic self-mapping, satisfying $T(A_i) \subseteq A_{i+1}, \forall i \in \overline{p}$, with p being nonempty closed subsets A_i of X, such that (X,d) is a complete metric space and $A_{pj+\ell} \equiv A_\ell, \forall \ell \in \overline{p-1}$, which all intersect, $\forall i \in \overline{p}$ satisfying the following particular contractive condition in* (2.1)*: $\alpha_i > 0; 1/2 > \delta_i = \delta_{ji} \neq 0\, (j = 1,2; i \in \overline{p})$ and the remaining constants are zero* *with* $K_i := 2\delta_i/(1-\alpha_i-\delta_i) < 1/3$. *Then,* $T : \bigcup_{i\in\overline{p}} A_i \rightarrow \bigcup_{i\in\overline{p}} A_i$ *is simultaneously a contractive and Kannan p-cyclic self-mapping which has a unique fixed point* $\overline{x} = \lim_{n\rightarrow\infty} T^n x$ *in* $\bigcap_{i\in\overline{p}} A_i, \forall x \in \bigcup_{i\in\overline{p}} A_i$.

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