Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 540274, 10 pages
doi:10.1155/2011/540274

Research Article

Stability of the Pexiderized Lobacevski Equation

Gwang Hui Kim

Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea

Correspondence should be addressed to Gwang Hui Kim, ghkim@kangnam.ac.kr

Received 16 April 2011; Accepted 11 June 2011

Academic Editor: Junjie Wei

The aim of this paper is to investigate the solution and the superstability of the Pexiderized Lobacevski equation $f((x+y)/2)^2 = g(x)h(y)$, where $f, g, h : G^2 \to \mathbb{C}$ are unknown functions on an Abelian semigroup $(G, +)$. The obtained result is a generalization of Găvruţa's result in 1994 and Kim's result in 2010.

1. Introduction

The stability problem of the functional equation was conjectured by Ulam [1] during the conference in the University of Wisconsin in 1940. In the next year, it was solved by Hyers [2] in the case of additive mapping, which is called the Hyers-Ulam stability. Thereafter, this problem was improved by Bourgin [3], Aoki [4], Rassias [5], Ger [6], and Găvruţa et al. [7, 8] in which Rassias' result is called the Hyers-Ulam-Rassias stability.

In 1979, Baker et al. [9] developed the superstability, which is that if f is a function from a vector space to $\mathbb{R}$ satisfying

$$\left| f(x+y) - f(x)f(y) \right| \le \varepsilon \tag{1.1}$$

for some fixed $\varepsilon > 0$, then either f is bounded or satisfies the exponential functional equation

$$f(x+y) = f(x)f(y). \tag{E}$$

In 1983, the superstability bounded by a constant for the sine functional equation

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = f(x)f(y) \tag{S}$$

was investigated by Cholewa [10] and was improved by Badora and Ger [11]. Recently, the superstability bounded by some function for the Pexider type sine functional equation

$$f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 = g(x)h(y) \tag{1.2}$$

has been investigated by Kim [12, 13].

In 1994, Găvruţa [14] proved the superstability of the Lobacevski equation

$$f\left(\frac{x+y}{2}\right)^2 = f(x)f(y) \tag{L}$$

under the condition bounded by a constant.

Kim [15] improved his result under the condition bounded by an unknown function. In there, author conjectured through an example that the Lobacevski equation (L) will have a solution as an exponential function. Namely, for a simple example of this equation, we can find the functional equation $(e^{(x+y)/2})^2 = e^x e^y$.

The aim of this paper is to investigate the solution and the superstability of the Pexiderized Lobacevski equation

$$f\left(\frac{x+y}{2}\right)^2 = g(x)h(y) \tag{PL}$$

under the condition bounded by a function. Namely, this has improved in the Pexider type for the results of Găvruţa and Kim.

Furthermore, the range of the function in all results is expanded to the Banach space.

The solution of (PL) will be represented as an exponential, namely, for a simple example of this equation, it will be considered as a geometric mean

$$f(x) = \sqrt{\alpha\beta}e^x = \sqrt{(\alpha e^x)(\beta e^x)} = \sqrt{g(x)h(x)}, \quad \text{where } \alpha, \beta > 0. \tag{1.3}$$

In this paper, let $(G, +)$ be a uniquely 2-divisible Abelian semigroup (i.e., for each $x \in G$, there exists a unique $y \in G$ such that $y + y = x$: such y will be denoted by $x/2$), $\mathbb{C}$ is the field of complex numbers, $\mathbb{R}$ the field of real numbers, and $\mathbb{R}_+$ the set of positive reals. We assume that $f, g, h : G \to \mathbb{C}$ are nonzero and nonconstant functions, ε is a nonnegative real constant, and $\varphi : G \to \mathbb{R}_+$ is a mapping.

2. Stability of the Pexiderized Lobacevski Equation (PL)

We will investigate the solution and the superstability of the Pexiderized Lobacevski equation (PL).

Theorem 2.1. *Suppose that* $f, g, h : G \to \mathbb{R}$ *satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g(x)h(y) \right| \le \varepsilon \tag{2.1}$$

for all $x, y \in G$.

Then, either there exist $C_1, C_2, C_3 > 0$ *such that*

$$|g(x)| \le C_1, \qquad |h(x)| \le C_2, \qquad |f(x)| \le C_3, \tag{2.2}$$

for all $x \in G$, *or else each function* g *and* h *satisfies* (L). *Here* g *and* h *are represented by*

$$g(x) = g(0)e(x), \qquad h(x) = h(0)e(x), \tag{2.3}$$

$$\left| f(x)^2 - g(0)h(0) \cdot e(x)^2 \right| \le \varepsilon, \tag{2.4}$$

where $e(x)$ *is an exponential function.*

Proof. Replacing x with y in (2.1), and then subtracting them and using triangle inequality, we have

$$|g(x)h(y) - g(y)h(x)| \le 2\varepsilon \quad \forall x, y \in G. \tag{2.5}$$

It follows from the inequality (2.5) that there exist constants $c_1, c_2, d_1, d_2 \ge 0$ such that

$$|g(x)| \le c_1|h(x)| + d_1, \tag{2.6}$$

$$|h(x)| \le c_2|g(x)| + d_2 \tag{2.7}$$

for all $x \in G$. It follows from (2.6) and (2.7) that g is bounded if and only if h is bounded. If either of g or h is bounded, then we obtain (2.2) from (2.1).

Now if $h(x)$ is unbounded, then we can choose $(y_n) \in G$ so that $|h(y_n)| \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (2.1), dividing by $|h(y_n)|$, and letting $n \to \infty$, we have

$$g(x) = \lim_{n\to\infty} \frac{f((x+y_n)/2)^2}{h(y_n)}, \quad \forall x \in G. \tag{2.8}$$

It follows from (2.1) and (2.8) that

$$\begin{aligned} g(x+y)g(z) &= \lim_{n\to\infty} \frac{f((x+y+y_n)/2)^2 g(z)}{h(y_n)} = \lim_{n\to\infty} \frac{g(x)h(y+y_n)g(z) + R_1}{h(y_n)} \\ &= \lim_{n\to\infty} \frac{g(x)f((y+z+y_n)/2)^2 + R_1 + R_2}{h(y_n)} = g(x)g(y+z) + \lim_{n\to\infty} \frac{R_1 + R_2}{h(y_n)}, \end{aligned} \tag{2.9}$$

where $|R_1| \le \varepsilon|g(z)|$, $|R_2| \le \varepsilon|g(x)|$, which implies

$$g(x+y)g(z) = g(x)g(y+z) \tag{2.10}$$

for all $x, y, z \in G$.

Letting $z = 0$ in (2.10), we get

$$g(x+y)g(0) = g(x)g(y) \tag{2.11}$$

for all $x, y \in G$, which implies that

$$g(x) = g(0)e_1(x), \tag{2.12}$$

where $g(0) \neq 0$ (since $g(x)$ is a nonzero and nonconstant function), and e_1 is an exponential.

Exchanging the roles of g and h, by the same proceeding, we have

$$h(x) = h(0)e_2(x), \tag{2.13}$$

where $h(0) \neq 0$, and e_2 is an exponential.

Putting (2.12) and (2.13) in (2.5), it implies

$$\left|e_1(x)e_2(y) - e_1(y)e_2(x)\right| \le \frac{2\varepsilon}{g(0)h(0)} = M \quad \forall x, y \in G. \tag{2.14}$$

Let $x = 0$ in (2.14). Since e_1 and e_2 are exponentials, this implies that $|e_1(y) - e_2(y)| \le M$ for all $y \in G$. Hence, from this and (2.14), we have

$$\begin{aligned} e_1(y)|e_1(x) - e_2(x)| &= \left|e_1(x)\left[e_1(y) - e_2(y)\right] + e_1(x)e_2(y) - e_1(y)e_2(x)\right| \\ &\le e_1(x)M + M, \end{aligned} \tag{2.15}$$

which is

$$|e_1(x) - e_2(x)| \le \frac{e_1(x)M + M}{e_1(y)}, \tag{2.16}$$

for all $x, y \in G$.

Since g is unbounded from (2.2), we can choose $(y_n) \in G$ so that $g(y_n) = g(0)e_1(y_n) \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (2.16), we get that $e_1(x) = e_2(x)$. Let it be denoted by $e(x)$. Then (2.12) and (2.13) state nothing but (2.3). Putting (2.3) with $x = y$ in (2.1), we get the inequality (2.4).

Finally, it is immediate that g and h in (2.3) satisfy (L), respectively. □

Corollary 2.2. *Suppose that* $f, g : G \to \mathbb{R}$ *satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g(x)g(y) \right| \le \varepsilon \tag{2.17}$$

for all $x, y \in G$.

Then, either g *is bounded or* g *satisfies* (L). *In particular,* g *is represented by*

$$g(x) = g(0)e(x), \tag{2.18}$$

where e *is exponential.*

Corollary 2.3. *Suppose that* $f, g : G \to \mathbb{R}$ *satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g(x)f(y) \right| \le \varepsilon \tag{2.19}$$

for all $x, y \in G$.

Then either there exist $C_1, C_2 > 0$ *such that*

$$|g(x)| \le C_1, \qquad |f(x)| \le C_2 \tag{2.20}$$

for all $x \in G$, *or else each function* f *and* g *satisfies* (L). *In particular,* f *and* g *are represented by*

$$f(x) = f(0)e(x), \qquad g(x) = g(0)e(x), \tag{2.21}$$

where $e : G \to \mathbb{R}$ *is exponential.*

Corollary 2.4. *Suppose that* $f : G \to \mathbb{R}$ *satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - f(x)f(y) \right| \le \varepsilon \tag{2.22}$$

for all $x, y \in G$.

Then either f *is bounded or* f *satisfies* (L). *In particular,* f *is represented by*

$$f(x) = f(0)e(x), \tag{2.23}$$

where $e : G \to \mathbb{R}$ *is exponential.*

In Corollary 2.4, it is founded in papers [14, 15] that f satisfies (L).

Theorem 2.5. *Suppose that $f, g, h : G \to \mathbb{R}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g(x)h(y) \right| \le \varphi(x) \tag{2.24}$$

for all $x, y \in G$.

Then, either h is bounded or g is an exponential by the multiplying of a scalar $g(0)$ and satisfies (L).

Proof. Suppose that $h(x)$ is unbounded. Then we can choose $(y_n) \in G$ such that $|h(y_n)| \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (2.24), dividing by $|h(y_n)|$, and letting $n \to \infty$, we have

$$g(x) = \lim_{n\to\infty} \frac{f(x+y_n)}{h(y_n)} \quad \forall x \in G. \tag{2.25}$$

Thus, it follows from (2.24) and (2.25) that

$$\begin{aligned} g(x+y)g(z) &= \lim_{n\to\infty} \frac{f((x+y+y_n)/2)^2 g(z)}{h(y_n)} = \lim_{n\to\infty} \frac{g(x)h(y+y_n)g(z) + R_1}{h(y_n)} \\ &= \lim_{n\to\infty} \frac{g(x)f((y+z+y_n)/2)^2 + R_1 + R_2}{h(y_n)} = g(x)g(y+z) + \lim_{n\to\infty} \frac{R_1+R_2}{h(y_n)}, \end{aligned} \tag{2.26}$$

where $|R_1| \le \varepsilon |g(z)|$, $|R_2| \le \varepsilon |g(x)|$, which implies

$$g(x+y)g(z) = g(x)g(y+z) \tag{2.27}$$

for all $x, y, z \in G$.

Letting $z = 0$ in (2.27), we get

$$g(x+y)g(0) = g(x)g(y) \tag{2.28}$$

for all $x, y, z \in G$. Namely, it means that g is an exponential function by the multiplying of a scalar $g(0)$ and satisfies (L). □

Theorem 2.6. *Suppose that $f, g, h : G \to \mathbb{R}$ satisfy the inequality*

$$\left| f\left(\frac{x+y}{2}\right)^2 - g(x)h(y) \right| \le \varphi(y) \tag{2.29}$$

for all $x, y \in G$.

Then, either g is bounded or h is an exponential by the multiplying of a scalar $h(0)$ and satisfies (L).

Proof. The proof runs along a slight change in the step-by-step procedure in Theorem 2.5. □

Remark 2.7. (i) As Corollaries 2.2–2.4 of Theorem 2.1, by replacing g and h with f in Theorems 2.5 and 2.6, we can obtain more corollaries for the following functional equations:

$$f\left(\frac{x+y}{2}\right)^2 = g(x)g(y),$$
$$f\left(\frac{x+y}{2}\right)^2 = f(x)g(y), \tag{2.30}$$
$$f\left(\frac{x+y}{2}\right)^2 = f(x)f(y),$$

in which the case of (2.30) is found in paper [15].

(ii) For the results obtained from each equation of the above (i), by applying $\varphi(y) = \varphi(x) = \varepsilon$, we can obtain the same number of corollaries.

3. Extension to Banach Algebra

All obtained results can be extended to the stability on the Banach algebras. We will illustrate only for the case of Theorem 2.1 among them.

Theorem 3.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach algebra. Assume that $f, g, h : G \to E$ satisfy the inequality*

$$\left\| f\left(\frac{x+y}{2}\right)^2 - g(x)h(y) \right\| \le \varepsilon \tag{3.1}$$

for all $x, y \in G$.

Then, for an arbitrary linear multiplicative functional $x^* \in E^*$, either there exist $C_1, C_2, C_3 > 0$ such that

$$\left|(x^* \circ g)(x)\right| \le C_1, \qquad |(x^* \circ h)(x)| \le C_2, \qquad \left|(x^* \circ f)(x)\right| \le C_3 \tag{3.2}$$

for all $x \in G$, or else each function g and h satisfies (L). *Here g and h are represented by*

$$g(x) = g(0)e(x), \qquad h(x) = h(0)e(x), \tag{3.3}$$
$$\left| f(x)^2 - g(0)h(0) \cdot e(x)^2 \right| \le \varepsilon, \tag{3.4}$$

where $e(x)$ is an exponential function.

Proof. Assume that (3.1) holds, and fix arbitrarily a linear multiplicative functional $x^* \in E$. As well known, we have $\|x^*\| = 1$, hence, for every $x, y \in G$, we have

$$\begin{aligned} \varepsilon &\geq \left\| f\left(\frac{x+y}{2}\right)^2 - g(x)h(y) \right\| \\ &= \sup_{\|y^*\|=1} \left| y^*\left(f\left(\frac{x+y}{2}\right)^2 - g(x)h(y) \right) \right| \\ &\geq \left| x^*\left(f\left(\frac{x+y}{2}\right)\right) - x^*(g(x))x^*(h(y)) \right|, \end{aligned} \tag{3.5}$$

which states that the superpositions $x^* \circ f$, $x^* \circ g$, and $x^* \circ h$ satisfy the inequality (2.1) of Theorem 2.1. Due to the same processing as from (2.5) to (2.7), to fix arbitrarily a linear multiplicative functional $x^* \in E$, indeed, we have

$$\left|(x^* \circ g)(x)(x^* \circ h)(y) - (x^* \circ g)(y)(x^* \circ h)(x)\right| \leq 2\varepsilon \quad \forall x, y \in G. \tag{3.6}$$

It follows from the inequality (3.6) that there exist constants $c_1, c_2, d_1, d_2 \geq 0$ such that

$$\begin{aligned} \left|(x^* \circ g)(x)\right| &\leq c_1|(x^* \circ h)(x)| + d_1, \\ |(x^* \circ h)(x)| &\leq c_2\left|(x^* \circ g)(x)\right| + d_2 \end{aligned} \tag{3.7}$$

for all $x \in G$. Since x^* is an arbitrarily linear multiplicative functional, it follows from (3.7) that g is bounded if and only if h is bounded. Assume that one of g or h is bounded. From (3.1), we arrive at (3.2).

By the assumption (3.2), an appeal to Theorem 2.1 shows that

$$(x^* \circ g)(x) = (x^* \circ g(0)e_1)(x), \tag{3.8}$$

$$(x^* \circ h)(x) = (x^* \circ h(0)e_2)(x), \tag{3.9}$$

$$\left|(x^* \circ f)(x) - (x^* \circ g(0)h(0)e_3)(x)\right| \leq \varepsilon, \tag{3.10}$$

where $e_1, e_2, e_3 : G \to \mathbb{R}$ are exponentials. In other words, bearing the linear multiplicativity of x^* in mind, for all $x \in G$, each difference derived from (3.8) and (3.9)

$$\begin{aligned} D(3.8)(x) &:= g(x) - (g(0)e_1)(x), \\ D(3.9)(x) &:= h(x) - (h(0)e_2)(x), \end{aligned} \tag{3.11}$$

falls into the kernel of x^*. Therefore, in view of the unrestricted choice of x^*, we infer that

$$D(3.8)(x), D(3.9)(x) \in \bigcap \{\ker x^* : x^* \text{ is a multiplicative member of } E^*\} \tag{3.12}$$

for all $x \in G$. Since the algebra E has been assumed to be semisimple, the last term of the previous formula coincides with the singleton $\{0\}$, that is,

$$g(x) - g(0)e_1(x) = 0, \quad h(x) - h(0)e_2(x) = 0, \quad x \in G. \tag{3.13}$$

Putting (3.13) in (3.6), following the same proceeding as after (2.13) in Theorem 2.1, then we arrive that $e_1(x) = e_2(x)$. Indeed, we have

$$\left|(x^* \circ g(0)e_1)(x)(x^* \circ h(0)e_2)(y) - (x^* \circ g(0)e_1)(y)(x^* \circ h(0)e_2)(x)\right| \leq 2\varepsilon \tag{3.14}$$

for all $x, y \in G$. This implies that

$$\left|(x^* \circ e_1)(x)(x^* \circ e_2)(y) - (x^* \circ e_1)(y)(x^* \circ e_2)(x)\right| \leq \frac{2\varepsilon}{g(0)h(0)} = M \quad \forall x, y \in G. \tag{3.15}$$

Letting $x = 0$ in (3.15), it implies $|(x^* \circ e_2)(y) - (x^* \circ e_1)(y)| \leq M/x^*(1) = M'$ for all $y \in G$. Thus, from this and (3.15), we have

$$\begin{aligned}
&\left|(x^* \circ e_1)(y)\right| |(x^* \circ e_1)(x) - (x^* \circ e_2)(x)| \\
&\quad = \left|(x^* \circ e_1)(x)\left[(x^* \circ e_1)(y) - (x^* \circ e_2)(y)\right]\right. \\
&\qquad \left. +(x^* \circ e_1)(x)(x^* \circ e_2)(y) - (x^* \circ e_1)(y)(x^* \circ e_2)(x)\right| \\
&\quad \leq |(x^* \circ e_1)(x)|M' + M,
\end{aligned} \tag{3.16}$$

which is

$$|(x^* \circ e_1)(x) - (x^* \circ e_2)(x)| \leq \frac{|(x^* \circ e_1)(x)|M' + M}{\left|(x^* \circ e_1)(y)\right|}, \tag{3.17}$$

for all $x, y \in G$.

Since $x^* \circ g$ is unbounded from (3.2), we can choose $(y_n) \in G$ so that $|(x^* \circ g)(y_n)| = |g(0)(x^* \circ e_1)(y_n)| \to \infty$ as $n \to \infty$. Letting $y = y_n$ in (3.17), which arrive that

$$(x^* \circ e_1)(x) = (x^* \circ e_2)(x). \tag{3.18}$$

Using the same logic as before, that is, bearing the linear multiplicativity of x^* in mind, the difference derived from (3.18), $D(3.18)(x) := e_1(x) - e_2(x)$, falls into the kernel of x^*. Then, the semisimplicity of E implies that $e_1(x) = e_2(x)$. Let it be denoted by $e(x)$, which arrive the claimed (3.3) and (3.4).

Since $e(x) : G \to \mathbb{R}$ is exponential, it is immediate from (3.3) that each function g and h satisfies (L). □

Remark 3.2. All results of Section 2 containing Remark 2.7 can be extended to the Banach space as Theorem 3.1.

Acknowledgment

This work was supported by Kangnam University Research Grant in 2010.

References

[1] S. M. Ulam, *Problems in Modern Mathematics*, chapter 6, John Wiley & Sons, New York, NY, USA, 1964.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[3] D. G. Bourgin, "Multiplicative transformations," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 36, pp. 564–570, 1950.
[4] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[5] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[6] R. Ger, "Superstability is not natural," *Rocznik Naukowo-Dydaktyczny*, vol. 159, no. 13, pp. 109–123, 1993.
[7] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[8] M. Frank, P. Găvruţa, and M. S. Moslehian, "Superstability of adjointable mappings on Hilbert C^*-modules," *Applicable Analysis and Discrete Mathematics*, vol. 3, no. 1, pp. 39–45, 2009.
[9] J. Baker, J. Lawrence, and F. Zorzitto, "The stability of the equation $f(x+y) = f(x)f(y)$," *Proceedings of the American Mathematical Society*, vol. 74, no. 2, pp. 242–246, 1979.
[10] P. W. Cholewa, "The stability of the sine equation," *Proceedings of the American Mathematical Society*, vol. 88, no. 4, pp. 631–634, 1983.
[11] R. Badora and R. Ger, "On some trigonometric functional inequalities," in *Functional Equations—Results and Advances*, vol. 3, pp. 3–15, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.
[12] G. H. Kim, "A stability of the generalized sine functional equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 886–894, 2007.
[13] G. H. Kim, "On the stability of the generalized sine functional equations," *Acta Mathematica Sinica*, vol. 25, no. 1, pp. 29–38, 2009.
[14] P. Găvruţa, "On the stability of some functional equations," in *Stability of Mappings of Hyers-Ulam Type*, pp. 93–98, Hadronic Press, Palm Harbor, Fla, USA, 1994.
[15] G. H. Kim, "Stability of the Lobacevski equation," *Journal of Nonlinear Science and its Applications*, vol. 4, no. 1, pp. 11–18, 2011.