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Research Article

Sharp Estimates of m-Linear p-Adic Hardy and Hardy-Littlewood-Pólya Operators

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The sharp estimates of the m-linear p-adic Hardy and Hardy-Littlewood-Pólya operators on Lebesgue spaces with power weights are obtained in this paper.

1. Introduction

In recent years, p-adic numbers are widely used in theoretical and mathematical physics (cf. [1–8]), such as string theory, statistical mechanics, turbulence theory, quantum mechanics, and so forth.

For a prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers. It is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the non-Archimedean p-adic norm $|\cdot|_p$. This norm is defined as follows: $|0|_p = 0$; If any nonzero rational number x is represented as $x = p^\gamma(m/n)$, where m and n are integers which are not divisible by p and γ is an integer, then $|x|_p = p^{-\gamma}$. It is not difficult to show that the norm satisfies the following properties:

$$|xy|_p = |x|_p|y|_p, \qquad |x+y|_p \le \max\left\{|x|_p, |y|_p\right\}. \tag{1.1}$$

From the standard p-adic analysis [6], we see that any nonzero p-adic number $x \in \mathbb{Q}_p$ can be uniquely represented in the canonical series

$$x = p^\gamma \sum_{j=0}^{\infty} a_j p^j, \quad \gamma = \gamma(x) \in \mathbb{Z}, \tag{1.2}$$

where a_j are integers, $0 \leq a_j \leq p-1$, $a_0 \neq 0$. The series (1.2) converges in the p-adic norm because $|a_j p^j|_p = p^{-j}$. Denote by $\mathbb{Q}_p^* = \mathbb{Q}_p \setminus \{0\}$ and $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

The space $\mathbb{Q}_p^n$ consists of points $x = (x_1, x_2, \ldots, x_n)$, where $x_j \in \mathbb{Q}_p$, $j = 1, 2, \ldots, n$. The p-adic norm on $\mathbb{Q}_p^n$ is

$$|x|_p := \max_{1 \leq j \leq n} |x_j|_p, \quad x \in \mathbb{Q}_p^n. \tag{1.3}$$

Denote by

$$B_\gamma(a) = \left\{x \in \mathbb{Q}_p^n : |x - a|_p \leq p^\gamma\right\}, \tag{1.4}$$

the ball with center at $a \in \mathbb{Q}_p^n$ and radius p^γ, and

$$S_\gamma(a) := \left\{x \in \mathbb{Q}_p^n : |x - a|_p = p^\gamma\right\} = B_\gamma(a) \setminus B_{\gamma-1}(a). \tag{1.5}$$

Since $\mathbb{Q}_p^n$ is a locally compact commutative group under addition, it follows from the standard analysis that there exists a Haar measure dx on $\mathbb{Q}_p^n$, which is unique up to positive constant multiple and is translation invariant. We normalize the measure dx by the equality

$$\int_{B_0(0)} dx = |B_0(0)|_H = 1, \tag{1.6}$$

where $|E|_H$ denotes the Haar measure of a measurable subset E of $\mathbb{Q}_p^n$. By simple calculation, we can obtain that

$$\left|B_\gamma(a)\right|_H = p^{\gamma n}, \qquad \left|S_\gamma(a)\right|_H = p^{\gamma n}\left(1 - p^{-n}\right), \tag{1.7}$$

for any $a \in \mathbb{Q}_p^n$. For a more complete introduction to the p-adic field, see [6] or [9].

The space $\underbrace{\mathbb{Q}_p^n \times \mathbb{Q}_p^n \times \cdots \times \mathbb{Q}_p^n}_{m}$ consists of points $(y_1, y_2, \ldots, y_m)$, where $y_i = (y_{i1}, y_{i2}, \ldots, y_{in}) \in \mathbb{Q}_p^n$, $i = 1, 2, \ldots, m$. The p-adic norm of m-tuple $(y_1, y_2, \ldots, y_m)$ is

$$\left|(y_1, y_2, \ldots, y_m)\right|_p := \max_{1 \leq i \leq m} \left|y_i\right|_p. \tag{1.8}$$

Recently, p-adic analysis has received a lot of attention due to its application in mathematical physics. There are numerous papers on p-adic analysis, such as [10, 11] about Riesz potentials, [12–16] about p-adic pseudodifferential equations, and so forth. The harmonic analysis on p-adic field has been drawing more and more concern (cf. [17–21] and references therein).

The well-known Hardy's integral inequality [22] tells us that for $1 < q < \infty$,

$$\|Hf\|_{L^q(\mathbb{R}^+)} \leq \frac{q}{q-1} \|f\|_{L^q(\mathbb{R}^+)}, \tag{1.9}$$

where the classical Hardy operator is defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)dt, \tag{1.10}$$

for nonnegative integral function f on $\mathbb{R}^+$, and the constant $q/(q-1)$ is the best possible. Thus the norm of Hardy operator on $L^q(\mathbb{R}^+)$ is

$$\|H\|_{L^q(\mathbb{R}^+)\to L^q(\mathbb{R}^+)} = \frac{q}{q-1}. \tag{1.11}$$

Faris [23] introduced the following n-dimensional Hardy operator, for nonnegative function f on $\mathbb{R}^n$,

$$\mathscr{H}f(x) := \frac{1}{\Omega_n|x|^n}\int_{|y|<|x|} f(y)dy, \quad x\in\mathbb{R}^n\setminus\{0\}, \tag{1.12}$$

where Ω_n is the volume of the unit ball in $\mathbb{R}^n$. Christ and Grafakos [24] obtained that the norm of $\mathscr{H}$ on $L^q(\mathbb{R}^n)$ is

$$\|\mathscr{H}\|_{L^q(\mathbb{R}^n)\to L^q(\mathbb{R}^n)} = \frac{q}{q-1}, \tag{1.13}$$

which is the same as that of the 1-dimension Hardy operator. In [25], Fu et al. introduced the m-linear Hardy operator, which is defined by

$$\mathscr{H}^m(f_1,\dots,f_m)(x) = \frac{1}{\Omega_{mn}|x|^{mn}}\int_{|(y_1,\dots,y_m)|<|x|} f_1(y_1)\cdots f_m(y_m)dy_1\cdots dy_m, \tag{1.14}$$

where $x\in\mathbb{R}^n\setminus\{0\}$ and $f_1,\dots,f_m$ are nonnegative locally integrable functions on $\mathbb{R}^n$. And they obtained the precise norms of $\mathscr{H}^m$ on Lebesgue spaces with power weight. The authors of [26] also got the best constants of m-linear Hilbert, Hardy and Hardy-Littlewood-Pólya operators on Lebesgue spaces.

The study of multilinear averaging operators in Euclidean spaces is a byproduct of the recent interest in multilinear singular integral operator theory. This subject was established by Coifman and Meyer [27] in 1975. In this article, we consider the sharp estimates of m-linear p-adic Hardy and Hardy-Littlewood-Pólya operators. In contrast with [25], we use a new technique in calculations based on the feature of p-adic field, and Theorem 3.1 is also new. They cannot be obtained immediately by [25]. In [28], we defined the p-adic Hardy operator.

Definition 1.1. For a function f on $\mathbb{Q}_p^n$, we define the p-adic Hardy operator as follows

$$\mathscr{H}^p f(x) = \frac{1}{|x|_p^n}\int_{B(0,|x|_p)} f(t)dt, \quad x\in\mathbb{Q}_p^n\setminus\{0\}, \tag{1.15}$$

where $B(0,|x|_p)$ is a ball in $\mathbb{Q}_p^n$ with center at $0\in\mathbb{Q}_p^n$ and radius $|x|_p$.

It is obvious that $|\mathcal{H}^p f| \le \mathcal{M}^p f$, where $\mathcal{M}^p$ is the Hardy-Littlewood maximal operator [17] defined by

$$\mathcal{M}^p f(x) = \sup_{\gamma\in\mathbb{Z}} \frac{1}{\left|B_\gamma(x)\right|_H} \int_{B_\gamma(x)} \left|f(y)\right| dy, \quad f \in L^1_{\text{loc}}\left(\mathbb{Q}_p^n\right). \tag{1.16}$$

The Hardy-Littlewood maximal operator plays an important role in harmonic analysis. The boundedness of $\mathcal{M}^p$ on $L^q(\mathbb{Q}_p^n)$ has been solved (see, e.g., [9]). But the best estimate of $\mathcal{M}^p$ on $L^q(\mathbb{Q}_p^n), q > 1$, even that of Hardy-Littlewood maximal operator on Euclidean spaces $\mathbb{R}^n$ is very difficult to obtain. Instead, we have obtained the sharp estimates of $\mathcal{H}^p$ (and p-adic Hardy-Littlewood-Pólya operator) elsewhere.

Definition 1.2. Let m be a positive integer and $f_1, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p^n$. The m-linear p-adic Hardy operator is defined by

$$\mathcal{H}_m^p(f_1, \ldots, f_m)(x) = \frac{1}{|x|_p^{mn}} \int_{|(y_1,\ldots,y_m)|_p \le |x|_p} f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m, \tag{1.17}$$

where $x \in \mathbb{Q}_p^n \setminus \{0\}$.

The Hardy-Littlewood-Pólya's linear operator [26] is defined by

$$Tf(x) = \int_0^\infty \frac{f(y)}{\max(x, y)} dy. \tag{1.18}$$

In [26], the authors obtained that the norm of Hardy-Littlewood-Pólya's operator on $L^q(\mathbb{R}^+)$ (see also [22, page 254]), $1 < q < \infty$, is

$$\|T\|_{L^q(\mathbb{R}^+) \to L^q(\mathbb{R}^+)} = \frac{q^2}{q-1}. \tag{1.19}$$

We define the p-adic Hardy-Littlewood-Pólya operator as (see [28])

$$T^p(x) = \int_{\mathbb{Q}_p} \frac{f(y)}{\max\left(|x|_p, \left|y\right|_p\right)} dy, \quad x \in \mathbb{Q}_p^*. \tag{1.20}$$

Definition 1.3. Let m be a positive integer and $f_1, \ldots, f_m$ be nonnegative locally integrable functions on $\mathbb{Q}_p^n$. The m-linear p-adic Hardy-Littlewood-Pólya operator is defined by

$$T_m^p(f_1, \ldots, f_m)(x) = \int_{\mathbb{Q}_p} \cdots \int_{\mathbb{Q}_p} \frac{f_1(y_1) \cdots f_m(y_m)}{\left[\max\left(|x|_p, \left|y_1\right|_p, \ldots, \left|y_m\right|_p\right)\right]^m} dy_1 \cdots dy_m, \quad x \in \mathbb{Q}_p^*. \tag{1.21}$$

We obtain the sharp estimates of the m-linear p-adic Hardy operator on Lebesgue spaces with power weights in Section 2. In Section 3, we get the best estimate of m-linear p-adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights.

In the following sequel, for $k \in \mathbb{Z}$, we denote $B_k = \{x \in \mathbb{Q}_p^n : |x|_p \le p^k\}$ and $S_k = \{x \in \mathbb{Q}_p^n : |x|_p = p^k\}$.

2. Sharp Estimates of m-Linear p-Adic Hardy Operator

Theorem 2.1. *Let $m \in \mathbb{Z}^+$, $f_i \in L^{q_i}(|x|_p^{\alpha_i q_i/q}dx)$, $1 < q_i < \infty$, $i = 1, 2, \ldots, m$, $1 \le q < \infty$, $1/q = \sum_{i=1}^m (1/q_i)$, $\alpha_i < qn(1 - (1/q_i))$ and $\alpha = \sum_{i=1}^m \alpha_i$. Then*

$$\left\|\mathcal{H}^p(f_1, \ldots, f_m)\right\|_{L^q(|x|_p^{\alpha}dx)} \le C_{\mathcal{H}} \left\|f_1\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)} \cdots \left\|f_m\right\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q}dx)}, \tag{2.1}$$

where

$$C_{\mathcal{H}} = \frac{\left(1 - p^{-n}\right)^m}{\prod_{i=1}^m \left(1 - p^{n/q_i + (\alpha_i/q) - n}\right)} \tag{2.2}$$

is the best constant.

When $\alpha = 0$, we get the sharp estimates of the m-linear p-adic Hardy operator on Lebesgue spaces.

Corollary 2.2. *Let $m \in \mathbb{Z}^+$, $1 < q_i < \infty$, $i = 1, 2, \ldots, m$, $1 \le q < \infty$ and $1/q = \sum_{i=1}^m 1/q_i$. Then*

$$\left\|\mathcal{H}_m^p\right\|_{L^{q_1}(\mathbb{Q}_p^n) \times \cdots \times L^{q_m}(\mathbb{Q}_p^n) \to L^q(\mathbb{Q}_p^n)} = \frac{\left(1 - p^{-n}\right)^m}{\prod_{i=1}^m \left(1 - p^{(n/q_i) - n}\right)}. \tag{2.3}$$

Proof of Theorem 2.1. Since the proof of the case when $m = 1$ is similar to and even simpler than that of the case when $m \ge 2$, for simplicity, we will only give the proof of case when $m \ge 2$. To make the proof clearer, we will discuss it in two parts.

(I) *When $m = 2$*

Firstly, we claim that the operator $\mathcal{H}^p$ and its restriction to the functions g satisfying $g(x) = g(|x|_p^{-1})$ have the same operator norm on $L^q(|x|_p^{\alpha}dx)$. In fact, set

$$g_i(x) = \frac{1}{1 - p^{-n}} \int_{|\xi_i|_p = 1} f_i\left(|x|_p^{-1}\xi_i\right) d\xi_i, \quad x \in \mathbb{Q}_p^n, \ i = 1, 2. \tag{2.4}$$

It's clear that $g_i(x) = g_i(|x|_p^{-1})$, $i = 1, 2$, and

$$\begin{aligned}
\mathscr{H}_2^p(g_1, g_2)(x) &= \frac{1}{|x|_p^{2n}} \int_{|(y_1,y_2)|_p \le |x|_p} g_1(y_1) g_2(y_2)\, dy_1 dy_2 \\
&= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(y_1,y_2)|_p \le |x|_p} \prod_{i=1}^{2} \left(\int_{|\xi_i|_p=1} f_i\left(|y_i|_p^{-1} \xi_i \right) d\xi_i \right) dy_1 dy_2 \\
&= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(y_1,y_2)|_p \le |x|_p} \prod_{i=1}^{2} \left(\int_{|z_i|_p=|y_i|_p} f_i(z_i) |y_i|_p^{-n} dz_i \right) dy_1 dy_2 \\
&= \frac{1}{(1-p^{-n})^2} \frac{1}{|x|_p^{2n}} \int_{|(z_1,z_2)|_p \le |x|_p} \prod_{i=1}^{2} \left(\int_{|y_i|_p=|z_i|_p} |y_i|_p^{-n} dy_i \right) f_1(z_1) f_2(z_2)\, dz_1 dz_2 \\
&= \frac{1}{|x|_p^{2n}} \int_{|(z_1,z_2)|_p \le |x|_p} f_1(z_1) f_2(z_2)\, dz_1 dz_2 \\
&= \mathscr{H}_2^p(f_1, f_2)(x).
\end{aligned} \tag{2.5}$$

By Hölder's inequality, we get

$$\begin{aligned}
\|g_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)} &= \left(\int_{\mathbb{Q}_p^n} \left| \frac{1}{1-p^{-n}} \int_{|\xi_i|_p=1} f_i\left(|x|_p^{-1} \xi_i \right) d\xi_i \right|^{q_i} |x|_p^{\alpha_i q_i/q} dx \right)^{1/q_i} \\
&\le \left\{ \int_{\mathbb{Q}_p^n} \frac{1}{(1-p^{-n})^{q_i}} \left(\int_{|\xi_i|_p=1} \left| f_i\left(|x|_p^{-1} \xi_i \right) \right|^{q_i} d\xi_i \right) \left(\int_{|\xi_i|_p=1} d\xi_i \right)^{q_i-1} |x|_p^{\alpha_i q_i/q} dx \right\}^{1/q_i} \\
&= \left\{ \int_{\mathbb{Q}_p^n} \frac{1}{1-p^{-n}} \left(\int_{|\xi_i|_p=1} \left| f_i\left(|x|_p^{-1} \xi_i \right) \right|^{q_i} d\xi_i \right) |x|_p^{\alpha_i q_i/q} dx \right\}^{1/q_i} \\
&= \frac{1}{(1-p^{-n})^{1/q_i}} \left\{ \int_{\mathbb{Q}_p^n} \left(\int_{|z_i|_p=|x|_p} |f_i(z_i)|^{q_i} dz_i \right) |x|_p^{(\alpha_i q_i/q)-n} dx \right\}^{1/q_i} \\
&= \frac{1}{(1-p^{-n})^{1/q_i}} \left\{ \int_{\mathbb{Q}_p^n} \left(\int_{|x|_p=|z_i|_p} |x|_p^{(\alpha_i q_i/q)-n} dx \right) |f_i(z_i)|^{q_i} dz_i \right\}^{1/q_i} \\
&= \|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}, \quad i = 1, 2.
\end{aligned} \tag{2.6}$$

Therefore,

$$\frac{\left\| \mathscr{H}_2^p(f_1, f_2) \right\|_{L^q(|x|_p^{\alpha} dx)}}{\|f_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \|f_2\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}} \le \frac{\left\| \mathscr{H}_2^p(g_1, g_2) \right\|_{L^q(|x|_p^{\alpha} dx)}}{\|g_1\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx)} \|g_2\|_{L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx)}}, \tag{2.7}$$

which implies the claim. In the following, without loss of generality, we may assume that $f_i \in L^{q_i}(|x|_p^{\alpha_i q_i/q}dx)$, $i=1,2$, which satisfy that $f_i(x)=f_i(|x|_p^{-1})$, $i=1,2$.

By changing of variables $y_i=|x|_p^{-1}z_i$, $i=1,2$, we have

$$\begin{aligned}\left\|\mathscr{H}_2^p(f_1,f_2)\right\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p^n}\left|\frac{1}{|x|_p^{2n}}\int_{|(y_1,y_2)|_p\le|x|_p} f_1(y_1)f_2(y_2)dy_1dy_2\right|^q|x|_p^\alpha dx\right)^{1/q}\\ &= \left(\int_{\mathbb{Q}_p^n}\left|\int_{|(z_1,z_2)|_p\le 1} f_1\left(|x|_p^{-1}z_1\right)f_2\left(|x|_p^{-1}z_2\right)dz_1dz_2\right|^q|x|_p^\alpha dx\right)^{1/q}\\ &= \left(\int_{\mathbb{Q}_p^n}\left|\int_{|(z_1,z_2)|_p\le 1} f_1\left(|z_1|_p^{-1}x\right)f_2\left(|z_2|_p^{-1}x\right)dz_1dz_2\right|^q|x|_p^\alpha dx\right)^{1/q}.\end{aligned} \tag{2.8}$$

Then using Minkowski's integral inequality and Hölder's inequality $((q/q_1)+(q/q_2)=1)$, we get

$$\begin{aligned}\left\|\mathscr{H}_2^p(f_1,f_2)\right\|_{L^q(|x|_p^\alpha dx)} &\le \int_{|(z_1,z_2)|_p\le 1}\left(\int_{\mathbb{Q}_p^n}\left|f_1\left(|z_1|_p^{-1}x\right)f_2\left(|z_2|_p^{-1}x\right)\right|^q|x|_p^\alpha dx\right)^{1/q}dz_1dz_2\\ &\le \int_{|(z_1,z_2)|_p\le 1}\prod_{i=1}^{2}\left(\int_{\mathbb{Q}_p^n}\left|f_i(|z_i|_p^{-1}x)\right|^{q_i}|x|_p^{\alpha_i q_i/q}dx\right)^{1/q_i}dz_1dz_2\\ &= \left(\int_{|(z_1,z_2)|_p\le 1}\prod_{i=1}^{2}|z_i|_p^{-(n/q_i)-(\alpha_i/q)}dz_1dz_2\right)\prod_{i=1}^{2}\|f_i\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q}dx)}.\end{aligned} \tag{2.9}$$

By calculation, we have

$$\begin{aligned}&\int_{|(z_1,z_2)|_p\le 1}\prod_{i=1}^{2}|z_i|_p^{-(n/q_i)-(\alpha_i/q)}dz_1dz_2\\ &\quad= \int_{|z_1|_p\le 1}\int_{|z_2|_p\le|z_1|_p}\prod_{i=1}^{2}|z_i|_p^{-(n/q_i)-(\alpha_i/q)}dz_1dz_2\\ &\qquad+\int_{|z_2|_p\le 1}\int_{|z_1|_p<|z_2|_p}\prod_{i=1}^{2}|z_i|_p^{-(n/q_i)-(\alpha_i/q)}dz_1dz_2\\ &\quad= \int_{|z_1|_p\le 1}|z_1|_p^{-(n/q_1)-(\alpha_1/q)}\left(\sum_{k=-\infty}^{\log_p|z_1|_p}\int_{S_k}|z_2|_p^{-(n/q_2)-(\alpha_2/q)}dz_2\right)dz_1\\ &\qquad+\int_{|z_2|_p\le 1}|z_2|_p^{-(n/q_2)-(\alpha_2/q)}\left(\sum_{k=-\infty}^{\log_p|z_2|_p-1}\int_{S_k}|z_1|_p^{-(n/q_1)-(\alpha_1/q)}dz_1\right)dz_2\end{aligned}$$

$$= \frac{(1-p^{-n})}{1-p^{(n/q_2)+(\alpha_2/q)-n}} \int_{|z_1|_p\le 1} |z_1|_p^{-(n/q)-(\alpha/q)+n} dz_1$$
$$+ \frac{(1-p^{-n})p^{(n/q_1)+(\alpha_1/q)-n}}{1-p^{(n/q_1)+(\alpha_1/q)-n}} \int_{|z_2|_p\le 1} |z_2|_p^{-(n/q)-(\alpha/q)+n} dz_2$$
$$= \frac{(1-p^{-n})^2}{\prod_{i=1}^{2}\left(1-p^{(n/q_i)+(\alpha_i/q)-n}\right)}. \tag{2.10}$$

Therefore,

$$\left\|\mathcal{H}_2^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)\times L^{q_2}(|x|_p^{\alpha_2 q_2/q}dx)\to L^q(|x|_p^{\alpha}dx)} \le \frac{(1-p^{-n})^2}{\prod_{i=1}^{2}\left(1-p^{(n/q_i)+(\alpha_i/q)-n}\right)}. \tag{2.11}$$

Now let us prove that our estimate is sharp. For $0<\epsilon<1$ and $|\epsilon|_p>1$, we take

$$f_i^{\epsilon}(x_i) = \begin{cases} 0, & |x_i|_p<1, \\ |x_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}, & |x_i|_p\ge 1, \end{cases} \quad i=1,2. \tag{2.12}$$

Then by calculation, we have

$$\left\|f_1^{\epsilon}\right\|^{q_1}_{L^{q_1}\left(|x|_p^{\alpha_1 q_1/q}dx\right)} = \left\|f_2^{\epsilon}\right\|^{q_2}_{L^{q_2}\left(|x|_p^{\alpha_2 q_2/q}dx\right)} = \frac{1-p^{-n}}{1-p^{-\epsilon q_2}}. \tag{2.13}$$

It is clear that when $|x|_p<1$, $\mathcal{H}_2^p(f_1^{\epsilon},f_2^{\epsilon})(x)=0$. But when $|x|_p\ge 1$,

$$\mathcal{H}_2^p(f_1^{\epsilon},f_2^{\epsilon})(x)$$
$$= |x|_p^{-(n/q)-(\alpha/q)-(q_2\epsilon/q)} \int_{|(y_1,y_2)|_p\le 1,|y_1|_p\ge 1/|x|_p,|y_2|_p\ge 1/|x|_p} \prod_{i=1}^{2}|y_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2. \tag{2.14}$$

Since $|\epsilon|_p > 1$, we get

$$\begin{aligned}
&\left\| \mathscr{H}_2^p (f_1^\epsilon, f_2^\epsilon) \right\|_{L^q(|x|_p^\alpha dx)} \\
&= \Bigg\{ \int_{|x|_p \geq 1} \Bigg(|x|_p^{-(n/q)-(\alpha/q)-(q_2\epsilon/q)} \\
&\qquad \times \int_{|(y_1,y_2)|_p \leq 1, |y_1|_p \geq 1/|x|_p, |y_2|_p \geq 1/|x|_p} \prod_{i=1}^{2} \left|y_i\right|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2 \Bigg)^q |x|_p^\alpha dx \Bigg\}^{1/q} \\
&\geq \Bigg\{ \int_{|x|_p \geq |\epsilon|_p} \Bigg(|x|_p^{-(n/q)-(\alpha/q)-(q_2\epsilon/q)} \\
&\qquad \times \int_{|(y_1,y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^{2} \left|y_i\right|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2 \Bigg)^q |x|_p^\alpha dx \Bigg\}^{1/q} \\
&= \left(\int_{|(y_1,y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^{2} \left|y_i\right|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2 \right) \left(\int_{|x|_p \geq |\epsilon|_p} |x|_p^{-n-\epsilon q_2} dx \right)^{1/q} \\
&= \left(\int_{|(y_1,y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^{2} \left|y_i\right|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2 \right) |\epsilon|_p^{-\epsilon q_2/q} \prod_{i=1}^{2} \| f_i^\epsilon \|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)}.
\end{aligned} \tag{2.15}$$

By the same calculation as that in (2.10), we obtain that

$$\begin{aligned}
&\int_{|(y_1,y_2)|_p \leq 1, |y_1|_p \geq 1/|\epsilon|_p, |y_2|_p \geq 1/|\epsilon|_p} \prod_{i=1}^{2} \left|y_i\right|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)} dy_1 dy_2 \\
&\qquad = \frac{(1-p^{-n})^2 \left[1 - \left(p|\epsilon|_p \right)^{(n/q)+(\alpha/q)+(q_2\epsilon/q)-2n} \right]}{\prod_{i=1}^{2} \left(1 - p^{(n/q_i)+(\alpha_i/q)+(q_2\epsilon/q_i)-n} \right)}.
\end{aligned} \tag{2.16}$$

Therefore,

$$\begin{aligned}
&\frac{|\epsilon|_p^{-\epsilon q_2/q} (1-p^{-n})^2 \left[1 - \left(p|\epsilon|_p \right)^{(n/q)+(\alpha/q)+(q_2\epsilon/q)-2n} \right]}{\prod_{i=1}^{2} \left(1 - p^{(n/q_i)+(\alpha_i/q)+(q_2\epsilon/q_i)-n} \right)} \\
&\qquad \leq \left\| \mathscr{H}_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \to L^q(|x|_p^\alpha dx)}.
\end{aligned} \tag{2.17}$$

Now take $\epsilon = p^{-k}$, $k = 1, 2, 3, \ldots$. Then $|\epsilon|_p = p^k > 1$. Letting k approach to ∞, then ϵ approaches to 0 and $|\epsilon|_p^{-\epsilon q_2/q}$ approaches to 1. Since $\alpha_i < qn(1 - (1/q_i))$, $i = 1, 2$, we have

$$\frac{(1-p^{-n})^2}{\prod_{i=1}^{2}\left(1-p^{(n/q_i)+(\alpha_i/q)-n}\right)} \le \left\|\mathscr{H}_2^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)\times L^{q_2}(|x|_p^{\alpha_2 q_2/q}dx)\to L^q(|x|_p^{\alpha}dx)}. \tag{2.18}$$

Then (2.11) and (2.18) imply that

$$\left\|\mathscr{H}_2^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)\times L^{q_2}(|x|_p^{\alpha_2 q_2/q}dx)\to L^q(|x|_p^{\alpha}dx)} = \frac{(1-p^{-n})^2}{\prod_{i=1}^{2}\left(1-p^{(n/q_i)+(\alpha_i/q)-n}\right)}. \tag{2.19}$$

(II) *When* $m \ge 3$

The proof of the upper bound in this case is similar to that of the previous case, and we can obtain that

$$\left\|\mathscr{H}_m^p(f_1,\ldots,f_m)\right\|_{L^q(|x|_p^{\alpha}dx)} \le C_{\mathscr{H}}\left\|f_1\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)}\cdots\left\|f_m\right\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q}dx)}, \tag{2.20}$$

where

$$C_{\mathscr{H}} = \int_{|(z_1,\ldots,z_m)|_p\le 1}\prod_{k=1}^{m}|z_k|_p^{-(n/q_k)-(\alpha_k/q)}dz_1\cdots dz_m. \tag{2.21}$$

Let

$$\begin{aligned}
D_1 &= \left\{(z_1,\ldots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n \mid |z_1|_p\le 1,\ |z_k|_p\le|z_1|_p,\ 1<k\le m\right\},\\
D_i &= \left\{(z_1,\ldots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n \mid |z_i|_p\le 1,\ \left|z_j\right|_p<|z_i|_p, |z_k|_p\le|z_i|_p,\ 1\le j<i<k\le m\right\},\\
D_m &= \left\{(z_1,\ldots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n \mid |z_m|_p\le 1,\ \left|z_j\right|_p<|z_m|_p,\ 1\le j<m\right\}.
\end{aligned} \tag{2.22}$$

It is clear that

$$\bigcup_{j=1}^{m}D_j = \left\{(z_1,\ldots,z_m)\in\mathbb{Q}_p^n\times\cdots\times\mathbb{Q}_p^n \mid |(z_1,\ldots,z_m)|_p\le 1\right\}, \tag{2.23}$$

and $D_i\cap D_j=\emptyset$. Then

$$C_{\mathscr{H}} = \sum_{j=1}^{m}\int_{D_j}\prod_{k=1}^{m}|z_k|_p^{-(n/q_k)-(\alpha_k/q)}dz_1\cdots dz_m := \sum_{j=1}^{m}I_j. \tag{2.24}$$

Now let us calculate I_j, $j = 1, 2, \ldots, m$, respectively,

$$\begin{aligned}
I_1 &= \int_{D_1} \prod_{k=1}^{m} |z_k|_p^{-(n/q_k)-(\alpha_k/q)} dz_1 \cdots dz_m \\
&= \int_{|z_1|_p \le 1} |z_1|_p^{-(n/q_1)-(\alpha_1/q)} \left(\prod_{k=2}^{m} \int_{|z_k|_p \le |z_1|_p} |z_k|_p^{-(n/q_k)-(\alpha_k/q)} dz_k \right) dz_1 \\
&= \int_{|z_1|_p \le 1} |z_1|_p^{-(n/q_1)-(\alpha_1/q)} \left(\prod_{k=2}^{m} \left(\sum_{j=-\infty}^{\log_p |z_1|_p} \int_{S_j} |z_k|_p^{-(n/q_k)-(\alpha_k/q)} dz_k \right) \right) dz_1 \\
&= \frac{(1-p^{-n})^{m-1}}{\prod_{k=2}^{m} (1 - p^{(n/q_k)+(\alpha_k/q)-n})} \int_{|z_1|_p \le 1} |z_1|_p^{-(n/q)-(\alpha/q)+(m-1)n} dz_1 \\
&= \frac{(1-p^{-n})^{m}}{(1 + p^{(n/q)+(\alpha/q)-mn}) \prod_{k=2}^{m} (1 - p^{(n/q_k)+(\alpha_k/q)-n})}.
\end{aligned} \tag{2.25}$$

Similarly, for $i = 2, \ldots, m-1$, we have

$$\begin{aligned}
I_i &= \int_{D_i} \prod_{k=1}^{m} |z_k|_p^{-(n/q_k)-(\alpha_k/q)} dz_1 \cdots dz_m \\
&= \int_{|z_i|_p \le 1} |z_i|_p^{-(n/q_i)-(\alpha_i/q)} \left(\prod_{j=1}^{i-1} \int_{|z_j|_p < |z_i|_p} |z_j|_p^{-(n/q_j)-(\alpha_j/q)} dz_j \right) \\
&\quad \times \left(\prod_{k=i+1}^{m} \int_{|z_k|_p \le |z_i|_p} |z_k|_p^{-(n/q_k)-(\alpha_k/q)} dz_k \right) dz_i \\
&= \frac{(1-p^{-n})^{m-1} \prod_{j=1}^{i-1} p^{(n/q_j)+(\alpha_j/q)-n}}{\prod_{1 \le k \le m, k \ne i} (1 - p^{(n/q_k)+(\alpha_k/q)-n})} \int_{|z_i|_p \le 1} |z_i|_p^{-(n/q)-(\alpha/q)+(m-1)n} dz_i \\
&= \frac{(1-p^{-n})^{m} \prod_{j=1}^{i-1} p^{(n/q_j)+(\alpha_j/q)-n}}{(1 + p^{(n/q)+(\alpha/q)-mn}) \prod_{1 \le k \le m, k \ne i} (1 - p^{(n/q_k)+(\alpha_k/q)-n})}, \\
I_m &= \int_{|z_m|_p \le 1} |z_m|_p^{-(n/q_m)-(\alpha_m/q)} \left(\prod_{j=1}^{m-1} \int_{|z_j|_p < |z_m|_p} |z_j|_p^{-(n/q_j)-(\alpha_j/q)} dz_j \right) dz_m \\
&= \frac{(1-p^{-n})^{m} \prod_{j=1}^{m-1} p^{(n/q_j)+(\alpha_j/q)-n}}{(1 + p^{(n/q)+(\alpha/q)-mn}) \prod_{j=1}^{m-1} (1 - p^{(n/q_j)+(\alpha_j/q)-n})}.
\end{aligned} \tag{2.26}$$

Therefore,

$$\begin{aligned} C_{\mathscr{H}} &= \frac{(1-p^{-n})^m}{\left(1+p^{(n/q)+(\alpha/q)-mn}\right)\prod_{k=2}^{m}\left(1-p^{(n/q_k)+(\alpha_k/q)-n}\right)} \\ &\quad + \sum_{i=2}^{m-1}\frac{(1-p^{-n})^m\prod_{j=1}^{m-1}p^{(n/q_j)+(\alpha_j/q)-n}}{\left(1+p^{(n/q)+(\alpha/q)-mn}\right)\prod_{k=1}^{m-1}\left(1-p^{(n/q_k)+(\alpha_k/q)-n}\right)} \\ &\quad + \frac{(1-p^{-n})^m\prod_{j=1}^{m-1}p^{(n/q_j)+(\alpha_j/q)-n}}{\left(1+p^{(n/q)+(\alpha/q)-mn}\right)\prod_{j=1}^{m-1}\left(1-p^{(n/q_j)+(\alpha_j/q)-n}\right)} \\ &= \frac{(1-p^{-n})^m}{\prod_{i=1}^{m}\left(1-p^{(n/q_i)+(n\alpha_i/q)-n}\right)}. \end{aligned} \tag{2.27}$$

To show that $C_{\mathscr{H}}$ is the best constant, we should prove that it is also the lower bound of the norm of $\mathscr{H}_m^p$ from $L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)\times\cdots\times L^{q_m}(|x|_p^{\alpha_m q_m/q}dx)$ to $L^q(|x|_p^{\alpha}dx)$. For $0<\epsilon<1$ and $|\epsilon|_p>1$, we take

$$f_i^{\epsilon}(x_i)=\begin{cases}0, & |x_i|_p<1,\\ |x_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_m\epsilon/q_i)}, & |x_i|_p\geq 1,\end{cases}\quad i=1,2,\ldots,m. \tag{2.28}$$

By simple calculation, we have

$$\left\|f_1^{\epsilon}\right\|^{q_1}_{L^{q_1}\left(|x|_p^{\alpha_1 q_1/q}dx\right)}=\cdots=\left\|f_m^{\epsilon}\right\|^{q_m}_{L^{q_m}\left(|x|_p^{\alpha_m q_m/q}dx\right)}=\frac{1-p^{-n}}{1-p^{-\epsilon q_m}}. \tag{2.29}$$

And when $|x|_p<1$, $\mathscr{H}_m^p(f_1^{\epsilon},\ldots,f_m^{\epsilon})(x)=0$. But when $|x|_p\geq 1$,

$$\begin{aligned} &\mathscr{H}_m^p(f_1^{\epsilon},\ldots,f_m^{\epsilon})(x) \\ &\quad=|x|_p^{-(n/q)-(\alpha/q)-(q_m\epsilon/q)}\int_{|(y_1,\ldots,y_m)|_p\leq 1,|y_1|_p\geq 1/|x|_p,\ldots,|y_m|_p\geq 1/|x|_p}\prod_{i=1}^{m}|y_i|_p^{-(n/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}dy_1\cdots dy_m. \end{aligned} \tag{2.30}$$

Then by the similar discussion to that in previous case, we can obtain that

$$\left\|\mathscr{H}_m^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)\times\cdots\times L^{q_m}(|x|_p^{\alpha_m q_m/q}dx)\to L^q(|x|_p^{\alpha}dx)}\geq C_{\mathscr{H}}. \tag{2.31}$$

Theorem 2.1 is established by (2.20), (2.27), and (2.31). □

3. Sharp Estimate of m-Linear p-Adic Hardy-Littlewood-Pólya Operator

We get the following best estimate of m-linear p-adic Hardy-Littlewood-Pólya operator on Lebesgue spaces with power weights.

Theorem 3.1. *Let $m \in \mathbb{Z}^+$, $f_i \in L_i^q(|x|_p^{\alpha_i q_i/q}dx)$, $1 < q_i < \infty$, $i = 1, 2, \ldots, m$, $1 \le q < \infty$, $1/q = \sum_{i=1}^m 1/q_i$, $\alpha_i < q(1-(1/q_i))$ and $\alpha = \sum_{i=1}^m \alpha_i$. Then*

$$\left\| T_m^p(f_1, \ldots, f_m) \right\|_{L^q(|x|_p^\alpha dx)} = C_T \left\| f_1 \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)} \cdots \left\| f_m \right\|_{L^{q_m}(|x|_p^{\alpha_m q_m/q}dx)}, \tag{3.1}$$

where

$$C_T = \frac{\left(1-p^{-1}\right)^m \left(1-q^{-m}\right)}{\left(1-p^{-(1/q)-(\alpha/q)}\right)\prod_{i=1}^m \left(1-p^{(1/q_i)+(\alpha_i/q)-1}\right)}, \tag{3.2}$$

is the best constant.

In particular, when $\alpha = 0$, we obtain the norm of the m-linear p-adic Hardy-Littlewood-Pólya operator on Lebesgue spaces.

Corollary 3.2. *Let $m \in \mathbb{Z}^+$, $1 < q_i < \infty$, $i = 1, 2, \ldots, m$, $1 \le q < \infty$ and $1/q = \sum_{i=1}^m 1/q_i$. Then*

$$\left\| T_m^p \right\|_{L^{q_1}(\mathbb{Q}_p)\times\cdots\times L^{q_m}(\mathbb{Q}_p)\to L^q(\mathbb{Q}_p)} = \frac{\left(1-p^{-1}\right)^m \left(1-q^{-m}\right)}{\left(1-p^{-1/q}\right)\prod_{i=1}^m \left(1-p^{(1/q_i)-1}\right)}. \tag{3.3}$$

Proof of Theorem 3.1. Just as the proof of Theorem 2.1, we will only give the proof of case when $m \ge 2$.

(I) *Case $m = 2$*

By definition and the change of variables $y_i = xz_i$, $i = 1, 2$, we have

$$\begin{aligned}
\left\| T_2^p(f_1, f_2) \right\|_{L^q(|x|_p^\alpha dx)} &= \left(\int_{\mathbb{Q}_p} \left| \iint_{\mathbb{Q}_p} \frac{f_1(y_1) f_2(y_2)}{\left[\max\left(|x|_p, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2 \right|^q |x|_p^\alpha dx \right)^{1/q} \\
&\le \left(\int_{\mathbb{Q}_p} \left(\iint_{\mathbb{Q}_p} \frac{|f_1(y_1) f_2(y_2)|}{\left[\max\left(|x|_p, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2 \right)^q |x|_p^\alpha dx \right)^{1/q} \\
&= \left(\int_{\mathbb{Q}_p} \left(\iint_{\mathbb{Q}_p} \frac{|f_1(xz_1) f_2(xz_2)|}{\left[\max\left(1, |z_1|_p, |z_2|_p\right)\right]^2} dz_1 dz_2 \right)^q |x|_p^\alpha dx \right)^{1/q}.
\end{aligned} \tag{3.4}$$

By Minkowski's integral inequality and Hölder's inequality $((q/q_1)+(q/q_2)=1)$, we get

$$\begin{aligned}\left\|T_2^p(f_1,f_2)\right\|_{L^q(|x|_p^\alpha dx)} &\le \iint_{\mathbb{Q}_p}\left(\int_{\mathbb{Q}_p}\left|f_1(xz_1)f_2(xz_2)\right|^q|x|_p^\alpha dx\right)^{1/q}\frac{1}{\left[\max\left(1,|z_1|_p,|z_2|_p\right)\right]^2}dz_1dz_2\\ &\le \iint_{\mathbb{Q}_p}\prod_{i=1}^{2}\left(\int_{\mathbb{Q}_p}\left|f_i(xz_i)\right|^{q_i}|x|_p^{\alpha_iq_i/q}dx\right)^{1/q_i}\frac{1}{\left[\max\left(1,|z_1|_p,|z_2|_p\right)\right]^2}dz_1dz_2\\ &\le \left(\iint_{\mathbb{Q}_p}\frac{\prod_{i=1}^{2}|z_i|_p^{-(1/q_i)-(\alpha_i/q)}}{\left[\max\left(1,|z_1|_p,|z_2|_p\right)\right]^2}dz_1dz_2\right)\prod_{i=1}^{2}\|f_i\|_{L^{q_i}(|x|_p^{\alpha_iq_i/q}dx)}.\end{aligned}\tag{3.5}$$

By calculation, we have

$$\begin{aligned}&\iint_{\mathbb{Q}_p}\frac{\prod_{i=1}^{2}|z_i|_p^{-(1/q_i)-(\alpha_i/q)}}{\left[\max\left(1,|z_1|_p,|z_2|_p\right)\right]^2}dz_1dz_2\\ &\quad=\int_{|z_1|_p\le 1}\int_{|z_2|_p\le 1}\prod_{i=1}^{2}|z_i|_p^{-(1/q_i)-(\alpha_i/q)}dz_1dz_2\\ &\qquad+\int_{|z_1|_p>1}\int_{|z_2|_p\le|z_1|_p}|z_1|_p^{-(1/q_1)-(\alpha_1/q)-2}|z_2|_p^{-(1/q_2)-(\alpha_2/q)}dz_1dz_2\\ &\qquad+\int_{|z_2|_p>1}\int_{|z_1|_p<|z_2|_p}|z_1|_p^{-(1/q_1)-(\alpha_1/q)}|z_2|_p^{-(1/q_2)-(\alpha_2/q)-2}dz_1dz_2\\ &\quad:=L_0+L_1+L_2.\end{aligned}\tag{3.6}$$

By definition,

$$\begin{aligned}L_0&=\int_{|z_1|_p\le 1}\int_{|z_2|_p\le 1}\prod_{i=1}^{2}|z_i|_p^{-(1/q_i)-(\alpha_i/q)}dz_1dz_2\\ &=\frac{\left(1-p^{-1}\right)^2}{\left(1-p^{(1/q_1)+(\alpha_1/q)-1}\right)\left(1-p^{(1/q_2)+(\alpha_2/q)-1}\right)},\\ L_1&=\int_{|z_1|_p>1}\int_{|z_2|_p\le|z_1|_p}|z_1|_p^{-(1/q_1)-(\alpha_1/q)-2}|z_2|_p^{-(1/q_2)-\alpha_2/q}dz_1dz_2\\ &=\frac{\left(1-p^{-1}\right)}{1-p^{(1/q_2)+(\alpha_2/q)-1}}\int_{|z_1|_p>1}|z_1|_p^{-(1/q)-(\alpha/q)-1}dz_1\\ &=\frac{\left(1-p^{-1}\right)^2p^{-(1/q)-(\alpha/q)}}{\left(1-p^{(1/q_2)+(\alpha_2/q)-1}\right)\left(1-p^{-(1/q)-(\alpha/q)}\right)}.\end{aligned}\tag{3.7}$$

Similarly,

$$L_2 = \int_{|z_2|_p>1} \int_{|z_1|_p<|z_2|_p} |z_1|_p^{-(1/q_1)-(\alpha_1/q)} |z_2|_p^{-(1/q_2)-(\alpha_2/q)-2} dz_1 dz_2$$
$$= \frac{(1-p^{-1})^2 p^{(1/q_1)+(\alpha_1/q)-1} p^{-(1/q)-(\alpha/q)}}{(1-p^{(1/q_1)+(\alpha_1/q)-1})(1-p^{-(1/q)-(\alpha/q)})}. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we get

$$\iint_{\mathbb{Q}_p} \frac{\prod_{i=1}^2 |z_i|_p^{-(1/q_i)-(\alpha_i/q)}}{\left[\max\left(1, |z_1|_p, |z_2|_p\right)\right]^2} dz_1 dz_2 = \frac{(1-p^{-1})^2 (1-q^{-2})}{(1-p^{-(1/q)-(\alpha/q)}) \prod_{i=1}^2 (1-p^{(1/q_i)+(\alpha_i/q)-1})}. \tag{3.9}$$

Then (3.5) and (3.9) imply that

$$\left\| T_2^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \to L^q(|x|_p^{\alpha} dx)} \le \frac{(1-p^{-1})^2 (1-p^{-2})}{(1-p^{-(1/q)-(\alpha/q)}) \prod_{i=1}^2 (1-p^{(1/q_i)+(\alpha_i/q)-1})}. \tag{3.10}$$

On the other hand, for $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we take

$$f_i^{\epsilon}(x_i) = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}, & |x_i|_p \ge 1, \end{cases} \quad i = 1, 2. \tag{3.11}$$

Then

$$\left\| f_1^{\epsilon} \right\|_{L^{q_1}\left(|x|_p^{\alpha_1 q_1/q} dx\right)}^{q_1} = \left\| f_2^{\epsilon} \right\|_{L^{q_2}\left(|x|_p^{\alpha_2 q_2/q} dx\right)}^{q_2} = \frac{1-p^{-1}}{1-p^{-\epsilon q_2}}. \tag{3.12}$$

Since $|\epsilon|_p > 1$, we have

$$\left\| T_2^p(f_1^{\epsilon}, f_2^{\epsilon}) \right\|_{L^q(|x|_p^{\alpha} dx)}$$
$$= \left(\int_{\mathbb{Q}_p} \left| \iint_{\mathbb{Q}_p} \frac{f_1^{\epsilon}(x_1) f_2^{\epsilon}(x_2)}{\left[\max\left(|x|_p, |x_1|_p, |x_2|_p\right)\right]^2} dx_1 dx_2 \right|^q |x|_p^{\alpha} dx \right)^{1/q}$$
$$\ge \left(\int_{|x|_p \ge 1} \left(\int_{|x_1|_p \ge 1} \int_{|x_2|_p \ge 1} \frac{\prod_{i=1}^2 |x_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}}{\left[\max\left(|x|_p, |x_1|_p, |x_2|_p\right)\right]^2} dx_1 dx_2 \right)^q |x|_p^{\alpha} dx \right)^{1/q}$$
$$= \left(\int_{|x|_p \ge 1} \left(\int_{|y_1|_p \ge 1/|x|_p} \int_{|y_2|_p \ge 1/|x|_p} \frac{\prod_{i=1}^2 |y_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}}{\left[\max\left(1, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2 \right)^q |x|_p^{-1-q_2\epsilon} dx \right)^{1/q}$$

$$\geq \left(\int_{|x|_p \geq |\epsilon|_p} \left(\int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^{2} |y_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}}{\left[\max\left(1, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2 \right)^q |x|_p^{-1-q_2\epsilon} dx \right)^{1/q}$$

$$= \prod_{i=1}^{2} \|f_i^\epsilon\|_{L^{q_i}(|x|_p^{\alpha_i q_i/q} dx)} |\epsilon|_p^{-q_2\epsilon/q} \int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^{2} |y_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}}{\left[\max\left(1, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2. \tag{3.13}$$

Therefore,

$$\begin{aligned} &\left\|T_2^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \to L^q(|x|_p^{\alpha} dx)} \\ &\quad \geq |\epsilon|_p^{-q_2\epsilon/q} \int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^{2} |y_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}}{\left[\max\left(1, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2. \end{aligned} \tag{3.14}$$

As the calculation of (3.6)–(3.8), we obtain that

$$\begin{aligned} &\int_{|y_1|_p \geq 1/|\epsilon|_p} \int_{|y_2|_p \geq 1/|\epsilon|_p} \frac{\prod_{i=1}^{2} |y_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}}{\left[\max\left(1, |y_1|_p, |y_2|_p\right)\right]^2} dy_1 dy_2 \\ &\quad = \frac{(1-p^{-1})^2 \prod_{i=1}^{2} \left[1 - (p|\epsilon|_p)^{(1/q_i)+(\alpha_i/q)+(q_2\epsilon/q_i)-1}\right]}{\prod_{i=1}^{2} \left(1 - p^{(1/q_i)+(\alpha_i/q)+(q_2\epsilon/q_i)-1}\right)} \\ &\qquad + \frac{(1-p^{-1})^2 p^{-(1/q)-(\alpha/q)-(q_2\epsilon/q)}}{\left(1 - p^{(1/q_2)+(\alpha_2/q)+\epsilon-1}\right)\left(1 - p^{-(1/q)-(\alpha/q)-(q_2\epsilon/q)}\right)} \\ &\qquad + \frac{(1-p^{-1})^2 p^{(1/q_1)+(\alpha_1/q)+(q_2\epsilon/q_1)-1} p^{-(1/q)-(\alpha/q)-(q_2\epsilon/q)}}{\left(1 - p^{(1/q_1)+(\alpha_1/q)+(q_2\epsilon/q_1)-1}\right)\left(1 - p^{-(1/q)-(\alpha/q)-(q_2\epsilon/q)}\right)}. \end{aligned} \tag{3.15}$$

Now take $\epsilon = p^{-k}, k \in \mathbb{Z}^+$ and let k approach to ∞, then by (3.9), (3.14), (3.15), and the fact that $\alpha_i < qn(1 - (1/q_i))$, $i = 1, 2$, we have

$$\begin{aligned} &\left\|T_2^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \to L^q(|x|_p^{\alpha} dx)} \\ &\quad \geq \frac{(1-p^{-1})^2 (1-q^{-2})}{\left(1 - p^{-(1/q)-(\alpha/q)}\right) \prod_{i=1}^{2} \left(1 - p^{(1/q_i)+(\alpha_i/q)-1}\right)}. \end{aligned} \tag{3.16}$$

Then by (3.10) and (3.16), we get

$$\left\|T_2^p\right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q} dx) \times L^{q_2}(|x|_p^{\alpha_2 q_2/q} dx) \to L^q(|x|_p^{\alpha} dx)} = \frac{(1-p^{-1})^2 (1-q^{-2})}{\left(1 - p^{-(1/q)-(\alpha/q)}\right) \prod_{i=1}^{2} \left(1 - p^{(1/q_i)+(\alpha_i/q)-1}\right)}. \tag{3.17}$$

(II) *Case* $m \geq 3$

The upper bound estimate for the norm can be obtained by the same way as that when $m = 2$, and we can obtain that

$$\left\| T_m^p(f_1, \ldots, f_m) \right\|_{L^q(|x|_p^\alpha dx)} \leq C_T \prod_{i=1}^{m} \| f_i \|_{L^{q_i}(|x|_p^{\alpha_i q_i / q} dx)}, \tag{3.18}$$

where

$$C_T = \int_{|(z_1, \ldots, z_m)|_p \leq 1} \frac{\prod_{j=1}^{m} |z_j|_p^{-(1/q_j)-(\alpha_j/q)}}{\left[\max\left(1, |z_1|_p, \ldots, |z_m|_p\right)\right]^m} dz_1 \cdots dz_m. \tag{3.19}$$

Let

$$\begin{aligned}
E_0 &= \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \\
E_1 &= \left\{ (z_1, \ldots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_1|_p > 1,\ |z_k|_p \leq |z_1|_p,\ 1 < k \leq m \right\}, \\
E_i &= \left\{ (z_1, \ldots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_i|_p > 1,\ \left|z_j\right|_p < |z_i|_p, |z_k|_p \leq |z_i|_p,\ 1 \leq j < i < k \leq m \right\}, \\
E_m &= \left\{ (z_1, \ldots, z_m) \in \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p \mid |z_m|_p > 1,\ \left|z_j\right|_p < |z_m|_p,\ 1 \leq j < m \right\}.
\end{aligned} \tag{3.20}$$

Obviously,

$$\bigcup_{k=1}^{m} E_k = \mathbb{Q}_p \times \cdots \times \mathbb{Q}_p, \qquad E_i \cap E_j = \emptyset, \quad i \neq j,\ 1 \leq i, j \leq m. \tag{3.21}$$

Then

$$C_T = \sum_{k=0}^{m} \int_{E_k} \frac{\prod_{j=1}^{m} |z_j|_p^{-(1/q_j)-(\alpha_j/q)}}{\left[\max\left(1, |z_1|_p, \ldots, |z_m|_p\right)\right]^m} dz_1 \cdots dz_m := \sum_{k=0}^{m} J_k. \tag{3.22}$$

Now let us calculate $J_k, k = 0, 1, \ldots, m$, respectively,

$$\begin{aligned}
J_0 &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \prod_{j=1}^{m} |z_j|_p^{-(1/q_j)-(\alpha_j/q)} dz_1 \cdots dz_m \\
&= \prod_{j=1}^{m} \int_{\mathbb{Z}_p} \left|z_j\right|_p^{-(1/q_j)-(\alpha_j/q)} dz_j = \prod_{j=1}^{m} \left(\sum_{k=-\infty}^{0} \int_{S_k} |z_j|_p^{-(1/q_j)-(\alpha_j/q)} dz_j \right) \\
&= \frac{(1 - p^{-1})^m}{\prod_{j=1}^{m} (1 - p^{(1/q_j)+(\alpha_j/q)-1})},
\end{aligned}$$

$$J_1 = \int_{|z_1|_p>1}\int_{|z_2|_p\le|z_1|_p}\cdots\int_{|z_m|_p\le|z_1|_p}|z_1|_p^{-(1/q_1)-(\alpha_1/q)-m}\prod_{j=2}^{m}|z_j|_p^{-(1/q_j)-(\alpha_j/q)}dz_1\cdots dz_m$$
$$= \int_{|z_1|_p>1}|z_1|_p^{-(1/q_1)-(\alpha_1/q)-m}\left(\prod_{j=2}^{m}\int_{|z_j|_p\le|z_1|_p}|z_j|_p^{-(1/q_j)-(\alpha_j/q)}dz_j\right)dz_1$$
$$= \int_{|z_1|_p>1}|z_1|_p^{-(1/q_1)-(\alpha_1/q)-m}\prod_{j=2}^{m}\left(\frac{(1-p^{-1})|z_1|_p^{-(1/q_j)-(\alpha_j/q)+1}}{1-p^{(1/q_j)+(\alpha_j/q)-1}}\right)dz_1$$
$$= \frac{(1-p^{-1})^{m-1}}{\prod_{j=2}^{m}(1-p^{(1/q_j)+(\alpha_j/q)-1})}\int_{|z_1|_p>1}|z_1|_p^{-(1/q)-(\alpha/q)-1}dz_1$$
$$= \frac{(1-p^{-1})^{m-1}p^{-(1/q)-(\alpha/q)}}{\prod_{j=2}^{m}(1-p^{(1/q_j)+(\alpha_j/q)-1})(1-p^{-(1/q)-(\alpha/q)})}. \tag{3.23}$$

Similar to J_1, for $1<i<m$, it is true that

$$J_i = \int_{|z_i|_p>1}|z_i|_p^{-(1/q_i)-(\alpha_i/q)-m}\left(\prod_{j=1}^{i-1}\int_{|z_j|_p<|z_i|_p}|z_j|_p^{-(1/q_j)-(\alpha_j/q)}dz_j\right)$$
$$\times\left(\prod_{k=i+1}^{m}\int_{|z_k|_p\le|z_i|_p}|z_k|_p^{-(1/q_k)-(\alpha_k/q)}dz_k\right)dz_i$$
$$= \frac{(1-p^{-1})^{m-1}\prod_{j=1}^{i-1}p^{(1/q_j)+(\alpha_j/q)-1}}{\prod_{1\le k\le m,k\ne i}(1-p^{(1/q_k)+(\alpha_k/q)-1})}\int_{|z_i|_p>1}|z_i|_p^{-(1/q)-(\alpha/q)-1}dz_i$$
$$= \frac{(1-p^{-1})^{m}\left(\prod_{j=1}^{i-1}p^{(1/q_j)+(\alpha_j/q)-1}\right)p^{-(1/q)-(\alpha/q)}}{(1-p^{-(1/q)-(\alpha/q)})\prod_{1\le k\le m,k\ne i}(1-p^{(1/q_k)+(\alpha_k/q)-1})}, \tag{3.24}$$
$$J_m = \int_{|z_m|_p>1}|z_i|_p^{-(1/q_i)-(\alpha_i/q)-m}\left(\prod_{j=1}^{m-1}\int_{|z_j|_p<|z_m|_p}|z_j|_p^{-(1/q_j)-(\alpha_j/q)}dz_j\right)dz_m$$
$$= \frac{(1-p^{-1})^{m-1}\prod_{j=1}^{m-1}p^{(1/q_j)+(\alpha_j/q)-1}}{\prod_{j=1}^{m-1}(1-p^{(1/q_k)+(\alpha_k/q)-1})}\int_{|z_m|_p>1}|z_m|_p^{-(1/q)-(\alpha/q)-1}dz_m$$
$$= \frac{(1-p^{-1})^{m}\left(\prod_{j=1}^{m-1}p^{(1/q_j)+(\alpha_j/q)-1}\right)p^{-(1/q)-(\alpha/q)}}{(1-p^{-(1/q)-(\alpha/q)})\prod_{j=1}^{m-1}(1-p^{(1/q_j)+(\alpha_j/q)-1})}.$$

Consequently, we have

$$C_T = \sum_{k=0}^{m} J_k = \frac{\left(1-p^{-1}\right)^m \left(1-q^{-m}\right)}{\left(1-p^{-(1/q)-(\alpha/q)}\right)\prod_{i=1}^{m}\left(1-p^{(1/q_i)+(\alpha_i/q)-1}\right)}. \tag{3.25}$$

To obtain that C_T is also the lower bound, for $0 < \epsilon < 1$ and $|\epsilon|_p > 1$, we define

$$f_i^{\epsilon} = \begin{cases} 0, & |x_i|_p < 1, \\ |x_i|_p^{-(1/q_i)-(\alpha_i/q)-(q_2\epsilon/q_i)}, & |x_i|_p \geq 1, \end{cases} \quad i = 1, 2, \ldots, m. \tag{3.26}$$

By the similar discussion to that in Case $m = 2$, we can also get that

$$\left\| T_m^p \right\|_{L^{q_1}(|x|_p^{\alpha_1 q_1/q}dx)\times\cdots\times L^{q_m}(|x|_p^{\alpha_m q_m/q}dx)\to L^q(|x|_p^{\alpha}dx)} \geq C_T. \tag{3.27}$$

Combining (3.18) with (3.27), we complete the proof of Theorem 3.1. □

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