Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 458275, 11 pages
doi:10.1155/2011/458275

Research Article

Oscillation of Second-Order Sublinear Impulsive Differential Equations

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Received 25 January 2011; Accepted 27 February 2011

Academic Editor: Josef Diblík

Oscillation criteria obtained by Kusano and Onose (1973) and by Belohorec (1969) are extended to second-order sublinear impulsive differential equations of Emden-Fowler type: $x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0$, $t \neq \theta_k$; $\Delta x'(t)|_{t=\theta_k} + q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)) = 0$; $\Delta x(t)|_{t=\theta_k} = 0$, $(0 < \alpha < 1)$ by considering the cases $\tau(t) \leq t$ and $\tau(t) = t$, respectively. Examples are inserted to show how impulsive perturbations greatly affect the oscillation behavior of the solutions.

1. Introduction

We deal with second-order sublinear impulsive differential equations of the form

$$\begin{gathered}
x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, \quad t \neq \theta_k, \\
\Delta x'(t)\big|_{t=\theta_k} + q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)) = 0, \\
\Delta x(t)|_{t=\theta_k} = 0,
\end{gathered} \tag{1.1}$$

where $0 < \alpha < 1$, $t \geq t_0$, and $k \geq k_0$ for some $t_0 \in \mathbb{R}_+$ and $k_0 \in \mathbb{N}$, $\{\theta_k\}$ is a strictly increasing unbounded sequence of positive real numbers,

$$\Delta z(t)|_{t=\theta} := z(\theta^+) - z(\theta^-), \quad z(\theta^\mp) := \lim_{t \to \theta^\mp} z(t). \tag{1.2}$$

Let $\mathrm{PLC}(J, R)$ denote the set of all real-valued functions u defined on J such that u is continuous for all $t \in J$ except possibly at $t = \theta_k$ where $u(\theta_k^\pm)$ exists and $u(\theta_k) := u(\theta_k^-)$.

We assume in the sequel that

(a) $p \in \mathrm{PLC}([t_0, \infty), \mathbb{R})$,

(b) $\{q_k\}$ is a sequence of real numbers,

(c) $\tau \in C([t_0, \infty), \mathbb{R}_+)$, $\tau(t) \le t$, $\lim_{t\to\infty}\tau(t) = \infty$.

By a solution of (1.1) on an interval $J \subset [t_0, \infty)$, we mean a function $x(t)$ which is defined on J such that $x, x', x'' \in \mathrm{PLC}(J)$ and which satisfies (1.1). Because of the requirement $\Delta x(t)|_{t=\theta_k} = 0$ every solution of (1.1) is necessarily continuous.

As usual we assume that (1.1) has solutions which are nontrivial for all large t. Such a solution of (1.1) is called oscillatory if it has no last zero and nonoscillatory otherwise.

In case there is no impulse, (1.1) reduces to Emden-Fowler equation with delay

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, \quad 0 < \alpha < 1, \tag{1.3}$$

and without delay

$$x'' + p(t)|x|^{\alpha-1}x = 0, \quad 0 < \alpha < 1. \tag{1.4}$$

The problem of oscillation of solutions of (1.3) and (1.4) has been considered by many authors. Kusano and Onose [1] see also [2, 3] proved the following necessary and sufficient condition for oscillation of (1.3).

Theorem 1.1. *If $p(t) \ge 0$, then a necessary and sufficient condition for every solution of* (1.3) *to be oscillatory is that*

$$\int^{\infty} [\tau(t)]^{\alpha} p(t)dt = \infty. \tag{1.5}$$

The condition $p(t) \ge 0$ is required only for the sufficiency part, and no similar criteria is available for $p(t)$ changing sign, except in the case $\tau(t) = t$. Without imposing a sign condition on $p(t)$, Belohorec [4] obtained the following sufficient condition for oscillation of (1.4).

Theorem 1.2. *If*

$$\int^{\infty} t^{\beta} p(t)dt = \infty \tag{1.6}$$

for some $\beta \in [0, \alpha]$, then every solution of (1.4) *is oscillatory.*

Compared to the large body of papers on oscillation of differential equations, there is only little known about the oscillation of impulsive differential equations; see [5–7] for equations with delay and [8–13] for equations without delay. For some applications of such equations, we may refer to [14–18]. The books [19, 20] are good sources for a general theory of impulsive differential equations.

The object of this paper is to extend Theorems 1.1 and 1.2 to impulsive differential equations of the form (1.1). The results show that the impulsive perturbations may greatly change the oscillatory behavior of the solutions. A nonoscillatory solution of (1.3) or (1.4) may become oscillatory under impulsive perturbations.

The following two lemmas are crucial in the proof of our main theorems. The first lemma is contained in [21] and the second one is extracted from [22].

Lemma 1.3. *If each* A_i *is continuous on* $[a,b]$*, then*

$$\int_a^b \sum_{s\leq\theta_i<b} A_i(s)ds = \sum_{a\leq\theta_i<b} \int_a^{\theta_i} A_i(s)ds. \tag{1.7}$$

Lemma 1.4. Fix $J=[a,b]$*, let* $u,\lambda \in C(J,\mathbb{R}_+)$*,* $h \in C(\mathbb{R}_+,\mathbb{R}_+)$*, and* $c\in\mathbb{R}_+$*, and let* $\{\lambda_k\}$ *a sequence of positive real numbers. If* $u(J)\subset I\subset \mathbb{R}_+$ *and*

$$u(t) \leq c + \int_a^t \lambda(s)h(u(s))ds + \sum_{a<\theta_k<t} \lambda_k h(u(\theta_k)), \quad t\in J, \tag{1.8}$$

then

$$u(t) \leq G^{-1}\left\{G(c) + \int_a^t \lambda(s)ds + \sum_{a<\theta_k<t}\lambda_k\right\}, \quad t\in[a,\beta), \tag{1.9}$$

where

$$\begin{gathered} G(u) = \int_{u_0}^u \frac{dx}{h(x)}, \quad u,u_0\in I, \\ \beta = \sup\left\{\nu\in J : G(c) + \int_a^t \lambda(s)ds + \sum_{a<\theta_k<t}\lambda_k \in G(I),\ a\leq t\leq \nu\right\}. \end{gathered} \tag{1.10}$$

2. The Main Results

We first establish a necessary and sufficient condition for oscillation of solutions of (1.1) when $\tau(t)\leq t$.

Theorem 2.1. *If*

$$\int^\infty [\tau(t)]^\alpha |p(t)|dt + \sum^\infty [\tau(\theta_k)]^\alpha |q_k| < \infty, \tag{2.1}$$

then (1.1) *has a solution* $x(t)$ *satisfying*

$$\lim_{t\to\infty}\frac{x(t)}{t} = a\neq 0. \tag{2.2}$$

Proof. Choose $t_1 \geq \max\{1, t_0\}$. In view of Lemma 1.3 by integrating (1.1) twice from t_0 to t, we obtain

$$x(t) = x(t_1) - x'(t_1)(t - t_1) - \sum_{t_1 \leq \theta_k < t} q_k |x(\tau(\theta_k))|^{\alpha-1} x((\tau(\theta_k)))(t - \theta_k) - \int_{t_1}^{t} (t - s)p(s)|x(\tau(s))|^{\alpha-1} x((\tau(s)))ds, \quad t \geq t_1. \tag{2.3}$$

Set

$$u(t) = c + \sum_{t_1 \leq \theta_k < t} |q_k| |x(\tau(\theta_k))|^{\alpha} + \int_{t_1}^{t} |p(s)| |x(\tau(s))|^{\alpha} \, ds, \quad t \geq t_1, \tag{2.4}$$

where $c = |x(t_1)| + |x'(t_1)|$. Then

$$|x(t)| \leq t u(t), \quad t \geq t_1. \tag{2.5}$$

Let $t_2 \geq t_1$ be such that $\tau(t) \geq t_1$ for all $t \geq t_2$. Replacing t by $\tau(t)$ in (2.5) and using the increasing character of $u(t)$, we see that

$$|x(\tau(t))| \leq \tau(t) u(t), \quad t \geq t_2. \tag{2.6}$$

From (2.4), we also see that

$$u'(t) = |p(t)| |x(\tau(t))|^{\alpha}, \quad t \neq \theta_k, \tag{2.7}$$

$$\Delta u(t)|_{t=\theta_k} = |q_k| |x(\tau(\theta_k))|^{\alpha} \tag{2.8}$$

for $t \geq t_2$ and $\theta_k \geq t_2$. Now, in view of (2.6) and (2.8), an integration of (2.7) from t_2 to t leads to

$$u(t) \leq c + \int_{t_2}^{t} |p(s)| [\tau(s)]^{\alpha} [u(s)]^{\alpha} ds + \sum_{t_2 \leq \theta_k < t} |q_k| [\tau(\theta_k)]^{\alpha} [u(\theta_k)]^{\alpha}. \tag{2.9}$$

Applying Lemma 1.4 with

$$h(x) = x^{\alpha}, \qquad \lambda(s) = |p(s)| [\tau(s)]^{\alpha}, \qquad \lambda_k = |q_k| [\tau(\theta_k)]^{\alpha}, \tag{2.10}$$

we easily see that

$$u(t) \leq G^{-1} \left\{ G(c) + \int_{t_2}^{t} |p(s)| [\tau(s)]^{\alpha} ds + \sum_{t_2 \leq \theta_k < t} |q_k| [\tau(\theta_k)]^{\alpha} \right\}. \tag{2.11}$$

Since

$$G(u) = \frac{u^{1-\alpha}}{1-\alpha} - \frac{u_0^{1-\alpha}}{1-\alpha}, \qquad G^{-1}(u) = \left[(1-\alpha)u + u_0^{1-\alpha}\right]^{1/(1-\alpha)}, \tag{2.12}$$

the inequality (2.11) becomes

$$u(t) \le \left[c^{1-\alpha} + (1-\alpha)\int_{t_1}^{t} \left|p(s)\right|[\tau(s)]^{\alpha} ds + (1-\alpha)\sum_{t_1\le\theta_k<t} \left|q_k\right|[\tau(\theta_k)]^{\alpha}\right]^{1/(1-\alpha)}, \tag{2.13}$$

from which, on using (2.1), we have

$$u(t) \le c_1, \quad t \ge t_2, \tag{2.14}$$

where

$$c_1 = \left[c^{1-\alpha} + (1-\alpha)\int_{t_1}^{\infty} \left|p(s)\right|[\tau(s)]^{\alpha} ds + (1-\alpha)\sum_{t_1\le\theta_k<\infty} \left|q_k\right|[\tau(\theta_k)]^{\alpha}\right]^{1/(1-\alpha)}. \tag{2.15}$$

In view of (2.5), (2.6), and (2.14) we see that

$$|x(t)| \le c_1 t, \quad |x(\tau(t))| \le c_1\tau(t), \quad t \ge t_2. \tag{2.16}$$

To complete the proof it suffices to show that $x'(t)$ approaches a nonzero limit as t tends to ∞. To see this we integrate (1.1) from t_2 to t to get

$$x'(t) = x'(t_1) - \int_{t_2}^{t} p(s)|x(\tau(s))|^{\alpha-1}x(\tau(s))ds - \sum_{t_2\le\theta_k<t} q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)). \tag{2.17}$$

Employing (2.16) we have

$$\begin{gathered}\int_{t_2}^{\infty} \left|p(s)x(\tau(s))\right|^{\alpha} ds \le c_1^{\alpha}\int_{t_2}^{\infty} \left|p(s)\right|[\tau(s)]^{\alpha} ds < \infty, \\ \sum_{t_2\le\theta_k<\infty} \left|q_k x(\tau(\theta_k))\right|^{\alpha} \le c_1^{\alpha}\sum_{t_2\le\theta_k<\infty} \left|q_k\right|[\tau(\theta_k)]^{\alpha} < \infty.\end{gathered} \tag{2.18}$$

Therefore, $\lim_{t\to\infty}x'(t) = L$ exists. Clearly, we can make $L \ne 0$ by requiring that

$$x'(t_2) > c_1^{\alpha}\left[\int_{t_2}^{\infty} \left|p(s)\right|[\tau(s)]^{\alpha} ds + \sum_{t_2\le\theta_k<\infty} \left|q_k\right|[\tau(\theta_k)]^{\alpha}\right], \tag{2.19}$$

which is always possible by arranging t_2. □

Theorem 2.2. *Suppose that p and $\{q_k\}$ are nonnegative. Then every solution of* (1.1) *is oscillatory if and only if*

$$\int^{\infty}[\tau(t)]^{\alpha}p(t)dt+\sum^{\infty}[\tau(\theta_k)]^{\alpha}q_k=\infty. \tag{2.20}$$

Proof. Let (2.20) fail to hold. Then, by Theorem 2.1 we see that there is a solution $x(t)$ which satisfies (2.2). Clearly, such a solution is nonoscillatory. This proves the necessity.

To show the sufficiency, suppose that (2.20) is valid but there is a nonoscillatory solution $x(t)$ of (1.1). We may assume that $x(t)$ is eventually positive; the case $x(t)$ being eventually negative is similar. Clearly, there exists $t_1 \geq t_0$ such that $x(\tau(t)) > 0$ for all $t \geq t_1$. From (1.1), we have that

$$x''(t)\leq 0 \quad \text{for } t\geq t_1,\ t\neq\theta_k. \tag{2.21}$$

Thus, $x'(t)$ is decreasing on every interval not containing $t=\theta_k$. From the impulse conditions in (1.1), we also have $\Delta x'(\theta_k)\leq 0$. Therefore, we deduce that $x'(t)$ is nondecreasing on $[t_1,\infty)$.

We may claim that $x'(t)$ is eventually positive. Because if $x'(t)<0$ eventually, then $x(t)$ becomes negative for large values of t. This is a contradiction.

It is now easy to show that

$$x(t)\geq (t-t_1)x'(t),\quad t\geq t_1. \tag{2.22}$$

Therefore,

$$x(t)\geq \frac{t}{2}x'(t),\quad t\geq t_2=2t_1. \tag{2.23}$$

Let $t_3\geq t_2$ be such that $\tau(t)\geq t_2$ for $t\geq t_3$. Using (2.23) and the nonincreasing character of $x'(t)$, we have

$$x(\tau(t))\geq \frac{\tau(t)}{2}x'(t),\quad t\geq t_3, \tag{2.24}$$

and so, by (1.1),

$$x''(t)+2^{-\alpha}p(t)[\tau(t)]^{\alpha}\left[x'(t)\right]^{\alpha}\leq 0,\quad t\neq\theta_k. \tag{2.25}$$

Dividing (2.25) by $[x'(t)]^{\alpha}$ and integrating from t_3 to t, we obtain

$$\begin{aligned}&\sum_{t_3\leq\theta_k<t}\left\{\left[x'(\theta_k)\right]^{1-\alpha}-\left[x'(\theta_k)-q_k[x(\tau(\theta_k))]^{\alpha}\right]^{1-\alpha}\right\}\\&\quad+\left[x'(t)\right]^{1-\alpha}-\left[x'(t_3)\right]^{1-\alpha}+(1-\alpha)2^{-\alpha}\int_{t_3}^{t}[\tau(t)]^{\alpha}p(s)ds\leq 0\end{aligned} \tag{2.26}$$

which clearly implies that

$$\sum_{t_3 \le \theta_k < t} a_k + (1-\alpha)2^{-\alpha} \int_{t_3}^{t} [\tau(t)]^{\alpha} p(s)ds \le \left[x'(t_3)\right]^{1-\alpha}, \tag{2.27}$$

where

$$a_k = \left[x'(\theta_k)\right]^{1-\alpha} \left[1 - \left(1 - \frac{q_k[x(\tau(\theta_k))]^{\alpha}}{x'(\theta_k)}\right)\right]^{1-\alpha}. \tag{2.28}$$

Since $1-(1-u)^{1-\alpha} \ge (1-\alpha)u$ for $u \in (0,\infty)$ and $0 < \alpha < 1$, by taking

$$u = \frac{q_k[x(\tau(\theta_k))]^{\alpha}}{x'(\theta_k)}, \tag{2.29}$$

we see from (2.28) that

$$a_k \ge (1-\alpha)\frac{q_k[x(\tau(\theta_k))]^{\alpha}}{[x'(\theta_k)]^{\alpha}}. \tag{2.30}$$

But, (2.24) gives

$$x(\tau(\theta_k)) \ge \frac{\tau(\theta_k)}{2} x'(\tau(\theta_k)) \ge \frac{\tau(\theta_k)}{2} x'(\theta_k), \tag{2.31}$$

and hence

$$a_k \ge (1-\alpha)2^{-\alpha}[\tau(\theta_k)]^{\alpha} q_k. \tag{2.32}$$

Finally, (2.27) and (2.32) result in

$$\int_{t_3}^{\infty} [\tau(t)]^{\alpha} p(t)dt + \sum_{t_3 < \theta_k < \infty} [\tau(\theta_k)]^{\alpha} q_k < \infty, \tag{2.33}$$

which contradicts (2.20). The proof is complete. □

Example 2.3. Consider the impulsive delay differential equation

$$\begin{gathered} x''(t) + (t-1)^{-2}|x(t-1)|^{-1/2}x(t-1) = 0, \quad t \ne k, \\ \Delta x'(t)\big|_{t=k} + (k-1)^{-1}|x(k-1)|^{-1/2}x(k-1) = 0, \\ \Delta x(t)|_{t=k} = 0, \end{gathered} \tag{2.34}$$

where $t \ge 2$ and $i \ge 2$.

We see that $\tau(t) = t - 1$, $\alpha = 1/2$, $p(t) = (t-1)^{-2}$, and $q_k = (k-1)^{-1}$, $\theta_k = k$. Since

$$\int^{\infty} (t-1)^{-3/2} dt + \sum^{\infty} (k-1)^{-1/2} = \infty, \tag{2.35}$$

applying Theorem 2.2 we conclude that every solution of (2.34) is oscillatory.

We note that if the equation is not subject to any impulse condition, then, since

$$\int^{\infty} (t-1)^{-5/2} dt < \infty, \tag{2.36}$$

the equation

$$x''(t) + (t-1)^{-2} |x(t-1)|^{-1/2} x(t-1) = 0 \tag{2.37}$$

has a nonoscillatory solution by Theorem 1.1.

Let us now consider (1.1) when $\tau(t) = t$. That is,

$$\begin{gathered} x'' + p(t)|x|^{\alpha-1}x = 0, \quad t \neq \theta_k, \\ \Delta x'\big|_{t=\theta_k} + q_k |x|^{\alpha-1} x = 0, \\ \Delta x|_{t=\theta_k} = 0, \end{gathered} \tag{2.38}$$

where $0 < \alpha < 1$ and p q_k are given by (a) and (b).

The following theorem is an extension of Theorem 1.2. Note that no sign condition is imposed on $p(t)$ and $\{q_k\}$.

Theorem 2.4. *If*

$$\int^{\infty} t^{\beta} p(t) dt + \sum^{\infty} \theta_k^{\beta} q_k = \infty \tag{2.39}$$

for some $\beta \in [0, \alpha]$, *then every solution of* (2.38) *is oscillatory.*

Proof. Assume on the contrary that (2.38) has a nonoscillatory solution $x(t)$ such that $x(t) > 0$ for all $t \geq t_0$ for some $t_0 \geq 0$. The proof is similar when $x(t)$ is eventually negative. We set

$$w(t) = \left(t^{-1} x(t)\right)^{1-\alpha}, \quad t \geq t_0. \tag{2.40}$$

It is not difficult to see that

$$w'(t) = (\alpha - 1) t^{\alpha-2} [x(t)]^{1-\alpha} + (1-\alpha) t^{\alpha-1} [x(t)]^{-\alpha} x'(t), \quad t \neq \theta_k, \tag{2.41}$$

and hence

$$\Delta w'\big|_{t=\theta_k} = (1-\alpha)q_k\theta_k^{\alpha-1}. \tag{2.42}$$

From (2.41), we have

$$\begin{aligned} t^{\beta-1-\alpha}\left(t^2 w'(t)\right)' &= (1-\alpha)t^{\beta}x''(t)x^{-\alpha}(t) \\ &\quad - \alpha(1-\alpha)t^{\beta-2}x^{-\alpha-1}(t)\left[tx'(t)-x(t)\right]^2, \end{aligned} \tag{2.43}$$

and so

$$t^{\beta-1-\alpha}\left(t^2 w'(t)\right)' \le (1-\alpha)t^{\beta}p(t), \quad t \ne \theta_k. \tag{2.44}$$

In view of (2.42), by a straightforward integration of (2.44), we have

$$\begin{aligned} \int_{t_0}^{t} s^{\beta-1-\alpha}\left(s^2 w'(s)\right)' ds &= s^{\beta-1-\alpha}s^2 w'(s)\Big|_{t_0}^{t} - \sum_{t_0\le\theta_k<t}\Delta\left(t^{\beta-\alpha+1}w'(t)\right)\Big|_{t=\theta_k} \\ &\quad - \int_{t_0}^{t}(\beta-1-\alpha)s^{\beta-\alpha}w'(s)ds \\ &= t^{\beta-\alpha+1}w'(t) - t_0^{\beta-\alpha+1}w'(t_0) - \sum_{t_0\le\theta_k<t}(1-\alpha)q_k\theta_k^{\beta} \\ &\quad -(\beta-\alpha-1)\left[s^{\beta-\alpha}w(s)\right]\Big|_{t_0}^{t} \\ &\quad + (\beta-\alpha)(\beta-\alpha-1)\int_{t_0}^{t}s^{\beta-1-\alpha}w(s)ds, \end{aligned} \tag{2.45}$$

which combined with (2.44) leads to

$$\begin{aligned} t^{\beta-\alpha+1}w'(t) &\le t_0^{\beta-\alpha+1}w'(t_0) - (\beta-\alpha+1)t_0^{\beta-\alpha}w(t_0) \\ &\quad + (1-\alpha)\left[\sum_{t_0\le\theta_k<t}\theta_k^{\beta}q_k + \int_{t_0}^{t}s^{\beta}p(s)ds\right]. \end{aligned} \tag{2.46}$$

Finally, by using (2.39) in the last inequality, we see that there is a $t_1 > t_0$ such that

$$w'(t) \le -t^{\alpha-\beta-1}, \quad t \ge t_1, \tag{2.47}$$

which, however, implies that $w(t) \to -\infty$ as $t \to \infty$, a contradiction with $x(t) > 0$. The proof is complete. □

Example 2.5. Consider the impulsive differential equation

$$x'' + t^{-7/3}|x|^{-1/2}x = 0, \quad t \neq k,$$
$$\Delta x'\big|_{t=k} + k^{-1/6}|x|^{-1/2}x = 0, \tag{2.48}$$
$$\Delta x|_{t=k} = 0,$$

where $t \geq 1$ and $i \geq 1$.

We have that $p(t) = t^{7/3}$, $\alpha = 1/2$, and $q_k = k^{-1/6}$, $\theta_k = k$. Taking $\beta = 1/3$ we see from (2.38) that

$$\int^{\infty} t^{-2}dt + \sum^{\infty} k^{-1/3} = \infty. \tag{2.49}$$

Since the conditions of Theorem 2.4 are satisfied, every solution of (2.48) is oscillatory.

Note that if the impulses are absent, then, since

$$\int^{\infty} t^{-2}dt < \infty, \tag{2.50}$$

the equation

$$x'' + t^{-7/3}|x|^{-1/2}x = 0 \tag{2.51}$$

is oscillatory by Theorem 1.2.

Acknowledgment

This work was partially supported by METU-B AP (project no: 01-01-2011-003).

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