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*Research Article*

# A Class of Analytic Functions with Missing Coefficients

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Let $T_n(A,B,\gamma,\alpha)$ $(-1 \le B < 1, B < A, 0 < \gamma \le 1$ and $\alpha > 0)$ denote the class of functions of the form $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ $(n \in N = \{1,2,3,\ldots\})$, which are analytic in the open unit disk $U$ and satisfy the following subordination condition $f'(z) + \alpha z f''(z) \prec ((1+Az)/(1+Bz))^{\gamma}$, for $(z \in U; A \le 1; 0 < \gamma < 1)$, $(1+Az)/(1+Bz)$, for $(z \in U; \gamma = 1)$. We obtain sharp bounds on $\mathrm{Re} f'(z)$, $\mathrm{Re} f(z)/z$, $|f(z)|$, and coefficient estimates for functions $f(z)$ belonging to the class $T_n(A,B,\gamma,\alpha)$. Conditions for univalency and starlikeness, convolution properties, and the radius of convexity are also considered.

## 1. Introduction

Let $A_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N = \{1,2,3,\ldots\}), \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Let $S_n$ and $S_n^*$ denote the subclasses of $A_n$ whose members are univalent and starlike, respectively.

For functions $f(z)$ and $g(z)$ analytic in $U$, we say that $f(z)$ is subordinate to $g(z)$ in $U$ and we write $f(z) \prec g(z)$ $(z \in U)$, if there exists an analytic function $w(z)$ in $U$ such that

$$|w(z)| \le |z|, \qquad f(z) = g(w(z)) \quad (z \in U). \tag{1.2}$$

Furthermore, if the function $g(z)$ is univalent in $U$, then

$$f(z) \prec g(z) \quad (z \in U) \Longleftrightarrow f(0) = g(0), \quad f(U) \subset g(U). \tag{1.3}$$

Throughout our present discussion, we assume that

$$n \in N, \quad -1 \leq B < 1, \quad B < A, \quad \alpha > 0, \quad \beta < 1, \quad 0 < \gamma \leq 1. \tag{1.4}$$

We introduce the following subclass of $A_n$.

*Definition 1.1.* A function $f(z) \in A_n$ is said to be in the class $T_n(A, B, \gamma, \alpha)$ if it satisfies

$$f'(z) + \alpha z f''(z) \prec h(z) \quad (z \in U), \tag{1.5}$$

where

$$h(z) = \begin{cases} \left(\dfrac{1 + Az}{1 + Bz}\right)^{\gamma}, & (A \leq 1;\, 0 < \gamma < 1), \\ \dfrac{1 + Az}{1 + Bz}, & (\gamma = 1). \end{cases} \tag{1.6}$$

The classes

$$T_1(1 - 2\beta, -1, 1, 1) = R(\beta) \quad (\beta = 0 \quad \text{or} \quad \beta < 1), \qquad T_1(A, 0, 1, \alpha) = \widetilde{R}(\alpha, A) \quad (A > 0) \tag{1.7}$$

have been studied by several authors (see [1–5]). Recently, Gao and Zhou [6] showed some mapping properties of the following subclass of $A_1$:

$$R(\beta, \alpha) = \left\{ f(z) \in A_1 : \operatorname{Re}\left\{ f'(z) + \alpha z f''(z) \right\} > \beta \ (z \in U) \right\}. \tag{1.8}$$

Note that

$$R(\beta, 1) = R(\beta), \qquad T_1(1 - 2\beta, -1, 1, \alpha) = R(\beta, \alpha). \tag{1.9}$$

For further information of the above classes (with $\gamma = 1$) and related analytic function classes, see Srivastava et al. [7], Yang and Liu [8], Kim [9], and Kim and Srivastava [10].

In this paper, we obtain sharp bounds on $\operatorname{Re} f'(z)$, $\operatorname{Re}(f(z)/z)$, $|f(z)|$, and coefficient estimates for functions $f(z)$ belonging to the class $T_n(A, B, \gamma, \alpha)$. Conditions for univalency and starlikeness, convolution properties, and the radius of convexity are also presented. One can see that the methods used in [6] do not work for the more general class $T_n(A, B, \gamma, \alpha)$ than $R(\beta, \alpha)$.

## 2. The bounds on $\mathbf{Re} f'(z)$, $\mathbf{Re}(f(z)/z)$, and $|f(z)|$ in $T_n(A,B,\gamma,\alpha)$

In this section, we let

$$\lambda_m(A,B,\gamma)=\begin{cases}\displaystyle\sum_{j=0}^{m}\binom{\gamma}{j}\binom{-\gamma}{m-j}A^jB^{m-j}, & (A\le 1; 0<\gamma<1),\\ (A-B)(-B)^{m-1}, & (\gamma=1),\end{cases}\tag{2.1}$$

where $m\in N$ and

$$\binom{\gamma}{j}=\begin{cases}\dfrac{\gamma(\gamma-1)\cdots(\gamma-j+1)}{j!}, & (j=1,2,\ldots,m),\\ 1, & (j=0).\end{cases}\tag{2.2}$$

With (2.1), it is easily seen that the function $h(z)$ given by (1.6) can be expressed as

$$h(z)=1+\sum_{m=1}^{\infty}\lambda_m(A,B,\gamma)z^m\quad(z\in U).\tag{2.3}$$

**Theorem 2.1.** *Let $f(z)\in T_n(A,B,\gamma,\alpha)$. Then, for $|z|=r<1$,*

$$\begin{aligned}\operatorname{Re} f'(z)&\ge 1+\sum_{m=1}^{\infty}\frac{(-1)^m\lambda_m(A,B,\gamma)}{\alpha nm+1}r^{nm},\\ \operatorname{Re} f'(z)&\le 1+\sum_{m=1}^{\infty}\frac{\lambda_m(A,B,\gamma)}{\alpha nm+1}r^{nm}.\end{aligned}\tag{2.4}$$

*The bounds in* (2.4) *are sharp for the function $f_n(z)$ defined by*

$$f_n(z)=z+\sum_{m=1}^{\infty}\frac{\lambda_m(A,B,\gamma)}{(nm+1)(\alpha nm+1)}z^{nm+1}\quad(z\in U).\tag{2.5}$$

*Proof.* The analytic function $h(z)$ given by (1.6) is convex (univalent) in $U$ (cf. [11]) and satisfies $h(\overline{z})=\overline{h(z)}$ $(z\in U)$. Thus, for $|\zeta|\le\sigma(\zeta\in C$ *and* $\sigma<1)$,

$$h(-\sigma)\le \operatorname{Re} h(\zeta)\le h(\sigma).\tag{2.6}$$

Let $f(z)\in T_n(A,B,\gamma,\alpha)$. Then, we can write

$$f'(z)+\alpha zf''(z)=h(w(z))\quad(z\in U),\tag{2.7}$$

where $w(z) = w_n z^n + w_{n+1} z^{n+1} + \cdots$ is analytic and $|w(z)| < 1$ for $z \in U$. By the Schwarz lemma, we know that $|w(z)| \le |z|^n$ $(z \in U)$. It follows from (2.7) that

$$\left(z^{1/\alpha} f'(z)\right)' = \frac{1}{\alpha} z^{(1/\alpha)-1} h(w(z)), \tag{2.8}$$

which leads to

$$f'(z) = \frac{1}{\alpha} z^{-1/\alpha} \int_0^z \zeta^{(1/\alpha)-1} h(w(\zeta)) d\zeta \tag{2.9}$$

or to

$$f'(z) = \frac{1}{\alpha} \int_0^1 t^{(1/\alpha)-1} h(w(tz)) dt \quad (z \in U). \tag{2.10}$$

Since

$$|w(tz)| \le (tr)^n \quad (|z| = r < 1;\ 0 \le t \le 1), \tag{2.11}$$

we deduce from (2.6) and (2.10) that

$$\frac{1}{\alpha} \int_0^1 t^{(1/\alpha)-1} h\left(-(tr)^n\right) dt \le \operatorname{Re} f'(z) \le \frac{1}{\alpha} \int_0^1 t^{(1/\alpha)-1} h\left((tr)^n\right) dt. \tag{2.12}$$

Now, by using (2.3) and (2.12), we can obtain (2.4).

Furthermore, for the function $f_n(z)$ defined by (2.5), we find that

$$f_n'(z) = 1 + \sum_{m=1}^{\infty} \frac{\lambda_m(A,B,\gamma)}{\alpha n m + 1} z^{nm}, \tag{2.13}$$

$$f_n'(z) + \alpha z f_n''(z) = 1 + \sum_{m=1}^{\infty} \lambda_m(A,B,\gamma) z^{nm} = h(z^n) \prec h(z) \quad (z \in U). \tag{2.14}$$

Hence, $f_n(z) \in T_n(A,B,\gamma,\alpha)$ and from (2.13), we see that the bounds in (2.4) are the best possible.

Hereafter, we write

$$T_n(A,B,1,\alpha) = T_n(A,B,\alpha). \tag{2.15}$$

□

**Corollary 2.2.** *Let $f(z) \in T_n(A,B,\alpha)$. Then, for $z \in U$,*

$$\operatorname{Re} f'(z) > 1 - (A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm+1}, \tag{2.16}$$

$$\operatorname{Re} f'(z) < 1 + (A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{\alpha nm+1} \quad (B \neq -1). \tag{2.17}$$

*The results are sharp.*

*Proof.* For $\gamma = 1$, it follows from (2.12) (used in the proof of Theorem 2.1) that

$$\begin{aligned}
\operatorname{Re} f'(z) &> \frac{1}{\alpha}\int_0^1 t^{(1/\alpha)-1}\left(\frac{1-At^n}{1-Bt^n}\right)dt, \\
\operatorname{Re} f'(z) &< \frac{1}{\alpha}\int_0^1 t^{(1/\alpha)-1}\left(\frac{1+At^n}{1+Bt^n}\right)dt \quad (B \neq -1),
\end{aligned} \tag{2.18}$$

for $z \in U$. From these, we have the desired results. □

The bounds in (2.16) and (2.17) are sharp for the function

$$f_n(z) = z + (A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{(nm+1)(\alpha nm+1)}z^{nm+1} \in T_n(A,B,\alpha). \tag{2.19}$$

**Theorem 2.3.** *Let $f(z) \in T_n(A,B,\gamma,\alpha)$. Then, for $|z| = r < 1$,*

$$\begin{aligned}
\operatorname{Re}\frac{f(z)}{z} &\geq 1 + \sum_{m=1}^{\infty}\frac{(-1)^m\lambda_m(A,B,\gamma)}{(nm+1)(\alpha nm+1)}r^{nm}, \\
\operatorname{Re}\frac{f(z)}{z} &\leq 1 + \sum_{m=1}^{\infty}\frac{\lambda_m(A,B,\gamma)}{(nm+1)(\alpha nm+1)}r^{nm}.
\end{aligned} \tag{2.20}$$

*The results are sharp.*

*Proof.* Noting that

$$f(z) = z\int_0^1 f'(uz)du, \qquad \operatorname{Re}\frac{f(z)}{z} = \int_0^1 \operatorname{Re} f'(uz)du \quad (z \in U), \tag{2.21}$$

an application of Theorem 2.1 yields (2.20). Furthermore, the results are sharp for the function $f_n(z)$ defined by (2.5). □

**Corollary 2.4.** *Let $f(z) \in T_n(A,B,\alpha)$. Then, for $z \in U$,*

$$\operatorname{Re}\frac{f(z)}{z} > 1-(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{(nm+1)(\alpha nm+1)},$$
$$\operatorname{Re}\frac{f(z)}{z} < 1+(A-B)\sum_{m=1}^{\infty}\frac{(-B)^{m-1}}{(nm+1)(\alpha nm+1)}. \tag{2.22}$$

*The results are sharp for the function $f_n(z)$ defined by* (2.19).

*Proof.* For $f(z) \in T_n(A,B,\alpha)$, it follows from (2.6) and (2.10) (with $\gamma = 1$) that

$$\frac{1}{\alpha}\int_0^1 t^{(1/\alpha)-1}\left(\frac{1-A(ut)^n}{1-B(ut)^n}\right)dt < \operatorname{Re} f'(uz) < \frac{1}{\alpha}\int_0^1 t^{(1/\alpha)-1}\left(\frac{1+A(ut)^n}{1+B(ut)^n}\right)dt, \tag{2.23}$$

for $z \in U$ and $0 < u \le 1$. Making use of (2.21) and (2.23), we can obtain (2.22). □

**Theorem 2.5.** *Let $f(z) \in T_1(A,B,\alpha)$ and $g(z) \in T_1(A_0,B_0,\alpha_0)$ $(-1 \le B_0 < 1, B_0 < A_0$ and $\alpha_0 > 0)$. If*

$$(A_0-B_0)\sum_{m=1}^{\infty}\frac{B_0^{m-1}}{(m+1)(\alpha_0 m+1)} \le \frac{1}{2}, \tag{2.24}$$

*then $(f * g)(z) \in T_1(A,B,\alpha)$, where the symbol $*$ stands for the familiar Hadamard product (or convolution) of two analytic functions in $U$.*

*Proof.* Since $g(z) \in T_1(A_0,B_0,\alpha_0)(-1 \le B_0 < 1, B_0 < A_0$ and $\alpha_0 > 0)$, it follows from Corollary 2.4 (with $n = 1$) and (2.24) that

$$\operatorname{Re}\frac{g(z)}{z} > 1-(A_0-B_0)\sum_{m=1}^{\infty}\frac{B_0^{m-1}}{(m+1)(\alpha_0 m+1)} \ge \frac{1}{2} \quad (z \in U). \tag{2.25}$$

Thus, $g(z)/z$ has the Herglotz representation

$$\frac{g(z)}{z} = \int_{|x|=1}\frac{d\mu(x)}{1-xz} \quad (z \in U), \tag{2.26}$$

where $\mu(x)$ is a probability measure on the unit circle $|x| = 1$ and $\int_{|x|=1} d\mu(x) = 1$.

For $f(z) \in T_1(A,B,\alpha)$, we have

$$(f * g)'(z) + \alpha z (f * g)''(z) = F(z) * \frac{g(z)}{z} \quad (z \in U), \tag{2.27}$$

where

$$F(z) = f'(z) + \alpha z f''(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{2.28}$$

In view of the function $(1 + Az)/(1 + Bz)$ is convex (univalent) in $U$, we deduce from (2.26) to (2.28) that

$$(f * g)'(z) + \alpha z (f * g)''(z) = \int_{|x|=1} F(xz) d\mu(x) \prec \frac{1 + Az}{1 + Bz} \quad (z \in U). \tag{2.29}$$

This shows that $(f * g)(z) \in T_1(A, B, \alpha)$. □

**Corollary 2.6.** *Let $f(z) \in T_1(A, B, \alpha)$, $g(z) \in R(\beta, 1)$ and*

$$\beta \geq -\frac{\pi^2 - 9}{12 - \pi^2}. \tag{2.30}$$

*Then, $(f * g)(z) \in T_1(A, B, \alpha)$.*

*Proof.* By taking $A_0 = 1 - 2\beta$, $B_0 = -1$ and $\alpha_0 = 1$, (2.24) in Theorem 2.5 becomes

$$2(1 - \beta) \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(m + 1)^2} = 2(1 - \beta)\left(1 - \frac{\pi^2}{12}\right) \leq \frac{1}{2}, \tag{2.31}$$

that is,

$$\beta \geq -\frac{\pi^2 - 9}{12 - \pi^2}. \tag{2.32}$$

Hence, the desired result follows as a special case from Theorem 2.5. □

*Remark 2.7.* R. Singh and S. Singh [4, Theorem 3] proved that, if $f(z)$ and $g(z)$ belong to $R(0, 1)$, then $(f * g)(z) \in R(0, 1)$. Obviously, for

$$-\frac{\pi^2 - 9}{12 - \pi^2} \leq \beta < 0, \tag{2.33}$$

Corollary 2.6 generalizes and improves Theorem 3 in [4].

**Theorem 2.8.** *Let $f(z) \in T_n(A, B, \gamma, \alpha)$ and $AB \leq 1$. Then, for $|z| = r < 1$,*

$$|f(z)| \leq r + \sum_{m=1}^{\infty} \frac{\lambda_m(A, B, \gamma)}{(nm + 1)(\alpha nm + 1)} r^{nm+1}. \tag{2.34}$$

*The result is sharp, with the extremal function $f_n(z)$ defined by* (2.5).

*Proof.* It is well known that for $\zeta \in C$ and $|\zeta| \leq \sigma < 1$,

$$\left| \frac{1 + A\zeta}{1 + B\zeta} - \frac{1 - AB\sigma^2}{1 - B^2\sigma^2} \right| \leq \frac{(A - B)\sigma}{1 - B^2\sigma^2}. \tag{2.35}$$

Since $AB \leq 1$, we have $1 - AB\sigma^2 > 0$ and so (2.35) leads to

$$\left| \frac{1 + A\zeta}{1 + B\zeta} \right|^{\gamma} \leq \left( \left| \frac{1 - AB\sigma^2}{1 - B^2\sigma^2} \right| + \frac{(A - B)\sigma}{1 - B^2\sigma^2} \right)^{\gamma} = \left( \frac{1 + A\sigma}{1 + B\sigma} \right)^{\gamma} \quad (|\zeta| \leq \sigma < 1). \tag{2.36}$$

By virtue of (1.6), (2.10), and (2.36), we have

$$\left| f'(uz) \right| \leq \frac{1}{\alpha} \int_0^1 t^{(1/\alpha)-1} |h(w(utz))| dt \leq \frac{1}{\alpha} \int_0^1 t^{(1/\alpha)-1} h\left( (ut|z|)^n \right) dt, \tag{2.37}$$

for $z \in U$ and $0 \leq u \leq 1$. Now, by using (2.3), (2.21) and (2.37), we can obtain (2.34). □

**Theorem 2.9.** *Let*

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in T_n(A, B, \gamma, \alpha). \tag{2.38}$$

*Then,*

$$|a_k| \leq \frac{\gamma(A - B)}{k(\alpha(k - 1) + 1)} \quad (k \geq n + 1). \tag{2.39}$$

*The result is sharp for each $k \geq n + 1$.*

*Proof.* It is known (cf. [12]) that, if

$$\varphi(z) = \sum_{k=1}^{\infty} b_k z^k \prec \psi(z) \quad (z \in U), \tag{2.40}$$

where $\varphi(z)$ is analytic in $U$ and $\psi(z) = z + \cdots$ is analytic and convex univalent in $U$, then $|b_k| \leq 1$ $(k \in N)$.

By (2.38), we have

$$\frac{f'(z) + \alpha z f''(z) - 1}{\gamma(A - B)} = \sum_{k=n+1}^{\infty} \frac{k(\alpha(k - 1) + 1)}{\gamma(A - B)} a_k z^{k-1} \prec \psi(z) \quad (z \in U), \tag{2.41}$$

where

$$\psi(z) = \frac{h(z) - 1}{\gamma(A - B)} = z + \cdots \tag{2.42}$$

and $h(z)$ is given by (1.6). Since the function $\psi(z)$ is analytic and convex univalent in $U$, it follows from (2.41) that

$$\frac{k(\alpha(k-1)+1)}{\gamma(A-B)}|a_k| \leq 1 \quad (k \geq n+1), \tag{2.43}$$

which gives (2.39).

Next, we consider the function

$$f_{k-1}(z) = z + \sum_{m=1}^{\infty} \frac{\lambda_m(A,B,\gamma)}{(m(k-1)+1)(\alpha m(k-1)+1)} z^{m(k-1)+1} \quad (z \in U; k \geq n+1). \tag{2.44}$$

It is easy to verify that

$$\begin{gathered} f'_{k-1}(z) + \alpha z f''_{k-1}(z) = h\left(z^{k-1}\right) \prec h(z) \quad (z \in U), \\ f_{k-1}(z) = z + \frac{\gamma(A-B)}{k(\alpha(k-1)+1)} z^k + \cdots . \end{gathered} \tag{2.45}$$

The proof of Theorem 2.9 is completed. □

## 3. The Univalency and Starlikeness of $T_n(A,B,\alpha)$

**Theorem 3.1.** *$T_n(A,B,\alpha) \subset S_n$ if and only if*

$$(A-B)\sum_{m=1}^{\infty} \frac{B^{m-1}}{\alpha n m+1} \leq 1. \tag{3.1}$$

*Proof.* Let $f(z) \in T_n(A,B,\alpha)$ and (3.1) be satisfied. Then, by (2.16) in Corollary 2.2, we see that $\operatorname{Re} f'(z) > 0 (z \in U)$. Thus, $f(z)$ is close-to-convex and univalent in $U$.

On the other hand, if

$$(A-B)\sum_{m=1}^{\infty} \frac{B^{m-1}}{\alpha n m+1} > 1, \tag{3.2}$$

then the function $f_n(z)$ defined by (2.19) satisfies $f'_n(0) = 1 > 0$ and

$$f'_n\left(re^{\pi i/n}\right) = 1-(A-B)\sum_{m=1}^{\infty} \frac{B^{m-1}}{\alpha n m+1} r^{nm} \longrightarrow 1-(A-B)\sum_{m=1}^{\infty} \frac{B^{m-1}}{\alpha n m+1} < 0 \tag{3.3}$$

as $r \to 1$. Hence, there exists a point $z_n = r_n e^{\pi i/n} (0 < r_n < 1)$ such that $f'_n(z_n) = 0$. This implies that $f_n(z)$ is not univalent in $U$ and so the theorem is proved. □

**Theorem 3.2.** *Let* (3.1) *in Theorem 3.1 be satisfied. If* $\alpha \geq 1$ *and*

$$(\alpha-1)\left(1-(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm+1}\right)+\frac{n\alpha}{2}\left(1-(A-B)\sum_{m=1}^{\infty}\frac{B^{m-1}}{(nm+1)(\alpha nm+1)}\right)\geq\frac{A-1}{1-B}, \tag{3.4}$$

*then* $T_n(A,B,\alpha)\subset S_n^*$.

*Proof.* We first show that

$$\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm+1}\geq\sum_{m=1}^{\infty}\frac{B^{m-1}}{(nm+1)(\alpha nm+1)}\quad(\alpha\geq 1). \tag{3.5}$$

Equation (3.5) is obvious when $B \geq 0$. For $0 > B \geq -1$, we have

$$\sum_{m=1}^{\infty}\frac{B^{m-1}}{\alpha nm+1}-\sum_{m=1}^{\infty}\frac{B^{m-1}}{(nm+1)(\alpha nm+1)}=\mu_1-\mu_2+\mu_3-\mu_4+\cdots+(-1)^{m-1}\mu_m+\cdots, \tag{3.6}$$

where

$$\mu_m=\frac{nm|B|^{m-1}}{(nm+1)(\alpha nm+1)}>0. \tag{3.7}$$

Since $|B| \leq 1$ and

$$\frac{d}{dx}\left(\frac{x}{(x+1)(\alpha x+1)}\right)=\frac{1-\alpha x^2}{(x+1)^2(\alpha x+1)^2}\leq 0\quad(x\geq 1;\alpha\geq 1), \tag{3.8}$$

$\{\mu_m\}$ is a monotonically decreasing sequence. Therefore, the inequality (3.5) follows from (3.6).

Let $f(z)\in T_n(A,B,\alpha)$. Then,

$$\mathrm{Re}\{f'(z)+\alpha zf''(z)\}>\frac{1-A}{1-B}\quad(z\in U). \tag{3.9}$$

Define $p(z)$ in $U$ by

$$p(z)=\frac{zf'(z)}{f(z)}. \tag{3.10}$$

In view of (3.1) in Theorem 3.1 is satisfied, the function $f(z)$ is univalent in $U$, and so $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots$ is analytic in $U$. Also it follows from (3.10) that

$$f'(z) + \alpha z f''(z) = (1-\alpha) f'(z) + \alpha \frac{f(z)}{z}\left[z p'(z) + \left(p(z)\right)^2\right]. \tag{3.11}$$

□

We want to prove now that $\operatorname{Re} p(z) > 0$ for $z \in U$. Suppose that there exists a point $z_0 \in U$ such that

$$\operatorname{Re} p(z) > 0 \quad (|z| < |z_0|), \qquad \operatorname{Re} p(z_0) = 0. \tag{3.12}$$

Then, applying a result of Miller and Mocanu [13, Theorem 4], we have

$$z_0 p'(z_0) + \left(p(z_0)\right)^2 \leq -\frac{n}{2} \operatorname{Re}\left(1 - p(z_0)\right) - \left(\operatorname{Im} p(z_0)\right)^2 \leq -\frac{n}{2}. \tag{3.13}$$

For $\alpha \geq 1$, we deduce from Corollaries 2.2 and 2.4, (3.1), (3.5), (3.11), (3.13), and (3.4) that

$$\begin{aligned} \operatorname{Re}\left\{f'(z_0) + \alpha z_0 f''(z_0)\right\} &\leq (1-\alpha)\left(1 - (A-B)\sum_{m=1}^{\infty} \frac{B^{m-1}}{\alpha n m + 1}\right) \\ &\quad - \frac{n\alpha}{2}\left(1 - (A-B)\sum_{m=1}^{\infty} \frac{B^{m-1}}{(nm+1)(\alpha n m + 1)}\right) \\ &\leq \frac{1-A}{1-B}. \end{aligned} \tag{3.14}$$

But this contradicts (3.9) at $z = z_0$. Therefore, we must have $\operatorname{Re} p(z) > 0$ $(z \in U)$ and the proof of Theorem 3.2 is completed.

*Remark 3.3.* In [6, Theorem 4(ii)], the authors gave the following: if $0 < \alpha < 1$ and $\beta_1$ is the solution of the equation

$$1 - \frac{3\alpha}{2} = \beta + (1-\beta)\sum_{m=2}^{\infty} \frac{(-1)^{m-1}\alpha + 2(\alpha - 1)m}{m(\alpha(m-1)+1)}, \tag{3.15}$$

then $R(\beta, \alpha) \subset S_1^*$ for $\beta \geq \beta_1$. However, this result is not true because the series in (3.15) diverges.

## 4. The Radius of Convexity

**Theorem 4.1.** *Let $f(z)$ belong to the class $T_n(\gamma)$ defined by*

$$T_n(\gamma) = T_n(1, -1, \gamma, 0) = \left\{ f(z) \in A_n : f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\gamma},\ (z \in U) \right\}, \tag{4.1}$$

*$0 < \delta \leq 1$ and $0 \leq \rho < 1$. Then,*

$$\operatorname{Re}\left\{(1-\delta)\left(f'(z)\right)^{1/\gamma} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \rho \quad \left(|z| < r_n(\gamma, \delta, \rho)\right), \tag{4.2}$$

*where $r_n(\gamma, \delta, \rho)$ is the root in $(0, 1)$ of the equation*

$$(1 - 2\delta + \rho)r^{2n} - 2(1 - \delta + n\delta\gamma)r^n + 1 - \rho = 0. \tag{4.3}$$

*The result is sharp.*

*Proof.* For $f(z) \in T_n(\gamma)$, we can write

$$\left(f'(z)\right)^{1/\gamma} = \frac{1 + z^n\varphi(z)}{1 - z^n\varphi(z)}, \tag{4.4}$$

where $\varphi(z)$ is analytic and $|\varphi(z)| \leq 1$ in $U$. Differentiating both sides of (4.4) logarithmically, we arrive at

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2n\gamma z^n\varphi(z)}{1 - \left(z^n\varphi(z)\right)^2} + \frac{2\gamma z^{n+1}\varphi'(z)}{1 - \left(z^n\varphi(z)\right)^2} \quad (z \in U). \tag{4.5}$$

Put $|z| = r < 1$ and $\left(f'(z)\right)^{1/\gamma} = u + iv$ $(u, v \in R)$. Then, (4.4) implies that

$$z^n\varphi(z) = \frac{u - 1 + iv}{u + 1 + iv}, \tag{4.6}$$

$$\frac{1 - r^n}{1 + r^n} \leq u \leq \frac{1 + r^n}{1 - r^n}. \tag{4.7}$$

With the help of the Carathéodory inequality

$$\left|\varphi'(z)\right| \leq \frac{1 - \left|\varphi(z)\right|^2}{1 - r^2}, \tag{4.8}$$

it follows from (4.5) and (4.6) that

$$\begin{aligned}
&\operatorname{Re}\left\{(1-\delta)\left(f'(z)\right)^{1/\gamma}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\} \\
&\quad\geq (1-\delta)u+\delta+2n\delta\gamma\operatorname{Re}\left\{\frac{z^n\varphi(z)}{1-\left(z^n\varphi(z)\right)^2}\right\}-2\delta\gamma\left|\frac{z^{n+1}\varphi'(z)}{1-\left(z^n\varphi(z)\right)^2}\right| \\
&\quad\geq (1-\delta)u+\delta+\frac{n\delta\gamma}{2}\left(u-\frac{u}{u^2+v^2}\right)+\frac{\delta\gamma}{2}\frac{(u-1)^2+v^2-r^{2n}\left((u+1)^2+v^2\right)}{r^{n-1}(1-r^2)(u^2+v^2)^{1/2}} \\
&\quad= F_n(u,v)\quad (say),
\end{aligned}\tag{4.9}$$

$$\frac{\partial}{\partial v}F_n(u,v)=\delta\gamma v G_n(u,v),\tag{4.10}$$

where $0<r<1, 0<\delta\leq 1$ and

$$\begin{aligned}
G_n(u,v) &= \frac{nu}{(u^2+v^2)^2}+\frac{1-r^{2n}}{r^{n-1}(1-r^2)(u^2+v^2)^{1/2}}+\frac{r^{2n}\left((u+1)^2+v^2\right)-\left((u-1)^2+v^2\right)}{2r^{n-1}(1-r^2)(u^2+v^2)^{3/2}} \\
&> 0
\end{aligned}\tag{4.11}$$

because of (4.6) and (4.7). In view of (4.10) and (4.11), we see that

$$\begin{aligned}
F_n(u,v) &\geq F_n(u,0) \\
&= (1-\delta)u+\delta+\frac{n\delta\gamma}{2}\left(u-\frac{1}{u}\right)+\frac{\delta\gamma}{2r^{n-1}(1-r^2)} \\
&\quad\times\left\{\left(1-r^{2n}\right)\left(u+\frac{1}{u}\right)-2\left(1+r^{2n}\right)\right\}.
\end{aligned}\tag{4.12}$$

□

Let us now calculate the minimum value of $F_n(u,0)$ on the closed interval $[(1-r^n)/(1+r^n),(1+r^n)/(1-r^n)]$. Noting that

$$\frac{1-r^{2n}}{r^{n-1}(1-r^2)}\geq n\quad (\text{see [8]})\tag{4.13}$$

and (4.7), we deduce from (4.12) that

$$\begin{aligned}\frac{d}{du}F_n(u,0) &= 1-\delta+\frac{\delta\gamma}{2}\left[\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}+n\right)-\frac{1}{u^2}\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}-n\right)\right]\\ &\geq 1-\delta+\frac{\delta\gamma}{2}\left[\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}+n\right)-\left(\frac{1+r^n}{1-r^n}\right)^2\left(\frac{1-r^{2n}}{r^{n-1}(1-r^2)}-n\right)\right]\\ &= 1-\delta+\frac{2\delta\gamma I_n(r)}{(1-r^n)^2},\end{aligned} \tag{4.14}$$

where

$$I_n(r)=\frac{n}{2}\left(1+r^{2n}\right)-r\left(1+r^2+\cdots+r^{2n-2}\right). \tag{4.15}$$

Also

$$I_n'(r)=n^2r^{2n-1}-\left(1+3r^2+\cdots+(2n-1)r^{2n-2}\right) \tag{4.16}$$

and $I_1'(r)=r-1<0$. Suppose that $I_n'(r)<0$. Then,

$$\begin{aligned}I_{n+1}'(r) &= (n+1)^2r^{2n+1}-(2n+1)r^{2n}-\left(1+3r^2+\cdots+(2n-1)r^{2n-2}\right)\\ &< n^2r^{2n}-\left(1+3r^2+\cdots+(2n-1)r^{2n-2}\right)<I_n'(r)<0.\end{aligned} \tag{4.17}$$

Hence, by virtue of the mathematical induction, we have $I_n'(r)<0$ for all $n\in N$ and $0\leq r<1$. This implies that

$$I_n(r)>I_n(1)=0 \quad (n\in N;\ 0\leq r<1). \tag{4.18}$$

In view of (4.14) and (4.18), we see that

$$\frac{d}{du}F_n(u,0)>0 \quad \left(\frac{1-r^n}{1+r^n}\leq u\leq\frac{1+r^n}{1-r^n}\right). \tag{4.19}$$

Further it follows from (4.9), (4.12), and (4.19) that

$$\begin{aligned} &\operatorname{Re}\left\{(1-\delta)\left(f'(z)\right)^{1/\gamma}+\delta\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}-\rho \\ &\geq F_n\left(\frac{1-r^n}{1+r^n},0\right)-\rho \\ &=(1-\delta)\frac{1-r^n}{1+r^n}+\delta\frac{1-2n\gamma r^n-r^{2n}}{1-r^{2n}}-\rho \\ &=\frac{J_n(r)}{1-r^{2n}}, \end{aligned} \tag{4.20}$$

where $0 \leq \rho < 1$ and

$$J_n(r)=(1-2\delta+\rho)r^{2n}-2(1-\delta+n\delta\gamma)r^n+1-\rho. \tag{4.21}$$

Note that $J_n(0)=1-\rho>0$ and $J_n(1)=-2n\delta\gamma<0$. If we let $r_n(\gamma,\delta,\rho)$ denote the root in $(0,1)$ of the equation $J_n(r)=0$, then (4.20) yields the desired result (4.2).

To see that the bound $r_n(\gamma,\delta,\rho)$ is the best possible, we consider the function

$$f(z)=\int_0^z\left(\frac{1-t^n}{1+t^n}\right)^{\gamma}dt \in T_n(\gamma). \tag{4.22}$$

It is clear that for $z=r\in(r_n(\gamma,\delta,\rho),1)$,

$$(1-\delta)\left(f'(r)\right)^{1/\gamma}+\delta\left(1+\frac{rf''(r)}{f'(r)}\right)-\rho=\frac{J_n(r)}{1-r^{2n}}<0, \tag{4.23}$$

which shows that the bound $r_n(\gamma,\delta,\rho)$ cannot be increased.

Setting $\delta=1$, Theorem 4.1 reduces to the following result.

**Corollary 4.2.** *Let $f(z)\in T_n(\gamma)$ and $0\leq\rho<1$. Then, $f(z)$ is convex of order $\rho$ in*

$$|z|<\left[\frac{\left((n\gamma)^2+(1-\rho)^2\right)^{1/2}-n\gamma}{1-\rho}\right]^{1/n}. \tag{4.24}$$

*The result is sharp.*

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