Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 408525, 20 pages
doi:10.1155/2011/408525

Research Article

Existence of Oscillatory Solutions of Singular Nonlinear Differential Equations

Irena Rachůnková, Lukáš Rachůnek, and Jan Tomeček

Department of Mathematics, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

Correspondence should be addressed to Irena Rachůnková, irena.rachunkova@upol.cz

Received 10 October 2010; Revised 25 February 2011; Accepted 23 March 2011

Academic Editor: Yuri V. Rogovchenko

Asymptotic properties of solutions of the singular differential equation $(p(t)u'(t))' = p(t)f(u(t))$ are described. Here, f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0,\infty)$ and has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$. Further conditions for f and p under which the equation has oscillatory solutions converging to 0 are given.

1. Introduction

For $k \in \mathbb{N}$, $k > 1$, and $L \in (0,\infty)$, consider the equation

$$u'' + \frac{k-1}{t}u' = f(u), \quad t \in (0,\infty), \tag{1.1}$$

where

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \qquad f(0) = f(L) = 0, \qquad f(x) < 0, \quad x \in (0,L), \tag{1.2}$$

$$\exists \overline{B} \in (-\infty,0) : f(x) > 0, \quad x \in \left[\overline{B},0\right). \tag{1.3}$$

Let us put

$$F(x) = -\int_0^x f(z)\mathrm{d}z \quad \text{for } x \in \mathbb{R}. \tag{1.4}$$

Moreover, we assume that f fulfils

$$F\left(\overline{B}\right) = F(L), \tag{1.5}$$

and denote

$$L_0 = \inf\left\{x < \overline{B} : f(x) > 0\right\} \geq -\infty. \tag{1.6}$$

Due to (1.2)–(1.4), we see that $F \in C^1(\mathbb{R})$ is decreasing and positive on $(L_0, 0)$ and increasing and positive on $(0, L]$.

Equation (1.1) arises in many areas. For example, in the study of phase transitions of Van der Waals fluids [1–3], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [4, 5], in the homogenous nucleation theory [6], and in relativistic cosmology for description of particles which can be treated as domains in the universe [7], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. Numerical simulations of solutions of (1.1), where f is a polynomial with three zeros, have been presented in [9–11]. Close problems about the existence of positive solutions can be found in [12–14].

In this paper, we investigate a generalization of (1.1) of the form

$$\left(p(t)u'\right)' = p(t)f(u), \quad t \in (0, \infty), \tag{1.7}$$

where f satisfies (1.2)–(1.5) and p fulfils

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p(0) = 0, \tag{1.8}$$

$$p'(t) > 0, \ t \in (0, \infty), \quad \lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0. \tag{1.9}$$

Equation (1.7) is singular in the sense that $p(0) = 0$. If $p(t) = t^{k-1}$, with $k > 1$, then p satisfies (1.8), (1.9), and (1.7) is equal to (1.1).

Definition 1.1. A function $u \in C^1[0, \infty) \cap C^2(0, \infty)$ which satisfies (1.7) for all $t \in (0, \infty)$ is called a *solution* of (1.7).

Consider a solution u of (1.7). Since $u \in C^1[0, \infty)$, we have $u(0), u'(0) \in \mathbb{R}$ and the assumption, $p(0) = 0$ yields $p(0)u'(0) = 0$. We can find $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $t \in (0, \delta)$. Integrating (1.7), we get

$$\left|u'(t)\right| = \left|\frac{1}{p(t)} \int_0^t p(s) f(u(s)) \mathrm{d}s\right| \leq \frac{M}{p(t)} \int_0^t p(s) \mathrm{d}s \leq Mt, \quad t \in (0, \delta). \tag{1.10}$$

Consequently, the condition

$$u'(0) = 0 \tag{1.11}$$

is necessary for each solution of (1.7). Denote

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\}. \tag{1.12}$$

Definition 1.2. Let u be a solution of (1.7). If $u_{\sup} < L$, then u is called a *damped* solution.

If a solution u of (1.7) satisfies $u_{\sup} = L$ or $u_{\sup} > L$, then we call u a bounding homoclinic solution or an escape solution. These three types of solutions have been investigated in [15–18]. Here, we continue the investigation of the existence and asymptotic properties of damped solutions. Due to (1.11) and Definition 1.2, it is reasonable to study solutions of (1.7) satisfying the initial conditions

$$u(0) = u_0 \in (L_0, L], \qquad u'(0) = 0. \tag{1.13}$$

Note that if $u_0 > L$, then a solution u of the problem (1.7), (1.13) satisfies $u_{\sup} > L$, and consequently u is not a damped solution. Assume that $L_0 > -\infty$, then $f(L_0) = 0$, and if we put $u_0 = L_0$, a solution u of (1.7), (1.13) is a constant function equal to L_0 on $[0, \infty)$. Since we impose no sign assumption on $f(x)$ for $x < L_0$, we do not consider the case $u_0 < L_0$. In fact, the choice of u_0 between two zeros L_0 and 0 of f has been motivated by some hydrodynamical model in [11].

A lot of papers are devoted to oscillatory solutions of nonlinear differential equations. Wong [19] published an account on a nonlinear oscillation problem originated from earlier works of Atkinson and Nehari. Wong's paper is concerned with the study of oscillatory behaviour of second-order Emden-Fowler equations

$$y''(x) + a(x)|y(x)|^{\gamma-1}y(x) = 0, \quad \gamma > 0, \tag{1.14}$$

where a is nonnegative and absolutely continuous on $(0, \infty)$. Both superlinear case ($\gamma > 1$) and sublinear case ($\gamma \in (0, 1)$) are discussed, and conditions for the function a giving oscillatory or nonoscillatory solutions of (1.14) are presented; see also [20]. Further extensions of these results have been proved for more general differential equations. For example, Wong and Agarwal [21] or Li [22] worked with the equation

$$\left(a(t)\left(y'(t)\right)^{\sigma}\right)' + q(t)f\left(y(t)\right) = 0, \tag{1.15}$$

where $\sigma > 0$ is a positive quotient of odd integers, $a \in C^1(\mathbb{R})$ is positive, $q \in C(\mathbb{R})$, $f \in C^1(\mathbb{R})$, $xf(x) > 0$, $f'(x) \geq 0$ for all $x \neq 0$. Kulenović and Ljubović [23] investigated an equation

$$\left(r(t)g\left(y'(t)\right)\right)' + p(t)f\left(y(t)\right) = 0, \tag{1.16}$$

where $g(u)/u \le m$, $f(u)/u \ge k > 0$, or $f'(u) \ge k$ for all $u \ne 0$. The investigation of oscillatory and nonoscillatory solutions has been also realized in the class of quasilinear equations. We refer to the paper [24] by Ho, dealing with the equation

$$\left(t^{n-1}\Phi_p(u')\right)' + t^{n-1}\sum_{i=1}^{N}\alpha_i t^{\beta_i}\Phi_{q_i}(u) = 0, \tag{1.17}$$

where $1 < p < n$, $\alpha_i > 0$, $\beta_i \ge -p$, $q_i > p-1$, $i = 1, \dots, N$, $\Phi_p(y) = |y|^{p-2}y$.

Oscillation results for the equation

$$\left(a(t)\Phi_p(x')\right)' + b(t)\Phi_q(x) = 0, \tag{1.18}$$

where $a, b \in C([0,\infty))$ are positive, can be found in [25]. We can see that the nonlinearity $f(y) = |y|^{\gamma-1}y$ in (1.14) is an increasing function on $\mathbb{R}$ having a unique zero at $y = 0$.

Nonlinearities in all the other (1.15)–(1.18) have similar globally monotonous behaviour. We want to emphasize that, in contrast to the above papers, the nonlinearity f in our (1.7) needs not be globally monotonous. Moreover, we deal with solutions of (1.7) starting at a singular point $t = 0$, and we provide an interval for starting values u_0 giving oscillatory solutions (see Theorems 2.3, 2.10, and 2.16). We specify a behaviour of oscillatory solutions in more details (decreasing amplitudes—see Theorems 2.10 and 2.16), and we show conditions which guarantee that oscillatory solutions converge to 0 (Theorem 3.1).

The paper is organized in this manner: Section 2 contains results about existence, uniqueness, and other basic properties of solutions of the problem (1.7), (1.13). These results which mainly concern damped solutions are taken from [18] and extended or modified a little. We also provide here new conditions for the existence of oscillatory solutions in Theorem 2.16. Section 3 is devoted to asymptotic properties of oscillatory solutions, and the main result is contained in Theorem 3.1.

2. Solutions of the Initial Problem (1.7), (1.13)

Let us give an account of this section in more details. The main objective of this paper is to characterize asymptotic properties of oscillatory solutions of the problem (1.7), (1.13). In order to present more complete results about the solutions, we start this section with the unique solvability of the problem (1.7), (1.13) on $[0,\infty)$ (Theorem 2.1). Having such global solutions, we have proved (see papers [15–18]) that oscillatory solutions of the problem (1.7), (1.13) can be found just in the class of damped solutions of this problem. Therefore, we give here one result about the existence of damped solutions (Theorem 2.3). Example 2.5 shows that there are damped solutions which are not oscillatory. Consequently, we bring results about the existence of oscillatory solutions in the class of damped solutions. This can be found in Theorem 2.10, which is an extension of Theorem 3.4 of [18] and in Theorem 2.16, which are new. Theorems 2.10 and 2.16 cover different classes of equations which is illustrated by examples.

Theorem 2.1 (existence and uniqueness). *Assume that* (1.2)–(1.5), (1.8), (1.9) *hold and that there exists* $C_L \in (0,\infty)$ *such that*

$$0 \le f(x) \le C_L \quad \textit{for } x \ge L \tag{2.1}$$

then the initial problem (1.7), (1.13) *has a unique solution* u. *The solution* u *satisfies*

$$\begin{aligned} u(t) &\ge u_0 \quad \textit{if} \ \ u_0 < 0, \\ u(t) &> \overline{B} \quad \textit{if} \ \ u_0 \ge 0, \end{aligned} \quad \textit{for } t \in [0,\infty). \tag{2.2}$$

Proof. Let $u_0 < 0$, then the assertion is contained in Theorem 2.1 of [18]. Now, assume that $u_0 \in [0, L]$, then the proof of Theorem 2.1 in [18] can be slightly modified. □

For close existence results, see also Chapters 13 and 14 of [26], where this kind of equations is studied.

Remark 2.2. Clearly, for $u_0 = 0$ and $u_0 = L$, the problem (1.7), (1.13) has a unique solution $u \equiv 0$ and $u \equiv L$, respectively. Since $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, no solution of the problem (1.7), (1.13) with $u_0 < 0$ or $u_0 \in (0, L)$ can touch the constant solutions $u \equiv 0$ and $u \equiv L$.

In particular, assume that $C \in \{0, L\}$, $a > 0$, u is a solution of the problem (1.7), (1.13) with $u_0 < L$, $u_0 \ne 0$, and (1.2), (1.8), and (1.9) hold. If $u(a) = C$, then $u'(a) \ne 0$, and if $u'(a) = 0$, then $u(a) \ne C$.

The next theorem provides an extension of Theorem 2.4 in [18].

Theorem 2.3 (existence of damped solutions). *Assume that* (1.2)–(1.5), (1.8), *and* (1.9) *hold, then for each* $u_0 \in [\overline{B}, L)$, *the problem* (1.7), (1.13) *has a unique solution. This solution is damped.*

Proof. First, assume that there exists $C_L > 0$ such that f satisfies (2.1), then, by Theorem 2.1, the problem (1.7), (1.13) has a unique solution u satisfying (2.2). Assume that u is not damped, that is,

$$\sup\{u(t) : t \in [0,\infty)\} \ge L. \tag{2.3}$$

By (1.3)–(1.5), the inequality $F(u_0) \le F(L)$ holds. Since u fulfils (1.7), we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)) \quad \text{for } t \in (0,\infty). \tag{2.4}$$

Multiplying (2.4) by u' and integrating between 0 and $t > 0$, we get

$$0 < \frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s = F(u_0) - F(u(t)), \quad t \in (0,\infty), \tag{2.5}$$

and consequently

$$0 < \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s \le F(u_0) - F(u(t)), \quad t \in (0,\infty). \tag{2.6}$$

By (2.3), we can find that $b \in (0,\infty]$ such that $u(b) \ge L$, $(u(\infty) = \limsup_{t\to\infty} u(t))$, and hence, according to (1.5),

$$0 < \int_0^b \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s \le F(u_0) - F(u(b)) \le F(B) - F(L) \le 0, \tag{2.7}$$

which is a contradiction. We have proved that $\sup\{u(t) : t \in [0,\infty)\} < L$, that is, u is damped. Consequently, assumption (2.1) can be omitted. □

Example 2.4. Consider the equation

$$u'' + \frac{2}{t}u' = u(u-1)(u+2), \tag{2.8}$$

which is relevant to applications in [9–11]. Here, $p(t) = t^2$, $f(x) = x(x-1)(x+2)$, $L_0 = -2$, and $L = 1$. Hence $f(\mathrm{x}) < 0$ for $x \in (0,1)$, $f(x) > 0$ for $x \in (-2,0)$, and

$$F(x) = -\int_0^x f(z)\mathrm{d}z = -\frac{x^4}{4} - \frac{x^3}{3} + x^2. \tag{2.9}$$

Consequently, F is decreasing and positive on $[-2,0)$ and increasing and positive on $(0,1]$. Since $F(1) = 5/12$ and $F(-1) = 13/12$, there exists a unique $\overline{B} \in (-1,0)$ such that $F(\overline{B}) = 5/12 = F(1)$. We can see that all assumptions of Theorem 2.3 are fulfilled and so, for each $u_0 \in [\overline{B},1)$, the problem (2.8), (1.13) has a unique solution which is damped. We will show later (see Example 2.11), that each damped solution of the problem (2.8), (1.13) is oscillatory.

In the next example, we will show that damped solutions can be nonzero and monotonous on $[0,\infty)$ with a limit equal to zero at ∞. Clearly, such solutions are not oscillatory.

Example 2.5. Consider the equation

$$u'' + \frac{3}{t}u' = f(u), \tag{2.10}$$

where

$$f(x) = \begin{cases} -x^3 & \text{for } x \le 1, \\ x-2 & \text{for } x \in (1,3), \\ 1 & \text{for } x \ge 3. \end{cases} \tag{2.11}$$

We see that $p(t) = t^3$ in (2.10) and the functions f and p satisfy conditions (1.2)–(1.5), (1.8), and (1.9) with $L = 2$. Clearly, $L_0 = -\infty$. Further,

$$F(x) = -\int_0^x f(z)\mathrm{d}z = \begin{cases} \dfrac{x^4}{4} & \text{for } x \le 1, \\ -\dfrac{x^2}{2} + 2x - \dfrac{5}{4} & \text{for } x \in (1,3), \\ -x + \dfrac{13}{4} & \text{for } x \ge 3. \end{cases} \tag{2.12}$$

Since $F(L) = F(2) = 3/4$, assumption (1.5) yields $F(\overline{B}) = \overline{B}^4/4 = 3/4$ and $\overline{B} = -3^{1/4}$. By Theorem 2.3, for each $u_0 \in [-3^{1/4}, 2)$, the problem (2.10), (1.13) has a unique solution u which is damped. On the other hand, we can check by a direct computation that for each $u_0 \le 1$ the function

$$u(t) = \frac{8u_0}{8 + u_0^2 t^2}, \quad t \in [0,\infty) \tag{2.13}$$

is a solution of equation (2.10) and satifies conditions (1.13). If $u_0 < 0$, then $u < 0$, $u' > 0$ on $(0,\infty)$, and if $u_0 \in (0,1]$, then $u > 0$, $u' < 0$ on $(0,\infty)$. In both cases, $\lim_{t\to\infty} u(t) = 0$.

In Example 2.5, we also demonstrate that there are equations fulfilling Theorem 2.3 for which all solutions with $u_0 < L$, not only those with $u_0 \in [\overline{B}, L)$, are damped. Some additional conditions giving, moreover, bounding homoclinic solutions and escape solutions are presented in [15–17].

In our further investigation of asymptotic properties of damped solutions the following lemmas are useful.

Lemma 2.6. *Assume* (1.2), (1.8), *and* (1.9). *Let u be a damped solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, L)$ which is eventually positive or eventually negative, then*

$$\lim_{t\to\infty} u(t) = 0, \qquad \lim_{t\to\infty} u'(t) = 0. \tag{2.14}$$

Proof. Let u be eventually positive, that is, there exists $t_0 \ge 0$ such that

$$u(t) > 0 \quad \text{for } t \in [t_0, \infty). \tag{2.15}$$

Denote $\theta = \inf\{t_0 \ge 0 : u(t) > 0,\ t \in [t_0,\infty)\}$.

Let $\theta > 0$, then $u(\theta) = 0$ and, by Remark 2.2, $u'(\theta) > 0$. Assume that $u' > 0$ on (θ,∞), then u is increasing on (θ,∞), and there exists $\lim_{t\to\infty} u(t) = \ell \in (0, L)$. Multiplying (2.4) by u', integrating between θ and t, and using notation (1.4), we obtain

$$\frac{u'^2(t)}{2} + \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s = F(u_0) - F(u(t)), \quad t \in (\theta,\infty). \tag{2.16}$$

Letting $t \to \infty$, we get

$$\lim_{t\to\infty}\frac{u'^2(t)}{2} = -\lim_{t\to\infty}\int_\theta^t \frac{p'(s)}{p(s)}u'^2(s)\mathrm{d}s + F(u_0) - F(\ell). \tag{2.17}$$

Since the function $\int_\theta^t (p'(s)/p(s))u'^2(s)\mathrm{d}s$ is positive and increasing, it follows that it has a limit at ∞, and hence there exists also $\lim_{t\to\infty}u'(t) \ge 0$. If $\lim_{t\to\infty}u'(t) > 0$, then $L > l = \lim_{t\to\infty}u(t) = \infty$, which is a contradiction. Consequently

$$\lim_{t\to\infty}u'(t) = 0. \tag{2.18}$$

Letting $t \to \infty$ in (2.4) and using (1.2), (1.9) and $\ell \in (0, L)$, we get $\lim_{t\to\infty}u''(t) = f(\ell) < 0$, and so $\lim_{t\to\infty}u'(t) = -\infty$, which is contrary to (2.18). This contradiction implies that the inequality $u' > 0$ on (θ, ∞) cannot be satisfied and that there exists $a > \theta$ such that $u'(a) = 0$. Since $u > 0$ on (a, ∞), we get by (1.2), (1.7), and (1.13) that $(pu')' < 0$ on (a, ∞). Due to $p(a)u'(a) = 0$, we see that $u' < 0$ on (a, ∞). Therefore, u is decreasing on (a, ∞) and $\lim_{t\to\infty}u(t) = \ell_0 \in [0, L)$. Using (2.16) with a in place of θ, we deduce as above that (2.18) holds and that $\lim_{t\to\infty}u''(t) = f(\ell_0) = 0$. Consequently, $\ell_0 = 0$. We have proved that (2.14) holds provided $\theta > 0$.

If $\theta = 0$, then we take $a = 0$ and use the above arguments. If u is eventually negative, we argue similarly. □

Lemma 2.7. *Assume* (1.2)–(1.5), (1.8), (1.9), *and*

$$p \in C^2(0, \infty), \quad \limsup_{t\to\infty}\left|\frac{p''(t)}{p'(t)}\right| < \infty, \tag{2.19}$$

$$\lim_{x\to 0+}\frac{f(x)}{x} < 0. \tag{2.20}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0 \in (0, L)$, then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \qquad u'(t) < 0 \quad \text{for } t \in (0, \delta_1]. \tag{2.21}$$

Proof. Assume that such δ_1 does not exist, then u is positive on $[0, \infty)$ and, by Lemma 2.6, u satisfies (2.14). We define a function

$$v(t) = \sqrt{p(t)}u(t), \quad t \in [0, \infty). \tag{2.22}$$

By (2.19), we have $v \in C^2(0, \infty)$ and

$$v'(t) = \frac{p'(t)u(t)}{2\sqrt{p(t)}} + \sqrt{p(t)}u'(t), \tag{2.23}$$

$$v''(t) = v(t)\left[\frac{1}{2}\frac{p''(t)}{p(t)} - \frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2 + \frac{f(u(t))}{u(t)}\right], \quad t \in (0, \infty). \tag{2.24}$$

By (1.9) and (2.19), we get

$$\lim_{t\to\infty}\left[\frac{1}{2}\frac{p''(t)}{p(t)}-\frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2\right]=\frac{1}{2}\lim_{t\to\infty}\frac{p''(t)}{p'(t)}\cdot\frac{p'(t)}{p(t)}=0. \tag{2.25}$$

Since u is positive on $(0,\infty)$, conditions (2.14) and (2.20) yield

$$\lim_{t\to\infty}\frac{f(u(t))}{u(t)}=\lim_{x\to 0+}\frac{f(x)}{x}<0. \tag{2.26}$$

Consequently, there exist $\omega>0$ and $R>0$ such that

$$\frac{1}{2}\frac{p''(t)}{p(t)}-\frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2+\frac{f(u(t))}{u(t)}<-\omega \quad \text{for } t\ge R. \tag{2.27}$$

By (2.22), v is positive on $(0,\infty)$ and, due to (2.24) and (2.27), we get

$$v''(t)<-\omega v(t)<0 \quad \text{for } t\ge R. \tag{2.28}$$

Thus, v' is decreasing on $[R,\infty)$ and $\lim_{t\to\infty}v'(t)=V$. If $V<0$, then $\lim_{t\to\infty}v(t)=-\infty$, contrary to the positivity of v. If $V\ge 0$, then $v'>0$ on $[R,\infty)$ and $v(t)\ge v(R)>0$ for $t\in[R,\infty)$. Then (2.28) yields $0>-\omega v(R)\ge-\omega v(t)>v''(t)$ for $t\in[R,\infty)$. We get $\lim_{t\to\infty}v'(t)=-\infty$ which contradicts $V\ge 0$. The obtained contradictions imply that u has at least one zero in $(0,\infty)$. Let $\delta_1>0$ be the first zero of u. Then $u>0$ on $[0,\delta_1)$ and, by (1.2) and (1.7), $u'<0$ on $(0,\delta_1)$. Due to Remark 2.2, we have also $u'(\delta_1)<0$. □

For negative starting value, we can prove a dual lemma by similar arguments.

Lemma 2.8. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.19) *and*

$$\lim_{x\to 0-}\frac{f(x)}{x}<0. \tag{2.29}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0\in(L_0,0)$, then there exists $\theta_1>0$ such that*

$$u(\theta_1)=0,\qquad u'(t)>0 \quad \textit{for } t\in(0,\theta_1]. \tag{2.30}$$

The arguments of the proof of Lemma 2.8 can be also found in the proof of Lemma 3.1 in [18], where both (2.20) and (2.29) were assumed. If one argues as in the proofs of Lemmas 2.7 and 2.8 working with a_1, A_1 and b_1, B_1 in place of 0, and u_0, one gets the next corollary.

Corollary 2.9. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.19), (2.20), *and* (2.29). *Let u be a solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, 0) \cup (0, L)$.*

(I) Assume that there exist $b_1 > 0$ and $B_1 \in (L_0, 0)$ such that

$$u(b_1) = B_1, \quad u'(b_1) = 0, \tag{2.31}$$

then there exists $\theta > b_1$ such that

$$u(\theta) = 0, \qquad u'(t) > 0 \quad \text{for } t \in (b_1, \theta]. \tag{2.32}$$

(II) Assume that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that

$$u(a_1) = A_1, \quad u'(a_1) = 0, \tag{2.33}$$

then there exists $\delta > a_1$ such that

$$u(\delta) = 0, \qquad u'(t) < 0 \quad \text{for } t \in (a_1, \delta]. \tag{2.34}$$

Note that if all conditions of Lemmas 2.7 and 2.8 are satisfied, then each solution of the problem (1.7), (1.13) with $u_0 \in (L_0, 0) \cup (0, L)$ has at least one simple zero in $(0, \infty)$. Corollary 2.9 makes possible to construct an unbounded sequence of all zeros of any damped solution u. In addition, these zeros are simple (see the proof of Theorem 2.10). In such a case, u has either a positive maximum or a negative minimum between each two neighbouring zeros. If we denote sequences of these maxima and minima by $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$, respectively, then we call the numbers $|A_n - B_n|$, $n \in \mathbb{N}$ amplitudes of u.

In [18], we give conditions implying that each damped solution of the problem (1.7), (1.13) with $u_0 < 0$ has an unbounded set of zeros and decreasing sequence of amplitudes. Here, there is an extension of this result for $u_0 \in (0, L)$.

Theorem 2.10 (existence of oscillatory solutions I). *Assume that* (1.2)–(1.5), (1.8), (1.9), (2.19), (2.20), *and* (2.29) *hold, Then each damped solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, 0) \cup (0, L)$ is oscillatory and its amplitudes are decreasing.*

Proof. For $u_0 < 0$, the assertion is contained in Theorem 3.4 of [18]. Let u be a damped solution of the problem (1.7), (1.13) with $u_0 \in (0, L)$. By (2.2) and Definition 1.2, we can find $L_1 \in (0, L)$ such that

$$\overline{B} < u(t) \leq L_1 \quad \text{for } t \in [0, \infty). \tag{2.35}$$

Step 1. Lemma 2.7 yields $\delta_1 > 0$ satisfying (2.21). Hence, there exists a maximal interval (δ_1, b_1) such that $u' < 0$ on (δ_1, b_1). If $b_1 = \infty$, then u is eventually negative and decreasing. On the other hand, by Lemma 2.6, u satisfies (2.14). But this is not possible. Therefore, $b_1 < \infty$ and there exists $B_1 \in (\overline{B}, 0)$ such that (2.31) holds. Corollary 2.9 yields $\theta_1 > b_1$ satisfying (2.32) with $\theta = \theta_1$. Therefore, u has just one negative local minimum $B_1 = u(b_1)$ between its first zero δ_1 and second zero θ_1.

Step 2. By (2.32) there exists a maximal interval (θ_1, a_1), where $u' > 0$. If $a_1 = \infty$, then u is eventually positive and increasing. On the other hand, by Lemma 2.6, u satisfies (2.14). We get a contradiction. Therefore $a_1 < \infty$ and there exists $A_1 \in (0, L)$ such that (2.33) holds. Corollary 2.9 yields $\delta_2 > a_1$ satisfying (2.34) with $\delta = \delta_2$. Therefore u has just one positive maximum $A_1 = u(a_1)$ between its second zero θ_1 and third zero δ_2.

Step 3. We can continue as in Steps 1 and 2 and get the sequences $\{A_n\}_{n=1}^{\infty} \subset (0, L)$ and $\{B_n\}_{n=1}^{\infty} \subset [u_0, 0)$ of positive local maxima and negative local minima of u, respectively. Therefore u is oscillatory. Using arguments of the proof of Theorem 3.4 of [18], we get that the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. In particular, we use (2.5) and define a Lyapunov function V_u by

$$V_u(t) = \frac{u'^2(t)}{2} + F(u(t)) = F(u_0) - \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s, \quad t \in (0, \infty), \tag{2.36}$$

then

$$V_u(t) > 0, \quad V_u'(t) = -\frac{p'(t)}{p(t)} u'^2(t) \le 0 \quad \text{for } t \in (0, \infty), \tag{2.37}$$

$$V_u'(t) < 0 \quad \text{for } t \in (0, \infty),\ t \ne a_n, b_n,\ n \in \mathbb{N}. \tag{2.38}$$

Consequently,

$$c_u := \lim_{t\to\infty} V_u(t) \ge 0. \tag{2.39}$$

So, sequences $\{V_u(a_n)\}_{n=1}^{\infty} = \{F(A_n)\}_{n=1}^{\infty}$ and $\{V_u(b_n)\}_{n=1}^{\infty} = \{F(B_n)\}_{n=1}^{\infty}$ are decreasing and

$$\lim_{n\to\infty} F(A_n) = \lim_{n\to\infty} F(B_n) = c_u. \tag{2.40}$$

Finally, due to (1.4), the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. Hence, the sequence of amplitudes $\{A_n - B_n\}_{n=1}^{\infty}$ is decreasing, as well. □

Example 2.11. Consider the problem (1.7), (1.13), where $p(t) = t^2$ and $f(x) = x(x-1)(x+2)$. In Example 2.4, we have shown that (1.2)–(1.5), (1.8), and (1.9) with $L_0 = -2$, $L = 1$ are valid. Since

$$\begin{gathered} \lim_{t\to\infty} \frac{p''(t)}{p'(t)} = \lim_{t\to\infty} \frac{1}{t} = 0, \\ \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} (x-1)(x+2) = -2 < 0, \end{gathered} \tag{2.41}$$

we see that (2.19), (2.20), and (2.29) are satisfied. Therefore, by Theorem 2.10, each damped solution of (2.8), (1.13) with $u_0 \in (-2,0) \cup (0,1)$ is oscillatory and its amplitudes are decreasing.

Example 2.12. Consider the problem (1.7), (1.13), where

$$p(t) = \frac{t^k}{1+t^\ell}, \quad k > \ell \geq 0,$$
$$f(x) = \begin{cases} x(x-1)(x+3), & \text{for } x \leq 0, \\ x(x-1)(x+4), & \text{for } x > 0, \end{cases} \tag{2.42}$$

then $L_0 = -3$, $L = 1$,

$$\lim_{t\to\infty} \frac{p''(t)}{p'(t)} = 0, \qquad \lim_{x\to 0-} \frac{f(x)}{x} = -3, \qquad \lim_{x\to 0+} \frac{f(x)}{x} = -4. \tag{2.43}$$

We can check that also all remaining assumptions of Theorem 2.10 are satisfied, and this theorem is applicable here.

Assume that f does not fulfil (2.20) and (2.29). It occurs, for example, if $f(x) = -|x|^\alpha \operatorname{sign} x$ with $\alpha > 1$ for x in some neighbourhood of 0, then Theorem 2.10 cannot be applied. Now, we will give another sufficient conditions for the existence of oscillatory solutions. For this purpose, we introduce the following lemmas.

Lemma 2.13. *Assume* (1.2)–(1.5), (1.8), (1.9), *and*

$$\int_1^\infty \frac{1}{p(s)} \mathrm{d}s = \infty, \tag{2.44}$$

$$\exists \epsilon > 0 : f \in C^1(0,\epsilon), \qquad f' \leq 0 \quad \text{on } (0,\epsilon). \tag{2.45}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0 \in (0, L)$, then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \qquad u'(t) < 0 \quad \textit{for } t \in (0, \delta_1]. \tag{2.46}$$

Proof. Assume that such δ_1 does not exist, then u is positive on $[0,\infty)$ and, by Lemma 2.6, u satisfies (2.14). In view of (1.7) and (1.2), we have $u' < 0$ on $(0,\infty)$. From (2.45), it follows that there exists $t_0 > 0$ such that

$$0 < u(t) < \epsilon, \quad \text{for } t \in [t_0, \infty). \tag{2.47}$$

Motivated by arguments of [27], we divide (1.7) by $f(u)$ and integrate it over interval $[t_0, t]$. We get

$$\int_{t_0}^{t} \frac{(p(s)u'(s))'}{f(u(s))} \mathrm{d}s = \int_{t_0}^{t} p(s)\mathrm{d}s \quad \text{for } t \in [t_0, \infty). \tag{2.48}$$

Using the per partes integration, we obtain

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_0}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s = \frac{p(t_0)u'(t_0)}{f(u(t_0))} + \int_{t_0}^{t} p(s)\mathrm{d}s, \quad t \in [t_0, \infty). \tag{2.49}$$

From (1.8) and (1.9), it follows that there exists $t_1 \in (t_0, \infty)$ such that

$$\frac{p(t_0)u'(t_0)}{f(u(t_0))} + \int_{t_0}^{t} p(s)\mathrm{d}s \geq 1, \quad t \in [t_1, \infty), \tag{2.50}$$

and therefore

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_0}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s \geq 1, \quad t \in [t_1, \infty). \tag{2.51}$$

From the fact that $f'(u(s)) \leq 0$ for $s > t_0$ (see (2.45)), we have

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s \geq 1, \quad t \in [t_1, \infty), \tag{2.52}$$

then

$$\frac{p(t)u'(t)}{f(u(t))} \geq 1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s > 0, \quad t \in [t_1, \infty), \tag{2.53}$$

$$\frac{p(t)u'(t)}{f(u(t))\left(1 - \int_{t_1}^{t} p(s)f'(u(s))u'^2(s)f^{-2}(u(s))\mathrm{d}s\right)} \geq 1, \quad t \in [t_1, \infty). \tag{2.54}$$

Multiplying this inequality by $-f'(u(t))u'(t)/f(u(t)) \geq 0$, we get

$$\left(\ln\left(1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s\right)\right)' \geq -\left(\ln\left|f(u(t))\right|\right)', \quad t \in [t_1, \infty), \tag{2.55}$$

and integrating it over $[t_1, t]$, we obtain

$$\ln\left(1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s\right) \geq \ln\left(\frac{f(u(t_1))}{f(u(t))}\right), \tag{2.56}$$

and therefore,

$$1 - \int_{t_1}^{t} \frac{p(s) f'(u(s)) u'^2(s)}{f^2(u(s))} \mathrm{d}s \geq \frac{f(u(t_1))}{f(u(t))}, \quad t \in [t_1, \infty). \tag{2.57}$$

According to (2.53), we have

$$\frac{p(t)u'(t)}{f(u(t))} \geq \frac{f(u(t_1))}{f(u(t))}, \quad t \in [t_1, \infty), \tag{2.58}$$

and consequently,

$$u'(t) \leq f(u(t_1)) \frac{1}{p(t)}, \quad t \in [t_1, \infty). \tag{2.59}$$

Integrating it over $[t_1, t]$, we get

$$u(t) \leq u(t_1) + f(u(t_1)) \int_{t_1}^{t} \frac{1}{p(s)} \mathrm{d}s, \quad t \in [t_1, \infty). \tag{2.60}$$

From (2.44), it follows that

$$\lim_{t \to \infty} u(t) = -\infty, \tag{2.61}$$

which is a contradiction. □

By similar arguments, we can prove a dual lemma.

Lemma 2.14. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.44), *and*

$$\exists \epsilon > 0 : f \in C^1(-\epsilon, 0), \quad f' \leq 0 \text{ on } (-\epsilon, 0). \tag{2.62}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0 \in (L_0, 0)$, then, there exists $\theta_1 > 0$ such that*

$$u(\theta_1) = 0, \qquad u'(t) > 0 \quad \text{for } t \in (0, \theta_1]. \tag{2.63}$$

Following ideas before Corollary 2.9, we get the next corollary.

Corollary 2.15. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.44), (2.45), *and* (2.62). *Let u be a solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, 0) \cup (0, L)$, then the assertions I and II of Corollary 2.9 are valid.*

Now, we are able to formulate another existence result for oscillatory solutions. Its proof is almost the same as the proof of Theorem 2.10 for $u_0 \in (L_0, 0)$ and the proof of Theorem 3.4 in [18] for $u_0 \in (0, L)$. The only difference is that we use Lemmas 2.13, 2.14, and Corollary 2.15, in place of Lemmas 2.7, 2.8, and Corollary 2.9, respectively.

Theorem 2.16 (existence of oscillatory solutions II). *Assume that* (1.2)–(1.5), (1.8), (1.9), (2.44), (2.45), *and* (2.62) *hold, then each damped solution of the problem* (1.7), (1.13) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *is oscillatory and its amplitudes are decreasing.*

Example 2.17. Let us consider (1.7) with

$$p(t) = t^{\alpha}, \quad t \in [0, \infty),$$
$$f(x) = \begin{cases} -|x|^{\lambda}\operatorname{sgn}x, & x \leq 1, \\ x - 2, & x \in (1, 3), \\ 1, & x \geq 3, \end{cases} \tag{2.64}$$

where λ and α are real parameters.

Case 1. Let $\lambda \in (1, \infty)$ and $\alpha \in (0, 1]$, then all assumptions of Theorem 2.16 are satisfied. Note that f satisfies neither (2.20) nor (2.29) and hence Theorem 2.10 cannot be applied.

Case 2. Let $\lambda = 1$ and $\alpha \in (0, \infty)$, then all assumptions of Theorem 2.10 are satisfied. If $\alpha \in (0, 1]$, then also all assumptions of Theorem 2.16 are fulfilled, but for $\alpha \in (1, \infty)$, the function p does not satisfy (2.44), and hence Theorem 2.16 cannot be applied.

3. Asymptotic Properties of Oscillatory Solutions

In Lemma 2.6 we show that if u is a damped solution of the problem (1.7), (1.13) which is not oscillatory then u converges to 0 for $t \to \infty$. In this section, we give conditions under which also oscillatory solutions converge to 0.

Theorem 3.1. *Assume that* (1.2)–(1.5), (1.8), *and* (1.9) *hold and that there exists* $k_0 > 0$ *such that*

$$\liminf_{t \to \infty} \frac{p(t)}{t^{k_0}} > 0, \tag{3.1}$$

then each damped oscillatory solution u of the problem (1.7), (1.13) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *satisfies*

$$\lim_{t \to \infty} u(t) = 0, \qquad \lim_{t \to \infty} u'(t) = 0. \tag{3.2}$$

Proof. Consider an oscillatory solution u of the problem (1.7), (1.13) with $u_0 \in (0, L)$.

Step 1. Using the notation and some arguments of the proof of Theorem 2.10, we have the unbounded sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$, such that

$$0 < \delta_1 < b_1 < \theta_1 < a_1 < \delta_2 < \cdots < \delta_n < b_n < \theta_n < a_n < \delta_{n+1} < \cdots, \tag{3.3}$$

where $u(\theta_n) = u(\delta_n) = 0$, $u(a_n) = A_n > 0$ is a unique local maximum of u in (θ_n, δ_{n+1}), $u(b_n) = B_n < 0$ is a unique local minimum of u in (δ_n, θ_n), $n \in \mathbb{N}$. Let V_u be given by (2.36) and then (2.39) and (2.40) hold and, by (1.2)–(1.4), we see that

$$\lim_{t\to\infty} u(t) = 0 \Longleftrightarrow c_u = 0. \tag{3.4}$$

Assume that (3.2) does not hold. Then $c_u > 0$. Motivated by arguments of [28], we derive a contradiction in the following steps.

Step 2 (estimates of u). By (2.36) and (2.39), we have

$$\lim_{n\to\infty} \frac{u'^2(\delta_n)}{2} = \lim_{n\to\infty} \frac{u'^2(\theta_n)}{2} = c_u > 0, \tag{3.5}$$

and the sequences $\{u'^2(\delta_n)\}_{n=1}^{\infty}$ and $\{u'^2(\theta_n)\}_{n=1}^{\infty}$ are decreasing. Consider $n \in \mathbb{N}$. Then $u'^2(\delta_n)/2 > c_u$ and there are α_n, β_n satisfying $a_n < \alpha_n < \delta_n < \beta_n < b_n$ and such that

$$u'^2(\alpha_n) = u'^2(\beta_n) = c_u, \quad u'^2(t) > c_u, \quad t \in (\alpha_n, \beta_n). \tag{3.6}$$

Since $V_u(t) > c_u$ for $t > 0$ (see (2.39)), we get by (2.36) and (3.6) the inequalities $c_u/2 + F(u(\alpha_n)) > c_u$ and $c_u/2 + F(u(\beta_n)) > c_u$, and consequently $F(u(\alpha_n)) > c_u/2$ and $F(u(\beta_n)) > c_u/2$. Therefore, due to (1.4), there exists $\tilde{c} > 0$ such that

$$u(\alpha_n) > \tilde{c}, \quad u(\beta_n) < -\tilde{c}, \quad n \in \mathbb{N}. \tag{3.7}$$

Similarly, we deduce that there are $\tilde{\alpha}_n, \tilde{\beta}_n$ satisfying $b_n < \tilde{\alpha}_n < \theta_n < \tilde{\beta}_n < a_{n+1}$ and such that

$$u(\tilde{\alpha}_n) < -\tilde{c}, \quad u\left(\tilde{\beta}_n\right) > \tilde{c}, \quad n \in \mathbb{N}. \tag{3.8}$$

The behaviour of u and inequalities (3.7) and (3.8) yield

$$|u(t)| > \tilde{c}, \qquad t \in [\beta_n, \tilde{\alpha}_n] \cup \left[\tilde{\beta}_n, \alpha_{n+1}\right], \; n \in \mathbb{N}. \tag{3.9}$$

Step 3 (estimates of $\beta_n - \alpha_n$). We prove that there exist $c_0, c_1 \in (0, \infty)$ such that

$$c_0 < \beta_n - \alpha_n < c_1, \quad n \in \mathbb{N}. \tag{3.10}$$

Assume on the contrary that there exists a subsequence satisfying $\lim_{\ell\to\infty}(\beta_\ell - \alpha_\ell) = 0$. By the mean value theorem and (3.7), there is $\xi_\ell \in (\alpha_\ell, \beta_\ell)$ such that $0 < 2\tilde{c} < u(\alpha_\ell) - u(\beta_\ell) = |u'(\xi_\ell)|(\beta_\ell - \alpha_\ell)$. Since $F(u(t)) \geq 0$ for $t \in [0, \infty)$, we get by (2.16) the inequality

$$\left|u'(t)\right| < \sqrt{2F(u_0)}, \quad t \in [0, \infty), \tag{3.11}$$

and consequently

$$0 < 2\tilde{c} \leq \sqrt{2F(u_0)} \lim_{\ell \to \infty} (\beta_\ell - \alpha_\ell) = 0, \tag{3.12}$$

which is a contradiction. So, c_0 satisfying (3.10) exists. Using the mean value theorem again, we can find $\tau_n \in (\alpha_n, \delta_n)$ such that $u(\delta_n) - u(\alpha_n) = u'(\tau_n)(\delta_n - \alpha_n)$ and, by (3.6),

$$\delta_n - \alpha_n = \frac{-u(\alpha_n)}{u'(\tau_n)} = \frac{u(\alpha_n)}{|u'(\tau_n)|} < \frac{A_1}{\sqrt{c_u}}. \tag{3.13}$$

Similarly, we can find $\eta_n \in (\delta_n, \beta_n)$ such that

$$\beta_n - \delta_n = \frac{u(\beta_n)}{u'(\eta_n)} = \frac{|u(\beta_n)|}{|u'(\eta_n)|} < \frac{|B_1|}{\sqrt{c_u}}. \tag{3.14}$$

If we put $c_1 = (A_1 + |B_1|)/\sqrt{c_u}$, then (3.10) is fulfilled. Similarly, we can prove

$$c_0 < \tilde{\beta}_n - \tilde{\alpha}_n < c_1, \quad n \in \mathbb{N}. \tag{3.15}$$

Step 4 (estimates of $\alpha_{n+1} - \alpha_n$). We prove that there exist $c_2 \in (0, \infty)$ such that

$$\alpha_{n+1} - \alpha_n < c_2, \quad n \in \mathbb{N}. \tag{3.16}$$

Put $m_1 = \min\{f(x) : B_1 \leq x \leq -\tilde{c}\} > 0$. By (3.9), $B_1 \leq u(t) < -\tilde{c}$ for $t \in [\beta_n, \tilde{\alpha}_n]$, $n \in \mathbb{N}$. Therefore,

$$f(u(t)) \geq m_1, \quad t \in [\beta_n, \tilde{\alpha}_n], \; n \in \mathbb{N}. \tag{3.17}$$

Due to (1.9), we can find $t_1 > 0$ such that

$$\frac{p'(t)}{p(t)} \sqrt{2F(u_0)} < \frac{m_1}{2}, \quad t \in [t_1, \infty). \tag{3.18}$$

Let $n_1 \in \mathbb{N}$ fulfil $\alpha_{n_1} \geq t_1$, then, according to (2.4), (3.11), (3.17), and (3.18), we have

$$u''(t) > -\frac{m_1}{2} + m_1 = \frac{m_1}{2}, \quad t \in [\beta_n, \tilde{\alpha}_n], \; n \geq n_1. \tag{3.19}$$

Integrating (3.19) from b_n to β_n and using (3.6), we get $2\sqrt{c_u} > m_1(b_n - \beta_n)$ for $n \geq n_1$. Similarly we get $2\sqrt{c_u} > m_1(\tilde{\alpha}_n - b_n)$ for $n \geq n_1$. Therefore

$$\frac{4}{m_1}\sqrt{c_u} > \tilde{\alpha}_n - \beta_n, \quad n \geq n_1. \tag{3.20}$$

By analogy, we put $m_2 = \min\{-f(x) : \widetilde{c} \le x \le A_1\} > 0$ and prove that there exists $n_2 \in \mathbb{N}$ such that

$$\frac{4}{m_2}\sqrt{c_u} > \alpha_{n+1} - \widetilde{\beta}_n, \quad n \ge n_2. \tag{3.21}$$

Inequalities (3.10), (3.15), (3.20), and (3.21) imply the existence of c_2 fulfilling (3.16).

Step 5 (construction of a contradiction). Choose $t_0 > c_1$ and integrate the equality in (2.37) from t_0 to $t > t_0$. We have

$$V_u(t) = V_u(t_0) - \int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau, \qquad t \ge t_0. \tag{3.22}$$

Choose $n_0 \in \mathbb{N}$ such that $\alpha_{n_0} > t_0$. Further, choose $n \in \mathbb{N}$, $n > n_0$ and assume that $t > \beta_n$, then, by (3.6),

$$\begin{aligned}\int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau &> \sum_{j=n_0}^{n} \int_{\alpha_j}^{\beta_j} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau \\ &> c_u \sum_{j=n_0}^{n} \int_{\alpha_j}^{\beta_j} \frac{p'(\tau)}{p(\tau)} \mathrm{d}\tau = c_u \sum_{j=n_0}^{n} [\ln p(\tau)]_{\alpha_j}^{\beta_j}.\end{aligned} \tag{3.23}$$

By virtue of (3.1) there exists $c_3 > 0$ such that $p(t)/t^{k_0} > c_3$ for $t \in [t_0, \infty)$. Thus, $\ln p(t) > \ln c_3 + k_0 \ln t$ and

$$\int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau > c_u \sum_{j=n_0}^{n} [\ln c_3 + k_0 \ln t]_{\alpha_j}^{\beta_j} = c_u k_0 \sum_{j=n_0}^{n} \ln \frac{\beta_j}{\alpha_j}. \tag{3.24}$$

Due to (3.10) and $c_1 < \alpha_{n_0}$, we have

$$1 < \frac{\beta_j}{\alpha_j} < 1 + \frac{c_1}{\alpha_j} < 2, \quad j = n_0, \ldots, n, \tag{3.25}$$

and the mean value theorem yields $\xi_j \in (1,2)$ such that

$$\ln \frac{\beta_j}{\alpha_j} = \left(\frac{\beta_j}{\alpha_j} - 1\right)\frac{1}{\xi_j} > \frac{\beta_j - \alpha_j}{2\alpha_j}, \quad j = n_0, \ldots, n. \tag{3.26}$$

By (3.10) and (3.16), we deduce

$$\frac{\beta_j - \alpha_j}{\alpha_j} > \frac{c_0}{\alpha_j}, \quad \alpha_j < jc_2 + \alpha_1, \; j = n_0, \ldots, n. \tag{3.27}$$

Thus,

$$\frac{\beta_j - \alpha_j}{\alpha_j} > \frac{c_0}{jc_2 + \alpha_1}, \quad j = n_0, \dots, n. \tag{3.28}$$

Using (3.24)–(3.28) and letting t to ∞, we obtain

$$\begin{aligned}\int_{t_0}^{\infty} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau &\geq c_u k_0 \sum_{n=n_0}^{\infty} \ln \frac{\beta_n}{\alpha_n} \geq \frac{1}{2} c_u k_0 \sum_{n=n_0}^{\infty} \frac{\beta_n - \alpha_n}{\alpha_n} \\ &\geq \frac{1}{2} c_u k_0 \sum_{n=n_0}^{\infty} \frac{c_0}{nc_2 + \alpha_1} = \infty.\end{aligned} \tag{3.29}$$

Using it in (3.22), we get $\lim_{t\to\infty} V_u(t) = -\infty$, which is a contradiction. So, we have proved that $c_u = 0$.

Using (2.4) and (3.4), we have

$$\lim_{t\to\infty} \left(\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s \right) = F(u_0) - F(0) = F(u_0). \tag{3.30}$$

Since the function $\int_0^t (p'(s)/p(s)) u'^2(s)\mathrm{d}s$ is increasing, there exists

$$\lim_{t\to\infty} \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s \leq F(u_0). \tag{3.31}$$

Therefore, there exists

$$\lim_{t\to\infty} u'^2(t) = \ell^2. \tag{3.32}$$

If $\ell > 0$, then $\lim_{t\to\infty} |u'(t)| = \ell$, which contradicts (3.4). Therefore, $\ell = 0$ and (3.2) is proved.

If $u_0 \in (L_0, 0)$, we argue analogously. □

Acknowledgments

The authors thank the referees for valuable comments and suggestions. This work was supported by the Council of Czech Government MSM 6198959214.

References

[1] V. Bongiorno, L. E. Scriven, and H. T. Davis, "Molecular theory of fluid interfaces," *Journal of Colloid and Interface Science*, vol. 57, pp. 462–475, 1967.

[2] H. Gouin and G. Rotoli, "An analytical approximation of density profile and surface tension of microscopic bubbles for Van Der Waals fluids," *Mechanics Research Communications*, vol. 24, no. 3, pp. 255–260, 1997.

[3] J. D. van der Waals and R. Kohnstamm, *Lehrbuch der Thermodynamik*, vol. 1, Mass and Van Suchtelen, Leipzig, Germany, 1908.
[4] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, vol. 28 of *Lecture Notes in Biomathematics*, Springer, Berlin, Germany, 1979.
[5] R. A. Fischer, "The wave of advance of advantegeous genes," *Journal of Eugenics*, vol. 7, pp. 355–369, 1937.
[6] F. F. Abraham, *Homogeneous Nucleation Theory*, Academic Press, New York, NY, USA, 1974.
[7] A. P. Linde, *Particle Physics and in Ationary Cosmology*, Harwood Academic, Chur, Switzerland, 1990.
[8] G. H. Derrick, "Comments on nonlinear wave equations as models for elementary particles," *Journal of Mathematical Physics*, vol. 5, pp. 1252–1254, 1964.
[9] F. Dell'Isola, H. Gouin, and G. Rotoli, "Nucleation of spherical shell-like interfaces by second gradient theory: numerical simulations," *European Journal of Mechanics, B/Fluids*, vol. 15, no. 4, pp. 545–568, 1996.
[10] G. Kitzhofer, O. Koch, P. Lima, and E. Weinmüller, "Efficient numerical solution of the density profile equation in hydrodynamics," *Journal of Scientific Computing*, vol. 32, no. 3, pp. 411–424, 2007.
[11] P. M. Lima, N. B. Konyukhova, A. I. Sukov, and N. V. Chemetov, "Analytical-numerical investigation of bubble-type solutions of nonlinear singular problems," *Journal of Computational and Applied Mathematics*, vol. 189, no. 1-2, pp. 260–273, 2006.
[12] H. Berestycki, P.-L. Lions, and L. A. Peletier, "An ODE approach to the existence of positive solutions for semilinear problems in $\mathbb{R}^N$," *Indiana University Mathematics Journal*, vol. 30, no. 1, pp. 141–157, 1981.
[13] D. Bonheure, J. M. Gomes, and L. Sanchez, "Positive solutions of a second-order singular ordinary differential equation," *Nonlinear Analysis*, vol. 61, no. 8, pp. 1383–1399, 2005.
[14] M. Conti, L. Merizzi, and S. Terracini, "Radial solutions of superlinear equations on $\mathbb{R}^N$. I. A global variational approach," *Archive for Rational Mechanics and Analysis*, vol. 153, no. 4, pp. 291–316, 2000.
[15] I. Rachůnková and J. Tomeček, "Bubble-type solutions of nonlinear singular problems," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 658–669, 2010.
[16] I. Rachůnková and J. Tomeček, "Strictly increasing solutions of a nonlinear singular differential equation arising in hydrodynamics," *Nonlinear Analysis*, vol. 72, no. 3–4, pp. 2114–2118, 2010.
[17] I. Rachůnková and J. Tomeček, "Homoclinic solutions of singular nonautonomous second-order differential equations," *Boundary Value Problems*, vol. 2009, Article ID 959636, 21 pages, 2009.
[18] I. Rachůnková, J. Tomeček, and J. Stryja, "Oscillatory solutions of singular equations arising in hydrodynamics," *Advances in Difference Equations*, vol. 2010, Article ID 872160, 13 pages, 2010.
[19] J. S. W. Wong, "Second-order nonlinear oscillations: a case history," in *Differential & Difference Equations and Applications*, pp. 1131–1138, Hindawi Publishing Corporation, New York, NY, USA, 2006.
[20] C. H. Ou and J. S. W. Wong, "On existence of oscillatory solutions of second order Emden-Fowler equations," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 670–680, 2003.
[21] P. J. Y. Wong and R. P. Agarwal, "Oscillatory behavior of solutions of certain second order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 2, pp. 337–354, 1996.
[22] W.-T. Li, "Oscillation of certain second-order nonlinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 217, no. 1, pp. 1–14, 1998.
[23] M. R. S. Kulenović and Ć. Ljubović, "All solutions of the equilibrium capillary surface equation are oscillatory," *Applied Mathematics Letters*, vol. 13, no. 5, pp. 107–110, 2000.
[24] L. F. Ho, "Asymptotic behavior of radial oscillatory solutions of a quasilinear elliptic equation," *Nonlinear Analysis*, vol. 41, no. 5-6, pp. 573–589, 2000.
[25] M. Bartušek, M. Cecchi, Z. Došlá, and M. Marini, "On oscillatory solutions of quasilinear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 320, no. 1, pp. 108–120, 2006.
[26] D. O'Regan, *Existence Theory for Nonlinear Ordinary differential Equations*, Kluwer Academic, Dodrecht, The Netherlands, 1997.
[27] I. Kiguradze and T. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Nauka, Moscow, Russia, 1990.
[28] I. Kiguradze and B. Shekhter, *Singular Boundary Value Problems for Ordinary Differential Equations of the Second Order*, Viniti, Moscow, Russia, 1987.