Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 361647, 15 pages
doi:10.1155/2011/361647

Research Article

Some Properties of Subclasses of Multivalent Functions

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Received 8 November 2010; Accepted 8 January 2011

Academic Editor: Ondřej Došlý

The authors introduce two new subclasses denoted by $\mathfrak{I}^*(\Omega, \lambda, p, \alpha)$ and $\mathfrak{I}^*_\varepsilon(\Omega, \lambda, p, \alpha)$ of the class $A(p, n)$ of p-valent analytic functions. They obtain coefficient inequality for the class $\mathfrak{I}^*(\Omega, \lambda, p, \alpha)$. They investigate various properties of classes $\mathfrak{I}^*(\Omega, \lambda, p, \alpha)$ and $\mathfrak{I}^*_\varepsilon(\Omega, \lambda, p, \alpha)$. Furthermore, they derive partial sums associated with the class $\mathfrak{I}^*_\varepsilon(\Omega, \lambda, p, \alpha)$.

1. Introduction and Definition

Let $A(p, n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad \left(n, p \in \mathbb{N} = \{1, 2, \ldots\}\right), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C},\ |z| < 1\}$. We write $A(1, 1) = A$.

A function $f \in A(p, n)$ is said to be in the class $S(p, n, \alpha)$ of p-valently star-like functions of order α if it satisfies the condition

$$\operatorname{Re}\left(\frac{z f'(z)}{f(z)}\right) > \alpha \quad \left(z \in \mathbb{C};\ 0 \le \alpha < p\right). \tag{1.2}$$

Furthermore, a function $f \in A(p,n)$ is said to be in the class $K(p,n,\alpha)$ of p-valently convex functions of order α if it satisfies the condition

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha \quad (z\in U;\ 0\le\alpha<p). \tag{1.3}$$

The classes $S(p,n,\alpha)$ and $K(p,n,\alpha)$ were studied by Owa [1]. The class $S^*(p,\alpha) := S(p,1,\alpha)$ was considered by Patil and Thakare [2].

We denote by $T(p,n)$ the subclass of the class $A(p,n)$ consisting of functions of the form

$$f(z)=z^p-\sum_{k=n}^{\infty}a_{p+k}z^{p+k} \quad (a_{p+k}\ge 0;\ n,p\in\mathbb{N}) \tag{1.4}$$

and define two further classes $T^*(p,n,\alpha)$ and $C(p,n,\alpha)$ by

$$T^*(p,n,\alpha) := S(p,n,\alpha)\cap T(p,n), \qquad C(p,n,\alpha) := K(p,n,\alpha)\cap T(p,n). \tag{1.5}$$

For the classes $T^*(p,\alpha) := S^*(p,\alpha)\cap T(p),\ C(p,\alpha) := K(p,\alpha)\cap T(p)$.

The following lemmas were given by Owa [3].

Lemma 1.1. *Let the function f be defined by*

$$f(z)=z^p-\sum_{k=1}^{\infty}a_{p+k}z^{p+k} \quad (a_{p+k}\ge 0;\ p\in\mathbb{N}). \tag{1.6}$$

Then, f is in the class $T^*(p,\alpha)$ if and only if

$$\sum_{k=1}^{\infty}(p+k-\alpha)a_{p+k}\le p-\alpha. \tag{1.7}$$

The result is sharp.

Lemma 1.2. *Let the function f be defined by* (1.6)*. Then, f is in the class $C(p,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}(p+k)(p+k-\alpha)a_{p+k}\le p(p-\alpha). \tag{1.8}$$

The result is sharp.

For a function f defined by (1.6) and in the class $T^*(p,\alpha)$, Lemma 1.1 yields

$$a_{p+1}\le\frac{p-\alpha}{p+1-\alpha}. \tag{1.9}$$

On the other hand, for a function f defined by (1.6) and in the class $C(p,\alpha)$, Lemma 1.2 yields

$$a_{p+1} \le \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}. \tag{1.10}$$

In view of the coefficient inequalities (1.9) and (1.10), it would seem to be natural to introduce and study here two further classes $T_\varepsilon^*(p,\alpha)$ and $C_\varepsilon(p,\alpha)$ of analytic and p-valent functions, where $T_\varepsilon^*(p,\alpha)$ denotes the subclass of $T^*(p,\alpha)$ consisting of functions of the form

$$f(z) = z^p - \frac{(p-\alpha)\varepsilon}{p+1-\alpha} z^{p+1} - \sum_{k=2}^{\infty} a_{p+k} z^{p+k} \quad \left(a_{p+k} \ge 0,\ p \in \mathbb{N},\ k \in \mathbb{N} - \{1\};\ 0 \le \alpha < p;\ 0 \le \varepsilon < 1\right) \tag{1.11}$$

and $C_\varepsilon(p,\alpha)$ denotes the subclass of $C(p,\alpha)$ consisting of functions of the form

$$\begin{aligned} f(z) &= z^p - \frac{p(p-\alpha)\varepsilon}{(p+1)(p+1-\alpha)} z^{p+1} \\ &\quad - \sum_{k=2}^{\infty} a_{p+k} z^{p+k} \quad \left(a_{p+k} \ge 0,\ p \in \mathbb{N},\ k \in \mathbb{N} - \{1\};\ 0 \le \alpha < p;\ 0 \le \varepsilon < 1\right). \end{aligned} \tag{1.12}$$

The classes $T_\varepsilon^*(p,\alpha)$ and $C_\varepsilon(p,\alpha)$ are studied by Aouf et al. [4].

The classes

$$T_\varepsilon^*(\alpha) := T_\varepsilon^*(1,\alpha), \qquad C_\varepsilon(\alpha) := C_\varepsilon(1,\alpha) \tag{1.13}$$

were considered earlier by Silverman and Silvia [5].

Now, we give the following equalities for the functions $f(z)$ belonging to the class $A(p,n)$:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = z\left(D^0 f(z)\right)' = z\left[pz^{p-1} + \sum_{k=n}^{\infty}(p+k)a_{p+k}z^{p+k-1}\right] = pz^p + \sum_{k=n}^{\infty}(p+k)a_{p+k}z^{p+k}, \\ D^2 f(z) &= D(Df(z)) = z\left(D^1 f(z)\right)' = z\left[pz^p + \sum_{k=n}^{\infty}(p+k)a_{p+k}z^{p+k}\right]' = p^2 z^p + \sum_{k=n}^{\infty}(p+k)^2 a_{p+k}z^{p+k}, \\ &\ \ \vdots \\ D^\Omega f(z) &= D\left(D^{\Omega-1} f(z)\right) = p^\Omega z^p + \sum_{k=n}^{\infty}(k+p)^\Omega a_{p+k}z^{p+k}. \end{aligned} \tag{1.14}$$

We define $\wp : A(p,n) \to A(p,n)$ such that

$$\wp(\Omega,\lambda,p) = \left(\frac{1}{p^{\Omega}} - \lambda\right) D^{\Omega} f(z) + \frac{\lambda}{p} z\left(D^{\Omega} f(z)\right)' \quad \left(0 \le \lambda \le \frac{1}{p^{\Omega}},\ \Omega \in \mathbb{N} \cup \{0\}\right). \tag{1.15}$$

A function $f(z) \in A(p,n)$ is said to be in the class $\Im(\Omega,\lambda,p,\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{z(\wp(\Omega,\lambda,p))'}{\wp(\Omega,\lambda,p)}\right\} = \operatorname{Re}\left\{z\frac{(1/p^{\Omega} + (1/p-1)\lambda)(D^{\Omega} f(z))' + (\lambda/p) z (D^{\Omega} f(z))''}{(1/p^{\Omega} - \lambda) D^{\Omega} f(z) + (\lambda/p) z (D^{\Omega} f(z))'}\right\} > \alpha, \tag{1.16}$$

for some α $(0 \le \alpha < p)$, $0 \le \lambda \le 1/p^{\Omega}$, $\Omega \in \mathbb{N} \cup \{0\}$ and for all $z \in U$.

If $\Omega = 0$ and $\lambda = 0$, we obtain the condition (1.2). Furthermore, we obtain the condition (1.3) for $\Omega = 0$ and $\lambda = 1$.

We denote by $T(p,n)$ the subclass of the class $A(p,n)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \ge 0;\ n,p \in \mathbb{N}), \tag{1.17}$$

and define the class $\Im^*(\Omega,\lambda,p,\alpha)$ by

$$\Im^*(\Omega,\lambda,p,\alpha) = \Im(\Omega,\lambda,p,\alpha) \cap T(p,n). \tag{1.18}$$

Furthermore, we denote by $\Im^*_{\varepsilon}(\Omega,\lambda,p,\alpha)$ the subclass of $\Im^*(\Omega,\lambda,p,\alpha)$ consisting of functions of the form

$$f(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} a_{p+k} z^{p+k} \quad (0 \le \varepsilon < 1). \tag{1.19}$$

The main object of the present paper is to investigate interesting properties and characteristics of the classes $\Im^*(\Omega,\lambda,p,\alpha)$ and $\Im^*_{\varepsilon}(\Omega,\lambda,p,\alpha)$. Also, the partial sums is defined for f function defined by (1.19).

2. A Coefficient Inequality for the Class $\Im^*(\Omega,\lambda,p,\alpha)$ and Some Theorems for the Class $\Im^*_{\varepsilon}(\Omega,\lambda,p,\alpha)$

First, we give a coefficient inequality for the class $\Im^*(\Omega,\lambda,p,\alpha)$.

Theorem 2.1. *Let the function f be defined by* (1.17)*. Then, f is in the class $\Im^*(\Omega,\lambda,p,\alpha)$ if and only if*

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \left(1 + \lambda k p^{\Omega-1}\right)(k+p-\alpha) a_{p+k} \le p - \alpha. \tag{2.1}$$

Proof. Suppose that $f(z) \in \Im^*(\Omega, \lambda, p, \alpha)$. Then, we find from (1.16) that

$$\begin{aligned} \operatorname{Re}\left\{ \frac{z(\wp(\Omega,\lambda,p))'}{\wp(\Omega,\lambda,p)} \right\} &= \operatorname{Re}\left\{ z\frac{(1/p^{\Omega} + (1/p-1)\lambda)(D^{\Omega}f(z))' + (\lambda/p)z(D^{\Omega}f(z))''}{(1/p^{\Omega} - \lambda)D^{\Omega}f(z) + (\lambda/p)z(D^{\Omega}f(z))'} \right\} \\ &= \operatorname{Re}\left\{ \frac{pz^p - \sum_{k=n}^{\infty}(k+p)^{\Omega+1}(1/p^{\Omega} + \lambda k/p)a_{p+k}z^{p+k}}{z^p - \sum_{k=n}^{\infty}(k+p)^{\Omega}(1/p^{\Omega} + \lambda k/p)a_{p+k}z^{p+k}} \right\} > \alpha. \end{aligned} \tag{2.2}$$

If we choose z to be real and let $z \to 1^-$, we get

$$\left\{ \frac{p - \sum_{k=n}^{\infty}(k+p)^{\Omega+1}(1/p^{\Omega} + \lambda k/p)a_{p+k}}{1 - \sum_{k=n}^{\infty}(k+p)^{\Omega}(1/p^{\Omega} + \lambda k/p)a_{p+k}} \right\} \geq \alpha \tag{2.3}$$

or, equivalently,

$$p - \sum_{k=n}^{\infty}(k+p)^{\Omega+1}\left(1/p^{\Omega} + \lambda k/p\right)a_{p+k} \geq \alpha\left\{ 1 - \sum_{k=n}^{\infty}(k+p)^{\Omega}\left(1/p^{\Omega} + \lambda k/p\right)a_{p+k} \right\}. \tag{2.4}$$

Thus, we have

$$\sum_{k=n}^{\infty}\left\{ (k+p)^{\Omega+1}\left(1/p^{\Omega} + \lambda k/p\right) - (k+p)^{\Omega}\left(1/p^{\Omega} + \lambda k/p\right) \right\}a_{p+k} \leq p - \alpha \tag{2.5}$$

or

$$\sum_{k=n}^{\infty}\left\{ \left(\frac{k+p}{p}\right)^{\Omega}\left(1 + \lambda k p^{\Omega-1}\right)(k+p-\alpha) \right\}a_{p+k} \leq p - \alpha. \tag{2.6}$$

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial U = \{z : z \in \mathbb{C},\ |z| = 1\}. \tag{2.7}$$

Then, we find from the definition (1.4) that

$$\begin{aligned} \left| \frac{z(\wp(\Omega,\lambda,p))'}{\wp(\Omega,\lambda,p)} - p \right| &= \left| z\frac{(1/p^{\Omega} + (1/p-1)\lambda)(D^{\Omega}f(z))' + (\lambda/p)z(D^{\Omega}f(z))''}{(1/p^{\Omega} - \lambda)D^{\Omega}f(z) + (\lambda/p)z(D^{\Omega}f(z))'} - p \right| \\ &= \left| \frac{pz^p - \sum_{k=n}^{\infty}((k+p)/p)^{\Omega}(k+p)(1 + \lambda k p^{\Omega-1})a_{p+k}z^{p+k}}{z^p - \sum_{k=n}^{\infty}((k+p)/p)^{\Omega}(1 + \lambda k p^{\Omega-1})a_{p+k}z^{p+k}} - p \right| \\ &\leq \frac{\sum_{k=n}^{\infty}k((k+p)/p)^{\Omega}(1 + \lambda k p^{\Omega-1})a_{p+k}}{1 - \sum_{k=n}^{\infty}((k+p)/p)^{\Omega}(1 + \lambda k p^{\Omega-1})a_{p+k}}. \end{aligned} \tag{2.8}$$

By means of inequality (2.1), we can write

$$\begin{aligned}
&\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k} \\
&\quad\le p-\alpha \Longrightarrow \sum_{k=n}^{\infty}k\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)a_{p+k} \\
&\quad\le p-\alpha-\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(p-\alpha)a_{p+k}
\end{aligned} \tag{2.9}$$

or

$$\sum_{k=n}^{\infty}k\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)a_{p+k} \le p-\alpha-(p-\alpha)\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)a_{p+k}. \tag{2.10}$$

Thus, we obtain

$$\begin{aligned}
&\left|z\frac{(1/p^{\Omega}+(1/p-1)\lambda)\left(D^{\Omega}f(z)\right)'+(\lambda/p)z\left(D^{\Omega}f(z)\right)''}{(1/p^{\Omega}-\lambda)D^{\Omega}f(z)+(\lambda/p)z\left(D^{\Omega}f(z)\right)'}-p\right| \\
&\quad=\left|\frac{\sum_{k=n}^{\infty}k((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})a_{p+k}z^{k}}{1-\sum_{k=n}^{\infty}((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})a_{p+k}z^{k}}\right| \\
&\quad\le\frac{p-\alpha-(p-\alpha)\sum_{k=n}^{\infty}((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})a_{p+k}}{1-\sum_{k=n}^{\infty}((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})a_{p+k}} \\
&\quad=p-\alpha.
\end{aligned} \tag{2.11}$$

This evidently completes the proof of Theorem 2.1. □

Now, we give a characterization theorem for the class $\Im_{\varepsilon}^{*}(\Omega,\lambda,p,\alpha)$.

Theorem 2.2. *Let the function f be defined by* (1.19). *Then, f is in the class $\Im_{\varepsilon}^{*}(\Omega,\lambda,p,\alpha)$ if and only if*

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k} \\
&\quad\le(p-\alpha)(1-\varepsilon)\quad\left(0\le\alpha<p\ ,\ 0\le\varepsilon<1,\ 0\le\lambda\le\frac{1}{p^{\Omega}}\right).
\end{aligned} \tag{2.12}$$

The result is sharp for the function f given by

$$f(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \frac{(p-\alpha)(1-\varepsilon)}{((p+k)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)} z^{p+k} \quad (k = n+1, n+2, \dots;\ n \in \mathbb{N}). \tag{2.13}$$

Proof. Using inequality (2.1), we have

$$\left(\frac{n+p}{p}\right)^{\Omega}\left(1+\lambda n p^{\Omega-1}\right)(n+p-\alpha)a_{p+n} + \sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k} \le p-\alpha \tag{2.14}$$

or

$$\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k} \le p-\alpha-\left(\frac{n+p}{p}\right)^{\Omega}\left(1+\lambda n p^{\Omega-1}\right)(n+p-\alpha)a_{p+n}. \tag{2.15}$$

Thus, by setting

$$a_{p+n} = \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)}, \tag{2.16}$$

we obtain

$$\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k} \le p-\alpha-\left(\frac{n+p}{p}\right)^{\Omega}\left(1+\lambda n p^{\Omega-1}\right)(n+p-\alpha)\frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} \tag{2.17}$$

or

$$\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k} \le (\mathrm{p}-\alpha)(1-\varepsilon). \tag{2.18}$$

A closure theorem for the class $\Im_{\varepsilon}^{*}(\Omega, \lambda, p, \alpha)$ is given by the following. □

Theorem 2.3. *Let*

$$f_s(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} a_{p+k,s} z^{p+k} \quad (a_{p+k,s} \geq 0;\ p,n \in \mathbb{N};\ 0 \leq \varepsilon < 1;\ s = 1,2,3,\ldots,m). \tag{2.19}$$

If $f_s \in \Im_{\varepsilon}^{*}(\Omega,\lambda,p,\alpha)$, then the function g given by

$$g(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} b_{p+k} z^{p+k}, \tag{2.20}$$

with

$$b_{p+k} := \frac{1}{m}\sum_{s=1}^{m} a_{p+k,s} \geq 0, \tag{2.21}$$

is also in the class $\Im_{\varepsilon}^{*}(\Omega,\lambda,p,\alpha)$.

Proof. Since $f_s \in \Im_{\varepsilon}^{*}(\Omega,\lambda,p,\alpha)$ for $s = 1,2,\ldots,m$, it follows from Theorem 2.2 that

$$\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k,s} \leq (p-\alpha)(1-\varepsilon) \quad (s = 1,2,\ldots,m). \tag{2.22}$$

By applying (2.22) and the definition (2.21), we write

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)b_{p+k} \\
&\quad = \sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)\left\{\frac{1}{m}\sum_{s=1}^{m} a_{p+k,s}\right\} \\
&\quad = \frac{1}{m}\sum_{s=1}^{m}\left\{\sum_{k=n+1}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha)a_{p+k,s}\right\} \\
&\quad \leq \frac{1}{m}\sum_{s=1}^{m}(p-\alpha)(1-\varepsilon) = (p-\alpha)(1-\varepsilon).
\end{aligned} \tag{2.23}$$

□

Theorem 2.4. *Let*

$$f_{p+n}(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n},$$
$$f_{p+k}(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \frac{(p-\alpha)(1-\varepsilon)}{((p+k)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)} z^{p+k}, \tag{2.24}$$

where $k \in \{n+1, n+2, \ldots\}$.

Then, f is in the class $\Im_{\varepsilon}^{*}(\Omega, \lambda, p, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=n}^{\infty} \eta_{p+k} f_{p+k}(z) \quad \left(\eta_{p+k} \geq 0;\ \sum_{k=n}^{\infty} \eta_{p+k} = 1 \right). \tag{2.25}$$

Proof. Suppose that f is given by (2.25), so that we find from (2.24) that

$$f(z) = z^p - \frac{(p-\alpha)\varepsilon}{((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \sum_{k=n+1}^{\infty} \frac{(p-\alpha)(1-\varepsilon)}{((p+k)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)} \eta_{p+k} z^{p+k}, \tag{2.26}$$

where the coefficients η_{p+k} are given with $\sum_{k=n}^{\infty} \eta_{p+k} = 1$, $\eta_{p+k} \geq 0$. Then, since,

$$\begin{aligned} \sum_{k=n+1}^{\infty} \left(\frac{k+p}{p} \right)^{\Omega} \left(1+\lambda k p^{\Omega-1}\right)(k+p-\alpha) \frac{(p-\alpha)(1-\varepsilon)}{((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)} \eta_{p+k} &= (p-\alpha)(1-\varepsilon) \sum_{k=n+1}^{\infty} \eta_{p+k} \\ &= (p-\alpha)(1-\varepsilon)(1-\eta_{p+n}) \\ &\leq (p-\alpha)(1-\varepsilon), \end{aligned} \tag{2.27}$$

we conclude from Theorem 2.2 that $f \in \Im_{\varepsilon}^{*}(\Omega, \lambda, p, \alpha)$.

Conversely, let us assume that the function f defined by (1.19) is in the class $\Im_{\varepsilon}^{*}(\Omega, \lambda, p, \alpha)$. Then,

$$a_{p+k} \leq \frac{(p-\alpha)(1-\varepsilon)}{((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)}, \tag{2.28}$$

which follows readily from (2.12) for $k \in \{n+1, n+2, \ldots\}$.

For $k \in \{n+1, n+2, \ldots\}$, setting

$$\eta_{p+k} = \frac{((k+p)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)}{(p-\alpha)(1-\varepsilon)} a_{p+k} \tag{2.29}$$

and $\eta_{p+n} = 1 - \sum_{k=n+1}^{\infty} \eta_{p+k}$, we thus arrive at (2.25). This completes the proof of Theorem 2.4. □

Theorem 2.5. *Let f be given by* (1.19) *and define the partial sums $s_m(z)$ by*

$$s_m(z) = \begin{cases} z^p - \dfrac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n}, & m = n, \\ z^p - \dfrac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)} z^{p+n} - \displaystyle\sum_{k=n+1}^{m} a_{p+k} z^{p+k}, & m = n+1, n+2, \ldots. \end{cases} \tag{2.30}$$

Suppose also that

$$\sum_{k=n+1}^{\infty} d_{p+k} a_{p+k} \leq 1 - \frac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)},$$
$$\left(\text{where } d_{p+k} = \frac{(p+k)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)}{p^{\Omega}(p-\alpha)(1-\varepsilon)};\ 0 \leq \alpha < p,\ 0 \leq \varepsilon < 1,\ 0 \leq \lambda \leq \frac{1}{p^{\Omega}}\right). \tag{2.31}$$

Then, for $m \geq n+1$, one has

$$\operatorname{Re}\left\{\frac{f(z)}{s_m(z)}\right\} > 1 - \frac{p^{\Omega}(p-\alpha)(1-\varepsilon)}{(p+m+1)^{\Omega}[1+\lambda(m+1)p^{\Omega-1}](p+m+1-\alpha)}, \tag{2.32}$$

$$\operatorname{Re}\left(\frac{s_m(z)}{f(z)}\right) > \frac{(p+m+1)^{\Omega}\{1+\lambda(m+1)p^{\Omega-1}\}(m+p+1-\alpha)}{p^{\Omega}(p-\alpha)(1-\varepsilon) + (p+m+1)^{\Omega}\{1+\lambda(m+1)p^{\Omega-1}\}(m+p+1-\alpha)}. \tag{2.33}$$

Each of the bounds in (2.32) *and* (2.33) *is the best possible.*

Proof. From (2.31) and Theorem 2.2, we have that $f \in \Im_{\varepsilon}^{*}(\Omega, \lambda, p, \alpha)$. By definition d_{p+k}, we can write

$$d_{p+k} = \frac{((p+k)/p)^{\Omega}(1+\lambda k p^{\Omega-1})(k+p-\alpha)}{(p-\alpha)(1-\varepsilon)}$$
$$= \left(1+\frac{k}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)\left[\frac{p-\alpha+k}{(p-\alpha)(1-\varepsilon)}\right] \tag{2.34}$$
$$= \left(1+\frac{k}{p}\right)^{\Omega}\left(1+\lambda k p^{\Omega-1}\right)\left[\frac{1}{1-\varepsilon}+\frac{k}{(p-\alpha)(1-\varepsilon)}\right] > 1.$$

Under the hypothesis of this theorem, we can see from (2.31) that $d_{p+k+1} > d_{p+k} > 1$, for $k = n+1, n+2 \ldots$. Therefore, we have

$$\sum_{k=n+1}^{m} a_{p+k} + d_{p+m+1}\sum_{k=m+1}^{\infty} a_{p+k} \le \sum_{k=n+1}^{\infty} d_{p+k}a_{p+k} \le 1 - \frac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(p+n-\alpha)}. \tag{2.35}$$

Using (1.19) and (2.30), we can write

$$\frac{f(z)}{s_m(z)} = \frac{z^p - \left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n} - \sum_{k=n+1}^{\infty} a_{p+k}z^{p+k}}{z^p - \left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n} - \sum_{k=n+1}^{m} a_{p+k}z^{p+k}}$$
$$= 1 - \frac{\sum_{k=m+1}^{\infty} a_{p+k}z^{p+k}}{z^p - \left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n} - \sum_{k=n+1}^{m} a_{p+k}z^{p+k}}. \tag{2.36}$$

Set

$$\Psi_1(z) = d_{p+m+1}\left[\frac{f(z)}{s_m(z)} - \left(1-\frac{1}{d_{p+m+1}}\right)\right]$$
$$= 1 - \frac{d_{p+m+1}\sum_{k=m+1}^{\infty} a_{p+k}z^{k}}{1 - \left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{n} - \sum_{k=n+1}^{m} a_{p+k}z^{k}}. \tag{2.37}$$

By applying (2.35) and (2.37), we find that

$$\begin{aligned}
&\left|\frac{\varphi_1(z)-1}{\varphi_1(z)+1}\right| \\
&\quad=\left|\frac{-d_{p+m+1}\sum_{k=m+1}^{\infty}a_{p+k}z^k}{2-\left(\dfrac{2p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)}\right)z^n-2\sum_{k=n+1}^{m}a_{p+k}z^k-d_{p+m+1}\sum_{k=m+1}^{\infty}a_{p+k}z^k}\right| \\
&\quad\le\frac{d_{p+m+1}\sum_{k=m+1}^{\infty}a_{p+k}z^{p+k}}{2\left[1-\left(\dfrac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)}\right)-\sum_{k=n+1}^{m}a_{p+k}\right]-d_{p+m+1}\sum_{k=m+1}^{\infty}a_{p+k}} \\
&\quad\le 1 \quad (z\in U;\ m\ge n+1),
\end{aligned}\tag{2.38}$$

which shows that $\operatorname{Re}\Psi_1(z)>0$. Thus, we obtain

$$\operatorname{Re}\Psi_1(z)=\operatorname{Re}\left\{d_{p+m+1}\left[\frac{f(z)}{s_m(z)}-\left(1-\frac{1}{d_{p+m+1}}\right)\right]\right\}>0\Longrightarrow\operatorname{Re}\left\{\frac{f(z)}{s_m(z)}\right\}>1-\frac{1}{d_{p+m+1}}\tag{2.39}$$

or

$$\operatorname{Re}\left\{\frac{f(z)}{s_m(z)}\right\}>1-\frac{p^{\Omega}(p-\alpha)(1-\varepsilon)}{(p+m+1)^{\Omega}\left[1+\lambda(m+1)p^{\Omega-1}\right](p+m+1-\alpha)}.\tag{2.40}$$

Let

$$\begin{aligned}
f(z)&=z^p-\frac{p^{\Omega}(p-\alpha)\varepsilon}{(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)}z^{p+n} \\
&\quad-\frac{1-p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda np^{\Omega-1})(n+p-\alpha)}{d_{p+m+1}}z^{p+m+1}.
\end{aligned}\tag{2.41}$$

Then, $f(z)$ satisfies the condition (2.31) and $f \in \Im_{\varepsilon}^{*}(\Omega,\lambda,p,\alpha)$. Thus, we can write

$$\begin{aligned}
\frac{f(z)}{s_m(z)} &= \frac{1-\left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^n}{1-\left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^n} \\
&\quad - \frac{\left(1-\left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)/d_{p+m+1}\right)z^{m+1}}{1-\left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^n} \\
&= 1-\frac{1-\left(\left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)/d_{p+m+1}\right)z^{m+1}}{1-\left(p^{\Omega}(p-\alpha)\varepsilon/(p+n)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^n}
\end{aligned} \tag{2.42}$$

and taking as $z \to 1^{-}$

$$\longrightarrow 1-\frac{1}{d_{p+m+1}} = 1-\frac{p^{\Omega}(p-\alpha)(1-\varepsilon)}{(p+m+1)^{\Omega}\left[1+\lambda(m+1)p^{\Omega-1}\right](p+m+1-\alpha)}, \tag{2.43}$$

which shows that the bound in (2.32) is the best possible.

By using definitions of $f(z)$ and $s_m(z)$, we can write

$$\begin{aligned}
\frac{s_m(z)}{f(z)} &= \frac{z^p-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n}-\sum_{k=n+1}^{m}a_{p+k}z^{p+k}}{z^p-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n}-\sum_{k=n+1}^{\infty}a_{p+k}z^{p+k}} \\
&= \frac{z^p-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n}}{z^p-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n}-\sum_{k=n+1}^{\infty}a_{p+k}z^{p+k}} \\
&\quad - \frac{\sum_{k=n+1}^{\infty}a_{p+k}z^{p+k}+\sum_{k=m+1}^{\infty}a_{p+k}z^{p+k}}{z^p-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{p+n}-\sum_{k=n+1}^{\infty}a_{p+k}z^{p+k}} \\
&= 1+\frac{\sum_{k=m+1}^{\infty}a_{p+k}z^{k}}{1-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{n}-\sum_{k=n+1}^{\infty}a_{p+k}z^{k}}.
\end{aligned} \tag{2.44}$$

If we put

$$\begin{aligned}
\varphi_2(z) &= (1+d_{p+m+1})\left\{\frac{s_m(z)}{f(z)}-\frac{d_{p+m+1}}{1+d_{p+m+1}}\right\} \\
&= 1+\left(\frac{(1+d_{p+m+1})\sum_{k=m+1}^{\infty}a_{p+k}z^{k}}{1-\left((p-\alpha)\varepsilon/((p+n)/p)^{\Omega}(1+\lambda n p^{\Omega-1})(n+p-\alpha)\right)z^{n}-\sum_{k=n+1}^{\infty}a_{p+k}z^{k}}\right)
\end{aligned} \tag{2.45}$$

and make use of (2.35), we can deduce that

$$\left|\frac{\psi_2(z)-1}{\psi_2(z)+1}\right|$$

$$=\left|\frac{\left(1+d_{p+m+1}\right)\sum_{k=m+1}^{\infty}a_{p+k}z^k}{2-\left(\dfrac{2(p-\alpha)\varepsilon}{\left(\dfrac{p+n}{p}\right)^{\Omega}\left(1+\lambda np^{\Omega-1}\right)(n+p-\alpha)}\right)z^n-2\sum_{k=n+1}^{m}a_{p+k}z^k+\left(d_{p+m+1}-1\right)\sum_{k=m+1}^{\infty}a_{p+k}z^k}\right|$$

$$\leq\frac{\left(1+d_{p+m+1}\right)\sum_{k=m+1}^{\infty}a_{p+k}}{2-\left(\dfrac{2(p-\alpha)\varepsilon}{\left(\dfrac{p+n}{p}\right)^{\Omega}\left(1+\lambda np^{\Omega-1}\right)(n+p-\alpha)}\right)-2\sum_{k=n+1}^{m}a_{p+k}-\left(d_{p+m+1}-1\right)\sum_{k=m+1}^{\infty}a_{p+k}}$$

$$=\frac{\left(1+d_{p+m+1}\right)\sum_{k=m+1}^{\infty}a_{p+k}}{2\left\{1-\left(\dfrac{(p-\alpha)\varepsilon}{\left(\dfrac{p+n}{p}\right)^{\Omega}\left(1+\lambda np^{\Omega-1}\right)(n+p-\alpha)}\right)-\sum_{k=n+1}^{m}a_{p+k}\right\}-\left(d_{p+m+1}-1\right)\sum_{k=m+1}^{\infty}a_{p+k}}$$

$$\leq\frac{\left(1+d_{p+m+1}\right)\sum_{k=m+1}^{\infty}a_{p+k}}{2d_{p+m+1}\sum_{k=m+1}^{\infty}a_{p+k}-\left(d_{p+m+1}-1\right)\sum_{k=m+1}^{\infty}a_{p+k}}=1, \tag{2.46}$$

$|(\psi_2(z)-1)/(\psi_2(z)+1)|\leq 1$ requires that $\operatorname{Re}\{\psi_2(z)\}>0$. Thus, we obtain

$$\operatorname{Re}\{\Psi_2(z)\}=\left(1+d_{p+m+1}\right)\operatorname{Re}\left[\frac{s_m(z)}{f(z)}-\left(\frac{d_{p+m+1}}{1+d_{p+m+1}}\right)\right]>0 \tag{2.47}$$

or

$$\operatorname{Re}\left\{\frac{s_m(z)}{f(z)}\right\}>\left(\frac{d_{p+m+1}}{1+d_{p+m+1}}\right). \tag{2.48}$$

It follows from the last inequality that assertion (2.33) of the Theorem 2.5 holds.

The bound in (2.33) is sharp with the extremal function given by (2.41). The proof of the theorem is thus completed. □

If $\Omega = 0$, $\lambda = 0$, and $n = 1$ are taken in Theorem 2.5, the following result is obtained given by Liu [6].

Corollary 2.6. *Let f be given by* (1.19) *and define the partial sums $s_m(z)$ by*

$$s_m(z) = \begin{cases} z^p - \dfrac{(p-\alpha)\varepsilon}{p+1-\alpha} z^{p+1}, & m = 1, \\ z^p - \dfrac{(p-\alpha)\varepsilon}{(n+p-\alpha)} z^{p+1} - \displaystyle\sum_{k=2}^{m} a_{p+k} z^{p+k}, & m = 2, 3, \ldots. \end{cases} \tag{2.49}$$

Suppose also that

$$\sum_{k=2}^{\infty} d_{p+k} a_{p+k} \leq 1 - \frac{(p-\alpha)\varepsilon}{p+1-\alpha} \quad \left(\text{where } d_{p+k} = \frac{k+p-\alpha}{(p-\alpha)(1-\varepsilon)};\ 0 \leq \alpha < p,\ 0 \leq \varepsilon < 1 \right). \tag{2.50}$$

Then, for $m \geq 2$, one has

$$\begin{gathered} \operatorname{Re}\left\{ \frac{f(z)}{s_m(z)} \right\} > \frac{(p-\alpha)\varepsilon + m + 1}{p+m+1-\alpha} \\ \operatorname{Re}\left(\frac{s_m(z)}{f(z)} \right) > \frac{p+m+1-\alpha}{(p-\alpha)(2-\varepsilon)+m+1}. \end{gathered} \tag{2.51}$$

Each of the bounds in (2.51) *is the best possible.*

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