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Research Article

Principal Functions of Non-Selfadjoint Sturm-Liouville Problems with Eigenvalue-Dependent Boundary Conditions

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We consider the operator L generated in $L^2(\mathbb{R}_+)$ by the differential expression $l(y) = -y'' + q(x)y$, $x \in \mathbb{R}_+ := [0, \infty)$ and the boundary condition $y'(0)/y(0) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2$, where q is a complex-valued function and $\alpha_i \in \mathbb{C}$, $i = 0, 1, 2$ with $\alpha_2 \neq 0$. In this paper we obtain the properties of the principal functions corresponding to the spectral singularities of L.

1. Introduction

Let T be a nonselfadjoint, closed operator in a Hilbert space H. We will denote the continuous spectrum and the set of all eigenvalues of T by $\sigma_c(T)$ and $\sigma_d(T)$, respectively. Let us assume that $\sigma_c(T) \neq \emptyset$.

Definition 1.1. If $\lambda = \lambda_0$ is a pole of the resolvent of T and $\lambda_0 \in \sigma_c(T)$, but $\lambda_0 \notin \sigma_d(T)$, then λ_0 is called a spectral singularity of T.

Let us consider the nonselfadjoint operator L_0 generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$l_0(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+, \tag{1.1}$$

and the boundary condition $y(0) = 0$, where q is a complex-valued function. The spectrum and spectral expansion of L_0 were investigated by Naĭmark [1]. He proved that the spectrum of L_0 is composed of continuous spectrum, eigenvalues, and spectral singularities. He showed that spectral singularities are on the continuous spectrum and are the poles of the resolvent kernel, which are not eigenvalues.

Lyance investigated the effect of the spectral singularities in the spectral expansion in terms of the principal functions of L_0 [2, 3]. He also showed that the spectral singularities play an important role in the spectral analysis of L_0.

The spectral analysis of the non-self-adjoint operator L_1 generated in $L^2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\int_0^{\infty} K(x)y(x)dx + \alpha y'(0) - \beta y(0) = 0, \tag{1.2}$$

in which $K \in L^2(\mathbb{R}_+)$ is a complex valued function and $\alpha, \beta \in \mathbb{C}$, was investigated in detail by Krall [4–8] In [4] he obtained the adjoint L_1^* of the operator L_1. Note that L_1^* deserves special interest, since it is not a purely differential operator. The eigenfunction expansions of L_1 and L_1^* were investigated in [5].

In [9] the results of Naimark were extended to the three-dimensional Schrödinger operators.

The Laurent expansion of the resolvents of the abstract non-self-adjoint operators in the neighborhood of the spectral singularities was studied in [10].

Using the boundary uniqueness theorems of analytic functions, the structure of the eigenvalues and the spectral singularities of a quadratic pencil of Schrödinger, Klein-Gordon, discrete Dirac, and discrete Schrödinger operators was investigated in [11–20]. By regularization of a divergent integral, the effect of the spectral singularities in the spectral expansion of a quadratic pencil of Schrödinger operators was obtained in [13]. In [19, 20] the spectral expansion of the discrete Dirac and Schrödinger operators with spectral singularities was derived using the generalized spectral function (in the sense of Marchenko [21]) and the analytical properties of the Weyl function.

Let L denote the operator generated in $L^2(\mathbb{R}_+)$ by the differential expression

$$l(y) = -y'' + q(x)y, \quad x \in \mathbb{R}_+ \tag{1.3}$$

and the boundary condition

$$\frac{y'(0)}{y(0)} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2, \tag{1.4}$$

where q is a complex-valued function and $\alpha_i \in \mathbb{C}$, $i = 0, 1, 2$ with $\alpha_2 \neq 0$. In this work we obtain the properties of the principal functions corresponding to the spectral singularities of L.

2. The Jost Solution and Jost Function

We consider the equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ \tag{2.1}$$

related to the operator L.

Now we will assume that the complex valued function q is almost everywhere continuous in $\mathbb{R}_+$ and satisfies the following:

$$\int_0^\infty x|q(x)|dx < \infty. \tag{2.2}$$

Let $\varphi(x,\lambda)$ and $e(x,\lambda)$ denote the solutions of (2.1) satisfying the conditions

$$\begin{gathered} \varphi(0,\lambda) = 1, \qquad \varphi'(0,\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2, \\ \lim_{x\to\infty} e(x,\lambda)e^{-i\lambda x} = 1, \quad \lambda \in \overline{\mathbb{C}}_+, \end{gathered} \tag{2.3}$$

respectively. The solution $e(x,\lambda)$ is called the Jost solution of (2.1). Note that, under the condition (2.2), the solution $\varphi(x,\lambda)$ is an entire function of λ and the Jost solution is an analytic function of λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$ and continuous in $\overline{\mathbb{C}}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}$.

In addition, Jost solution has a representation ([22])

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{2.4}$$

where the kernel $K(x,t)$ satisfies

$$\begin{aligned} K(x,t) = {}& \frac{1}{2}\int_{(x+t)/2}^\infty q(s)ds + \frac{1}{2}\int_x^{(x+t)/2}\int_{t+x-s}^{t+s-x} q(s)K(s,u)du\,ds \\ &+ \frac{1}{2}\int_{(x+t)/2}^\infty \int_s^{t+s-x} q(s)K(s,u)du\,ds \end{aligned} \tag{2.5}$$

and $K(x,t)$ is continuously differentiable with respect to its arguments. We also have

$$|K(x,t)| \leq cw\left(\frac{x+t}{2}\right), \tag{2.6}$$

$$|K_x(x,t)|, |K_t(x,t)| \leq \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + cw\left(\frac{x+t}{2}\right), \tag{2.7}$$

where $w(x) = \int_x^\infty |q(s)|ds$ and $c > 0$ is a constant.

Let $\hat{e}^\pm(x,\lambda)$ denote the solutions of (2.1) subject to the conditions

$$\lim_{x\to\infty} e^{\pm i\lambda x}\hat{e}^\pm(x,\lambda) = 1, \quad \lim_{x\to\infty} e^{\pm i\lambda x}\hat{e}_x^\pm(x,\lambda) = \pm i\lambda, \quad \lambda \in \overline{\mathbb{C}}_\pm. \tag{2.8}$$

Then

$$\begin{gathered} W\left[e(x,\lambda), \hat{e}^\pm(x,\lambda)\right] = \mp 2i\lambda, \quad \lambda \in \mathbb{C}_\pm, \\ W[e(x,\lambda), e(x,-\lambda)] = -2i\lambda, \quad \lambda \in \mathbb{R} = (-\infty,\infty), \end{gathered} \tag{2.9}$$

where $W[f_1, f_2]$ is the Wronskian of f_1 and f_2, ([23]).

We will denote the Wronskian of the solutions $\varphi(x,\lambda)$ with $e(x,\lambda)$ and $e(x,-\lambda)$ by $E^{+}(\lambda)$ and $E^{-}(\lambda)$, respectively, where

$$\begin{aligned} E^{+}(\lambda) &:= e'(0,\lambda) - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)e(0,\lambda), \quad \lambda \in \overline{\mathbb{C}}_{+}, \\ E^{-}(\lambda) &:= e'(0,-\lambda) - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)e(0,-\lambda), \quad \lambda \in \overline{\mathbb{C}}_{-}, \end{aligned} \tag{2.10}$$

and $\overline{\mathbb{C}}_{-} = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \leq 0\}$. Therefore E^{+} and E^{-} are analytic in $\mathbb{C}_{+}$ and $\mathbb{C}_{-} = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda < 0\}$, respectively, and continuous up to real axis.

The functions E^{+} and E^{-} are called Jost functions of L.

3. Eigenvalues and Spectral Singularities of L

Let

$$G(x,t;\lambda) = \begin{cases} G^{+}(x,t;\lambda), & \lambda \in \mathbb{C}_{+}, \\ G^{-}(x,t;\lambda), & \lambda \in \mathbb{C}_{-} \end{cases} \tag{3.1}$$

be the Green function of L (obtained by the standard techniques), where

$$\begin{aligned} G^{+}(x,t;\lambda) &= \begin{cases} -\dfrac{\varphi(t,\lambda)e(x,\lambda)}{E^{+}(\lambda)}, & 0 \leq t \leq x \\ -\dfrac{\varphi(x,\lambda)e(t,\lambda)}{E^{+}(\lambda)}, & x \leq t < \infty \end{cases} \\ G^{-}(x,t;\lambda) &= \begin{cases} -\dfrac{\varphi(t,\lambda)e(x,-\lambda)}{E^{-}(\lambda)}, & 0 \leq t \leq x \\ -\dfrac{\varphi(x,\lambda)e(t,-\lambda)}{E^{-}(\lambda)}, & x \leq t < \infty. \end{cases} \end{aligned} \tag{3.2}$$

We will denote the set of eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. From (3.1)–(3.2)

$$\begin{aligned} \sigma_d(L) &= \{\lambda : \lambda \in \mathbb{C}_{+}, E^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_{-}, E^{-}(\lambda) = 0\}, \\ \sigma_{ss}(L) &= \{\lambda : \lambda \in \mathbb{R}^{*}, E^{+}(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{R}^{*}, E^{-}(\lambda) = 0\}, \end{aligned} \tag{3.3}$$

where $\mathbb{R}^{*} = \mathbb{R} \setminus \{0\}$.

From (3.3) we obtain that to investigate the structure of the eigenvalues and the spectral singularities of L, we need to discuss the structure of the zeros of the functions E^{+} and E^{-} in $\overline{\mathbb{C}}_{+}$ and $\overline{\mathbb{C}}_{-}$, respectively.

Definition 3.1. The multiplicity of zero of the function E^{+}(or E^{-}) in $\overline{\mathbb{C}}_{+}$(or $\overline{\mathbb{C}}_{-}$) is called the multiplicity of the corresponding eigenvalue and spectral singularity of L.

We see from (2.9) that the functions

$$\begin{aligned}
\varphi^{+}(x,\lambda) &= \frac{\widehat{E}^{+}(\lambda)}{2i\lambda}e(x,\lambda) - \frac{E^{+}(\lambda)}{2i\lambda}\widehat{e}^{+}(x,\lambda), \quad \lambda \in \mathbb{C}_{+}, \\
\varphi^{-}(x,\lambda) &= \frac{\widehat{E}^{-}(\lambda)}{2i\lambda}e(x,-\lambda) - \frac{E^{-}(\lambda)}{2i\lambda}\widehat{e}^{-}(x,\lambda), \quad \lambda \in \mathbb{C}_{-}, \\
\varphi(x,\lambda) &= \frac{E^{+}(\lambda)}{2i\lambda}e(x,-\lambda) - \frac{E^{-}(\lambda)}{2i\lambda}e(x,\lambda), \quad \lambda \in \mathbb{R}^{*}
\end{aligned} \tag{3.4}$$

are the solutions of the boundary value problem

$$\begin{gathered}
-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_{+}, \\
\frac{y'(0)}{y(0)} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2,
\end{gathered} \tag{3.5}$$

where

$$\widehat{E}^{\pm}(\lambda) = \widehat{e}^{\pm'}(0,\lambda) - \left(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\right)\widehat{e}^{\pm}(0,\lambda). \tag{3.6}$$

Now let us assume that

$$q \in AC(\mathbb{R}_{+}), \quad \lim_{x\to\infty} q(x) = 0, \quad \sup_{x\in\mathbb{R}_{+}}\left[e^{\varepsilon\sqrt{x}}\left|q'(x)\right|\right] < \infty, \quad \varepsilon > 0. \tag{3.7}$$

Theorem 3.2 (see [24]). *Under the condition* (3.7) *the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.*

4. Principal Functions of L

In this section we assume that (3.7) holds. Let $\lambda_1, \ldots, \lambda_j$ and $\lambda_{j+1}, \ldots, \lambda_k$ denote the zeros of E^{+} in $\mathbb{C}_{+}$ and E^{-} in $\mathbb{C}_{-}$ (which are the eigenvalues of L) with multiplicities $m_1, \ldots, m_j$ and $m_{j+1}, \ldots, m_k$, respectively. It is obvious that from definition of the Wronskian

$$\left\{\frac{d^n}{d\lambda^n}W\left[\varphi^{+}(x,\lambda), e(x,\lambda)\right]\right\}_{\lambda=\lambda_p} = \left\{\frac{d^n}{d\lambda^n}E^{+}(\lambda)\right\}_{\lambda=\lambda_p} = 0 \tag{4.1}$$

for $n = 0, 1, \ldots, m_p - 1, p = 1, 2, \ldots, j$, and

$$\left\{\frac{d^n}{d\lambda^n}W\left[\varphi^{-}(x,\lambda), e(x,-\lambda)\right]\right\}_{\lambda=\lambda_p} = \left\{\frac{d^n}{d\lambda^n}E^{-}(\lambda)\right\}_{\lambda=\lambda_p} = 0 \tag{4.2}$$

for $n = 0, 1, ..., m_p - 1, p = j + 1, ..., k$.

Theorem 4.1. *The fallowing formulae:*

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}A_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.3}$$

$n=0,1,\ldots,m_p-1, p=1,2,\ldots,j$, *where*

$$A_m(\lambda_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}\widehat{E}^+(\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.4}$$

$$\left\{\frac{\partial^n}{\partial\lambda^n}\varphi^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}B_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.5}$$

$n=0,1,\ldots,m_p-1, p=j+1,\ldots,k$, *where*

$$B_m(\lambda_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}\widehat{E}^-(\lambda)\right\}_{\lambda=\lambda_p} \tag{4.6}$$

holds.

Proof. We will proceed by mathematical induction, we prove first (4.3). Let $n=0$. From (4.1) we get

$$\varphi^+(x,\lambda_p)=a_0(\lambda_p)\cdot e(x,\lambda_p), \tag{4.7}$$

where $a_0(\lambda_p)\neq 0$. Let us assume that for $1\le n_0\le m_p-2$, (4.3) holds; that is,

$$\left\{\frac{\partial^{n_0}}{\partial\lambda^{n_0}}\varphi^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n_0}A_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,\lambda)\right\}_{\lambda=\lambda_p}. \tag{4.8}$$

Now we will prove that (4.3) holds for n_0+1. If $y(x,\lambda)$ is a solution of (2.1), then $(\partial^n/\partial\lambda^n)y(x,\lambda)$ satisfies

$$\left[-\frac{d^2}{dx^2}+q(x)-\lambda^2\right]\frac{\partial^n}{\partial\lambda^n}y(x,\lambda)=2\lambda n\frac{\partial^{n-1}}{\partial\lambda^{n-1}}y(x,\lambda)+n(n-1)\frac{\partial^{n-2}}{\partial\lambda^{n-2}}y(x,\lambda). \tag{4.9}$$

Writing (4.9) for $\varphi^+(x,\lambda)$ and $e(x,\lambda)$, and using (4.8), we find

$$\left[-\frac{d^2}{dx^2}+q(x)-\lambda^2\right]f_{n_0+1}(x,\lambda_p)=0, \tag{4.10}$$

where

$$f_{n_0+1}(x,\lambda_p) = \left\{ \frac{\partial^{n_0+1}}{\partial\lambda^{n_{0+1}}}\varphi^{+}(x,\lambda) \right\}_{\lambda=\lambda_p} - \sum_{m=0}^{n_0+1} A_m(\lambda_p) \left\{ \frac{\partial^m}{\partial\lambda^m} e(x,\lambda) \right\}_{\lambda=\lambda_p}. \tag{4.11}$$

From (4.1) we have

$$W[f_{n_0+1}(x,\lambda_p), e(x,\lambda_p)] = \left\{ \frac{d^{n_0+1}}{d\lambda^{n_0+1}} W[\varphi^{+}(x,\lambda), e(x,\lambda)] \right\}_{\lambda=\lambda_p} = 0. \tag{4.12}$$

Hence there exists a constant $a_{n_0+1}(\lambda_p)$ such that

$$f_{n_0+1}(x,\lambda_p) = a_{n_0+1}(\lambda_p) e(x,\lambda_p). \tag{4.13}$$

This shows that (4.3) holds for $n = n_0 + 1$.

Similarly we can prove that (4.5) holds. □

Definition 4.2. Let $\lambda = \lambda_0$ be an eigenvalue of L. If the functions

$$y_0(x,\lambda_0), y_1(x,\lambda_0), \ldots, y_s(x,\lambda_0) \tag{4.14}$$

satisfy the equations

$$l(y_0) - \lambda_0 y_0 = 0, \qquad l(y_j) - \lambda_0 y_j - y_{j-1} = 0, \quad j = 1,2,\ldots,s, \tag{4.15}$$

then the function $y_0(x,\lambda_0)$ is called the eigenfunction corresponding to the eigenvalue $\lambda = \lambda_0$ of L. The functions $y_1(x,\lambda_0),\ldots,y_s(x,\lambda_0)$ are called the associated functions corresponding $\lambda = \lambda_0$. The eigenfunctions and the associated functions corresponding to $\lambda = \lambda_0$ are called the principal functions of the eigenvalue $\lambda = \lambda_0$.

The principal functions of the spectral singularities of L are defined similarly.

Now using (4.3) and (4.5) define the functions

$$U_{n,p}(x) = \left\{ \frac{\partial^n}{\partial\lambda^n}\varphi^{+}(x,\lambda) \right\}_{\lambda=\lambda_p} = \sum_{m=0}^{n} A_m(\lambda_p) \left\{ \frac{\partial^m}{\partial\lambda^m} e(x,\lambda) \right\}_{\lambda=\lambda_p}, \tag{4.16}$$

$n = 0,1,\ldots,m_p - 1, \ldots p = 1,2,\ldots,j$

and

$$U_{n,p}(x) = \left\{ \frac{\partial^n}{\partial\lambda^n}\varphi^{-}(x,\lambda) \right\}_{\lambda=\lambda_p} = \sum_{m=0}^{n} B_m(\lambda_p) \left\{ \frac{\partial^m}{\partial\lambda^m} e(x,-\lambda) \right\}_{\lambda=\lambda_p}, \tag{4.17}$$

$n = 0,1,\ldots,m_p - 1, p = j+1,\ldots,k.$

Then for $\lambda = \lambda_p, p = 1, 2, \ldots, j, j+1, \ldots, k$,

$$\begin{gathered} l(U_{0,p}) = 0, \\ l(U_{1,p}) + \frac{1}{1!}\frac{\partial}{\partial\lambda} l(U_{0,p}) = 0, \\ l(U_{n,p}) + \frac{1}{1!}\frac{\partial}{\partial\lambda} l(U_{n-1,p}) + \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2} l(U_{n-2,p}) = 0, \end{gathered} \tag{4.18}$$

$n = 2, 3, \ldots, m_p - 1$,

hold, where $l(u) = -u'' + q(x)u - \lambda^2 u$ and $(\partial^m/\partial\lambda^m) l(u)$ denotes the differential expressions whose coefficients are the m-th derivatives with respect to λ of the corresponding coefficients of the differential expression $l(u)$. Equation (4.18) shows that $U_{0,p}$ is the eigenfunction corresponding to the eigenvalue $\lambda = \lambda_p; U_{1,p}, U_{2,p}, \ldots, U_{m_p-1,p}$ are the associated functions of $U_{0,p}$ ([25, 26]).

$U_{0,p}, U_{1,p}, \ldots, U_{m_p-1,p}, p = 1, 2, \ldots, j, j+1, \ldots, k$ are called the principal functions corresponding to the eigenvalue $\lambda = \lambda_p, p = 1, 2, \ldots, j, j+1, \ldots, k$ of L.

Theorem 4.3. *One has*

$$U_{n,p} \in L^2(\mathbb{R}_+), \quad n = 0, 1, \ldots, m_p - 1, \; p = 1, 2, \ldots, j, \; j+1, \ldots, k. \tag{4.19}$$

Proof. Let $0 \le n \le m_p - 1$ and $1 \le p \le j$. Using (2.6) and (3.7) we obtain that

$$|K(x,t)| \le c e^{-\varepsilon\sqrt{(x+t)/2}}. \tag{4.20}$$

From (2.4) we get

$$\left| \left\{ \frac{\partial^n}{\partial\lambda^n} e(x,\lambda) \right\}_{\lambda=\lambda_p} \right| \le x^n e^{-x \operatorname{Im}\lambda_p} + c \int_x^\infty t^n e^{-\varepsilon\sqrt{(x+t)/2}} e^{-t \operatorname{Im}\lambda_p} dt, \tag{4.21}$$

where $c > 0$ is a constant. Since $\operatorname{Im}\lambda_p > 0$ for the eigenvalues $\lambda_p, p = 1, \ldots, j$, of L, (4.21) implies that

$$\left\{ \frac{\partial^n}{\partial\lambda^n} e(x,\lambda) \right\}_{\lambda=\lambda_p} \in L^2(\mathbb{R}_+), \quad n = 0, 1, \ldots, m_p - 1, \; p = 1, 2, \ldots, j. \tag{4.22}$$

The proof of theorem is obtained from (4.16) and (4.22). In a similar way using (4.17) we may also prove the results for $0 \le n \le m_p - 1$ and $j+1 \le p \le k$. $\square$

Let $\mu_1, \ldots, \mu_v$, and $\mu_{v+1}, \ldots, \mu_l$ be the zeros of E^+ and E^- in $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ (which are the spectral singularities of L) with multiplicities $n_1, \ldots, n_v$ and $n_{v+1}, \ldots, n_l$, respectively.

Similar to (4.3) and (4.5) we can show the following:

$$\left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} C_m(\mu_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,\lambda)\right\}_{\lambda=\mu_p}, \tag{4.23}$$

$n = 0,1,\ldots,n_p-1, p = 1,2,\ldots,v,$

where

$$\begin{gathered} C_m(\mu_p) = -\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}E^-(\lambda)\right\}_{\lambda=\mu_p}, \\ \left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} D_m(\mu_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,-\lambda)\right\}_{\lambda=\mu_p}, \end{gathered} \tag{4.24}$$

$n = 0,1,\ldots,n_p-1, p = v+1,\ldots,l,$

where

$$D_m(\mu_p) = \binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}E^+(\lambda)\right\}_{\lambda=\mu_p}. \tag{4.25}$$

Now define the generalized eigenfunctions and generalized associated functions corresponding to the spectral singularities of L by the following:

$$v_{n,p}(x) = \left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} C_m(\mu_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,\lambda)\right\}_{\lambda=\mu_p}, \tag{4.26}$$

$n = 0,1,\ldots,n_p-1, p = 1,2,\ldots,v,$

$$v_{n,p}(x) = \left\{\frac{\partial^n}{\partial\lambda^n}\psi(x,\lambda)\right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} D_m(\mu_p)\left\{\frac{\partial^m}{\partial\lambda^m}e(x,-\lambda)\right\}_{\lambda=\mu_p}, \tag{4.27}$$

$n = 0,1,\ldots,n_p-1, p = v+1,\ldots,l.$

Then $v_{n,p}, n = 0,1,\ldots,n_p-1, p = 1,2,\ldots,v,v+1,\ldots,l,$ also satisfy the equations analogous to (4.18).

$v_{0,p}, v_{1,p},\ldots,v_{n_p-1,p}, p = 1,2,\ldots,v,v+1,\ldots,l$ are called the principal functions corresponding to the spectral singularities $\lambda = \mu_p, p = 1,2,\ldots,v,v+1,\ldots,l$ of L.

Theorem 4.4. *One has*

$$v_{n,p} \notin L^2(\mathbb{R}_+), \quad n = 0,1,\ldots,n_p-1, \; p = 1,2,\ldots,v,v+1,\ldots,l. \tag{4.28}$$

Proof. If we consider (4.21) for the principal functions corresponding to the spectral singularities $\lambda = \mu_p, p = 1,2,\ldots,v,v+1,\ldots,l,$ of L and consider that $\operatorname{Im}\lambda_p = 0$ for the spectral singularities, then we have (4.28), by (4.26) and (4.27). □

Now introduce the Hilbert spaces

$$H_n = \left\{ f : \int_0^\infty (1+x)^{2n} |f(x)|^2 dx < \infty \right\}, \quad n = 1, 2, \ldots,$$
$$H_{-n} = \left\{ g : \int_0^\infty (1+x)^{-2n} |g(x)|^2 dx < \infty \right\}, \quad n = 1, 2, \ldots, \tag{4.29}$$

with

$$\|f\|_n^2 = \int_0^\infty (1+x)^{2n} |f(x)|^2 dx; \qquad \|g\|_{-n}^2 = \int_0^\infty (1+x)^{-2n} |g(x)|^2 dx, \tag{4.30}$$

respectively. Then

$$H_{n+1} \subsetneqq H_n \subsetneqq L^2(\mathbb{R}_+) \subsetneqq H_{-n} \subsetneqq H_{-(n+1)}, \quad n = 1, 2, \ldots, \tag{4.31}$$

and H_{-n} is isomorphic to the dual of H_n.

Theorem 4.5. *One has*

$$v_{n,p} \in H_{-(n+1)}, \quad n = 0, 1, \ldots, n_p - 1, \; p = 1, 2, \ldots, v, v+1, \ldots, l. \tag{4.32}$$

Proof. From (2.4) we have

$$\int_0^\infty (1+x)^{-2(n+1)} \left| (ix)^n e^{i\mu_p x} \right|^2 dx < \infty,$$
$$\int_0^\infty (1+x)^{-2(n+1)} \left| \int_x^\infty (it)^n K(x,t) e^{i\mu_p t} dt \right|^2 dx < \infty. \tag{4.33}$$

Using (4.26), (4.33) we obtain

$$v_{n,p} \in H_{-(n+1)}, \quad n = 0, 1, \ldots, n_p - 1, \quad p = 1, 2, \ldots, v. \tag{4.34}$$

In a similar way, we find

$$v_{n,p} \in H_{-(n+1)}, \quad n = 0, 1, \ldots, n_p - 1, \; p = v+1, \ldots, l. \tag{4.35}$$

□

Let us choose n_0 so that

$$n_0 = \max\{n_1, n_2 \ldots, n_v, n_{v+1}, \ldots, n_l\}. \tag{4.36}$$

By Theorem 4.5 and (4.31) we get following theorem

Theorem 4.6. *One has*

$$v_{n,p} \in H_{-n_0}, \quad n = 0, 1, \ldots, n_p - 1, \; p = 1, 2, \ldots, v, v+1, \ldots, l. \tag{4.37}$$

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