Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2011, Article ID 310957, 15 pages
doi:10.1155/2011/310957

*Research Article*

# Existence and Multiplicity of Periodic Solutions Generated by Impulses

**Liu Yang[1,2] and Haibo Chen[2]**

[1] *Department of Mathematics and Computing Sciences, Hengyang Normal University, Hunan, Hengyang 421008, China*
[2] *Department of Mathematics, Central South University, Hunan, Changsha 421008, China*

Correspondence should be addressed to Haibo Chen, math_chb@mail.csu.edu.cn

Received 30 October 2010; Accepted 23 March 2011

Academic Editor: Martin D. Schechter



We investigate the existence and multiplicity of periodic solutions for a class of second-order differential systems with impulses. By using variational methods and critical point theory, we obtain such a system possesses at least one nonzero, two nonzero, or infinitely many periodic solutions generated by impulses under different conditions, respectively. Recent results in the literature are generalized and significantly improved.

## 1. Introduction

Consider the following second-order impulsive differential equations:

$$\begin{aligned} &\ddot{u}(t) + V_u(t, u(t)) = 0, \quad t \in (s_{k-1}, s_k), \\ &\Delta \dot{u}(s_k) = \lambda g_k(u(s_k)), \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{$P_\lambda$}$$

where $s_k$, $k = 1, 2, \ldots, m$, are instants in which the impulses occur and $0 = s_0 < s_1 < s_2 < \cdots < s_m < s_{m+1} = T$, $\Delta \dot{u}(s_k) = \dot{u}(s_k^+) - \dot{u}(s_k^-)$ with $\dot{u}(s_k^\pm) = \lim_{t \to s_k^\pm} \dot{u}(t)$, $g_k(u) = \operatorname{grad}_u G_k(u)$, $G_k \in C^1(\mathbb{R}^N, \mathbb{R})$, $V \in C^1([0,T] \times \mathbb{R}^N, \mathbb{R})$, $V_u(t, u(t)) = \operatorname{grad}_u V(t, u)$, $\lambda > 0$ is a constant.

Impulsive differential equations can be used to describe the dynamics of processes which possess abrupt changes at certain instants. Up to now, impulsive differential systems have been widely applied in many science fields such as control theory, biology, mechanics, see [1–7] and references therein. For general theory of impulsive differential equations, we refer the readers to the monographs as [8–10].

The existence of the solutions is one of the most important topics of impulsive differential systems. Many classical methods and tools have been used to study them, such as coincidence degree theory, fixed point theory, and the method of upper and lower solutions. See [11–15] and references therein.

Recently, some authors creatively applied variational method to deal with impulsive problems, see [16–21]. The variational method is opening a new approach for dealing with discontinuity problems such as impulses. However, when the problems studied in [16–21] degenerate to the cases without impulses, plenty of the corresponding results can also be obtained. In other words, the effect of impulses was not seen evidently. As pointed out in [22], these results, in some sense, mean that the nonlinear term $V_u$ plays a more important role than the impulsive terms $g_k$ do in guaranteeing the existence of solutions. Due to this point, Zhang and Li [22] studied the existence of solutions for impulsive differential systems generated by impulses.

*Definition 1.1* (see [22]). A solution is called a solution generated by impulses if this solution is nontrivial when impulsive terms are not zero, but it is trivial when impulsive term is zero.

For example, if problem $(P_\lambda)$ does not possess non-zero solution when $g_k \equiv 0$ for all $1 \le k \le m$, then nonzero solution for problem $(P_\lambda)$ is called solution generated by impulses. In detail, Zhang and Li [22] obtained the following theorem.

**Theorem A** (see [22]). *Assume that $V$, $W$ satisfy the following conditions:*

$(V_1)$ *$V$ is continuous differentiable and there exist positive constants $b_1, b_2 > 0$ such that $b_1|u|^2 \le -V(t,u) \le b_2|u|^2$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$;*

$(V_2)$ *$-V(t,u) \le -V_u(t,u)u \le -2V(t,u)$ for all $(t,u) \in [0,T] \times \mathbb{R}^N$;*

$(g_1)$ *there exists a $\theta > 2$ such that $g_k(u)u \le \theta G_k(u) < 0$, for $u \in \mathbb{R}^N \setminus \{0\}$ and $k = 1,2,\ldots,m$.*

*Then, problem $(P_1)$ possesses at least one non-zero solution generated by impulses.*

Motivated by the facts mentioned above, in this paper, we will further study the existence of solution for problem $(P_\lambda)$ generated by impulses under more general conditions. In addition, we will investigate the multiple solutions and infinitely many solutions generated by impulses.

Now, we state our results.

**Theorem 1.2.** *Assume that $(g_1)$ and the following $(V_1')$, $(V_2')$ hold.*

$(V_1')$ *$V$ is continuous differentiable and there exist positive constants $b > 0$ and $\gamma \in (1,2]$ such that*

$$-V(t,u) \ge b|u|^\gamma \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N; \tag{1.1}$$

$(V_2')$ *there exists a constant $\varrho \in [2,\theta)$ such that*

$$-V(t,u) \le -V_u(t,u)u \le -\varrho V(t,u) \quad \forall (t,u) \in [0,T] \times \mathbb{R}^N. \tag{1.2}$$

*Then, problem $(P_1)$ possesses at least one non-zero solution generated by impulses.*

*Remark 1.3.* It is easy to see that $(V_1')$, $(V_2')$ are weaker than $(V_1)$, $(V_2)$. Therefore, Theorem 1.2 improves Theorem A.

Indeed, taking

$$V(t,u) = -|u|^{\gamma} - |u|^{\varsigma}, \quad 1 < \gamma < \varsigma \leq 2, \quad G_k(u) = -|u|^{\theta}, \tag{1.3}$$

all conditions in Theorem 1.4 are satisfied, but conditions in Theorem A cannot be satisfied.

**Theorem 1.4.** *Assume that* $(V_1)$, $(V_2)$ *hold. Moreover, the following conditions are satisfied:*

$(V_3)$ $V_u(t,u_1 - u_2) = V_u(t,u_1) - V_u(t,u_2)$;

$(H_1)$ $\sum_{k=1}^{m} G_k(u) \leq 0$ *and*

$$\max\left\{\limsup_{u\to 0}\left(\frac{-\sum_{k=1}^{m} G_k(u)}{|u|^2}\right), \limsup_{|u|\to+\infty}\left(\frac{-\sum_{k=1}^{m} G_k(u)}{|u|^2}\right)\right\} < A, \qquad 0 < AC_1^2 < b_3, \tag{1.4}$$

*where* $b_3 = \min\{1/2, b_1\}$ *and* $C_1$ *is a constant which will be defined in Section 2;*

$(H_2)$ *there exists a constant vector* $\xi = (\xi^1, \xi^2, \ldots, \xi^N) \in \mathbb{R}^N \setminus \{0\}$ *such that* $\sum_{k=1}^{m} G_k(\xi) < \int_0^T V(t,\xi)dt$.

*Then, there exists* $B > 0$ *such that problem* $(P_1)$ *possesses at least two non-zero solutions generated by impulses; moreover, their norms are less than* $B$.

*Remark 1.5.* Compared with Theorem A, Theorem 1.4 deals with the multiple solutions generated by impulses. In Theorem A and Theorem 1.2, impulses are superquadratic. However, some impulses which are subquadratic can satisfy the conditions of Theorem 1.4.

*Example 1.6.* Let $V(t,u) = -|u|^2$. It is easy to see that conditions $(V_1)$, $(V_2)$, and $(V_3)$ hold. Let $T = 1$, $m = 1$, $\xi = (1, 0, \ldots, 0)$, and

$$G_1(u) = \begin{cases} -2|u|^3, & |u| < 1, \\ -6|u| + 4, & |u| \geq 1. \end{cases} \tag{1.5}$$

Then, we have

$$\begin{gathered} \limsup_{u\to 0}\frac{-G_1(u)}{|u|^2} = \limsup_{|u|\to+\infty}\frac{-G_1(u)}{|u|^2} = 0, \\ -2 = G_1(\xi) < \int_0^T V(t,\xi)dt = -1. \end{gathered} \tag{1.6}$$

Hence, conditions $(H_1)$ and $(H_2)$ in Theorem 1.4 are satisfied.

**Theorem 1.7.** *Assume that* $(V_1)$, $(V_2)$ *hold. Moreover, the following conditions are satisfied:*

$(H_3)$ *let*

$$\alpha = \liminf_{\rho \to +\infty} \frac{\max_{|u|\le\rho}\left[-\sum_{k=1}^{m} G_k(u)\right]}{\rho^2},$$
$$\beta = \limsup_{|u| \to +\infty} \frac{-\sum_{k=1}^{m} G_k(u)}{|u|^2}, \tag{1.7}$$

*and* $Tb_2C_1^2\alpha < b_3\beta$.

Then, for every $\lambda \in \Lambda := (Tb_2/\beta, b_3/C_1^2\alpha)$, problem $(P_\lambda)$ possesses an unbounded sequence of solutions generated by impulses.

*Example 1.8.* Let $V(t,u) = (-1/2)|u|^2$, $T = 1$, $m = 1$, $N = 1$,

$$a_n = \frac{2n!(n+2)! - 1}{4(n+1)!}, \qquad b_n = \frac{2n!(n+2)! + 1}{4(n+1)!}, \tag{1.8}$$

$$g_1(u) = \begin{cases} \pi(n+1)!\left[(n+1)!^2 - n!^2\right]\sin(2\pi(n+1)!(u-a_n)), & |u| \in \bigcup_{n\in\mathbb{N}}[a_n, b_n], \\ 0, & \text{otherwise}, \end{cases} \tag{1.9}$$

and $g_2 = \max\{0, 2u\}$. Let

$$g = g_1 + g_2, \qquad -G = \int_0^u g(t)\,dt. \tag{1.10}$$

Then, $-G$ is a $C^1$ function with $-G' = g$. From the computation of Example 3.2 in [23], one has $C_1 = \sqrt{2}$, $b_2 = 1/2$, $b_3 = 1/2$ in this case and

$$\lim_{n\to+\infty} \frac{-G(a_n)}{a_n^2} = 1, \qquad \lim_{n\to+\infty} \frac{-G(b_n)}{b_n^2} = 5. \tag{1.11}$$

Moreover,

$$\liminf_{|u|\to+\infty} \frac{-G(u)}{u^2} = 1, \qquad \limsup_{|u|\to+\infty} \frac{-G(u)}{u^2} = 5. \tag{1.12}$$

In addition, $-G$ is nondecreasing. By the monotonicity of $-G$, one has

$$\max_{|u|\le\rho}[-G(u)] = -G(\rho). \tag{1.13}$$

Hence, we have

$$\alpha = 1, \qquad \beta = 5. \tag{1.14}$$

Then, for $\lambda \in [1/10, 1/4]$, problem

$$\begin{gathered} \ddot{u}(t) - u = 0, \quad t \neq s_1, \\ \Delta \dot{u}(s_1) = \lambda g(u(s_1)), \\ u(0) - u(1) = \dot{u}(0) - \dot{u}(1) = 0 \end{gathered} \tag{1.15}$$

possesses an unbounded sequence of solutions generated by impulses.

*Remark 1.9.* In Theorem 1.7, substituting $\rho \to +\infty$ and $|u| \to +\infty$ with $\rho \to 0^+$ and $|u| \to 0^+$, applying part (g) of Lemma 2.4 instead of part (f) in the proof, we can obtain a sequence of pairwise distinct solutions generated by impulses.

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the proof of our main results.

## 2. Preliminaries

In order to prove our main results, we give some definitions and lemmas that will be used in the proof of our main results. Let

$$H_T^1 = \left\{ u : [0, T] \to \mathbb{R}^N \mid u \text{ is absoltely continuous}, \ u(0) = u(T), \ \dot{u} \in L^2([0, T], \mathbb{R}^n) \right\}. \tag{2.1}$$

Then, $H_T^1$ is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_0^T [(u(t), v(t)) + (\dot{u}(t), \dot{v}(t))] dt, \quad \forall u, v \in H_T^1, \tag{2.2}$$

where $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^N$. The corresponding norm is defined by

$$\|u\| = \left( \int_0^T \left[ |\dot{u}(t)|^2 + |u(t)|^2 \right] dt \right)^{1/2}, \quad \forall u \in H_T^1. \tag{2.3}$$

Let $\|\cdot\|_2 = (\int_0^T |u(t)|^2)^{1/2}$ denote the norm of Banach space of $L^p([0, T], \mathbb{R}^N)$. Since $(H_T^1, \|\cdot\|)$ is compactly embedded in $C([0, T], \mathbb{R}^N)$ (see [24]), we claim that there exists a positive constant $C_1$ such that

$$\|u\|_\infty \leq C_1 \|u\|, \tag{2.4}$$

where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$.

To study the problem $(P_\lambda)$, we consider the functional $I$ defined by

$$I(u) = \int_0^T \left[\frac{1}{2}|\dot{u}|^2 - V(t,u)\right]dt + \lambda\sum_{k=1}^{m} G_k(u(s_k)). \tag{2.5}$$

Similar to the proof of Lemma 1 of [22] (see also [20, 21]), we can easily prove the following Lemma 2.1.

**Lemma 2.1.** *Suppose $V \in C^1[0,T]\times\mathbb{R}^N \to \mathbb{R}$, $G_k \in C^1(\mathbb{R}^N,\mathbb{R}^N)$, $k = 1,2,\ldots,m$. Then, $I$ is Frechét differentiable with*

$$I'(u)v = \int_0^T [\dot{u}\dot{v} - V_u(t,u)v]dt + \lambda\sum_{k=1}^{m} g_k(u(s_k))v(s_k), \tag{2.6}$$

*for any $u$ and $v$ in $H_T^1$. Furthermore, $u$ is a solution of $(P_\lambda)$ if and only if $u$ is a critical point of $I$ in $H_T^1$.*

**Lemma 2.2** (see [24, 25]). *Let $E$ be a real Banach space and $I \in C^1(E,\mathbb{R})$ satisfying the (P.S.) condition. Suppose $I(0) = 0$ and*

*$(I_1)$ there are constants $\rho,\beta > 0$ such that $I|_{\partial B_\rho} \geq \beta$, where $B_\rho = \{x \in E \mid \|x\| < \rho\}$,*

*$(I_2)$ there is an $e \in E \setminus B_\rho$ such that $I(e) \leq 0$.*

*Then, $I$ possesses a critical value $c \geq \beta$.*

If $E$ is a real Banach space, denote by $\mathcal{W}_X$ the class of all functionals $\Phi : E \to \mathbb{R}$ possessing the following property: if $\{u_n\}$ is a sequence in $E$ converging weakly to $u$ and $\liminf_{n\to\infty}\Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to $u$. For example, if $E$ is uniformly convex and $g : [0,+\infty) \to \mathbb{R}$ is a continuous, strictly increasing function, then, by a classical results, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

**Lemma 2.3** (see [26]). *Let $E$ be a separable and reflexive real Banach space; let $\Phi : E \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous $C^1$ functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of $E$, and whose derivative admits a continuous inverse on $E^*$; $J : E \to \mathbb{R}$ a $C^1$ functional with compact derivative. Assume that $\Phi$ has a strict local minimum $u_0$ with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\begin{gathered}\alpha' = \max\left\{0, \limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0}\frac{J(u)}{\Phi(u)}\right\},\\ \beta' = \sup_{u\in\Phi^{-1}(0,+\infty)}\frac{J(u)}{\Phi(u)},\end{gathered} \tag{2.7}$$

*assume that $\alpha' < \beta'$. Then, for each compact interval $[a,b] \subset (1/\beta', 1/\alpha')$ (with the conventions $1/0 = +\infty, 1/+\infty = 0$), there exists $B > 0$ with the following property: for every $\lambda \in [a,b]$ and every*

*$C^1$ functional $\Psi : E \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation*

$$\Phi'(x) = \lambda J'(x) + \mu \Psi'(x) \tag{2.8}$$

*has at least three solutions in E whose norms are less than B.*

Now, we recall a result which insures infinitely critical points. For all $r > \inf_X \Phi$, we put

$$\begin{gathered}\varphi(r) = \inf_{x \in \Phi^{-1}((-\infty, r))} \frac{\left(\sup_{x \in \Phi^{-1}((-\infty, r))} \Psi(x)\right) - \Psi(x)}{r - \Phi(x)}, \\ \gamma := \liminf_{r \to +\infty} \varphi(r), \quad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).\end{gathered} \tag{2.9}$$

**Lemma 2.4** (see [27]). *Let X be a reflexive real Banach space, let $\Phi : X \to R$ be a (strongly) continuous, coercive, and sequentially weakly lower semicontinuous and Gâteaux differentiable function, and let $\Psi : X \to R$ be a sequentially weakly upper semicontinuous and Gâteaux differentiable function. One has*

(e) *for every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional $\Phi - \lambda\Psi$ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of $\Phi - \lambda\Psi$ in X.*

(f) *if $\gamma < +\infty$, then, for each $\lambda \in (0, 1/\gamma)$, the following alternative holds:*

*either*

(f1) *$\Phi - \lambda\Psi$ possesses a global minimum*

*or*

(f2) *there is a sequence $\{x_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that*

$$\lim_{n \to +\infty} \Phi(x_n) = +\infty. \tag{2.10}$$

(g) *if $\delta < +\infty$, then, for each $\lambda \in (0, 1/\delta)$, the following alternative holds:*

*either*

(g1) *here is a global minimum of $\Phi$ which is a local minimum of $\Phi - \lambda\Psi$*

*or*

(g2) *there is a sequence of pairwise distinct critical points (local minima) of $\Phi - \lambda\Psi$, with $\lim_{n\to+\infty} \Phi(x_n) = \inf_X \Phi$ which weakly converges to a global minimum of $\Phi$.*

## 3. Proof of the Main Results

In this section, we give the proof of our main results in turn. $C_i$, $i = 1, 2, \ldots$ denote different constants.

*Proof of Theorem 1.2.* We firstly show that $I$ satisfies the $P.S.$ condition. Assume that $\{u_j\}_{j\in\mathbb{N}} \subset H_T^1$ is a sequence such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'(u_j) \to 0$ as $j \to +\infty$. Then, there exists a constant $C_2 > 0$ such that

$$\left|I(u_j)\right| \le C_2, \quad \left|I'(u_j)\right| \le C_2, \quad \forall j \in \mathbb{N}. \tag{3.1}$$

By (3.1), (2.4), (2.5), (2.6), $(V_1')$, $(V_2')$, and $(g_1)$, we have

$$\begin{aligned}
2C_2 + C_3\|u_j\| &\ge 2I(u_j) - \frac{2}{\theta}I'(u_j)u_j \\
&= \left(1-\frac{2}{\theta}\right)\int_0^T \left|\dot{u}_j\right|^2 dt \\
&\quad + \int_0^T \left[\frac{2}{\theta}V_u(t,u_j)u_j - 2V(t,u_j)\right] dt \\
&\quad + 2\sum_{k=1}^m G_k(u_j(s_k)) - \frac{2}{\theta}\sum_{k=1}^m g_k(u_j(s_k))u_j(s_k) \\
&\ge \left(1-\frac{2}{\theta}\right)\int_0^T \left|\dot{u}_j\right|^2 dt + 2\left(1-\frac{\varrho}{\theta}\right)\int_0^T b\left|u_j\right|^\gamma dt \\
&\ge \left(1-\frac{2}{\theta}\right)\int_0^T \left|\dot{u}_j\right|^2 dt \\
&\quad + 2\left(1-\frac{\varrho}{\theta}\right)bC_1^{\gamma-2}\|u_j\|^{\gamma-2}\int_0^T \left|u_j\right|^2 dt \\
&\ge \min\left\{\left(1-\frac{2}{\theta}\right), 2\left(1-\frac{\rho}{\theta}\right)bC_1^{\gamma-2}\|u_j\|^{\gamma-2}\right\}\|u_j\|^2 \\
&\ge \min\left\{\left(1-\frac{2}{\theta}\right)\|u_j\|^2, 2\left(1-\frac{\varrho}{\theta}\right)bC_1^{\gamma-2}\|u_j\|^{\gamma}\right\}.
\end{aligned} \tag{3.2}$$

It follows from (3.2) that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $H_T^1$. In a similar way to Lemma 2 in [22], we can prove that $\{u_j\}_{j\in\mathbb{N}}$ has a convergent subsequence in $H_T^1$. Hence, $I$ satisfies the $P.S.$ condition.

Second, we verify $(I_1)$ of Lemma 2.2. By $(g_1)$ and (2.4), there exists a $\delta_1 > 0$ such that, for any $\|u\|_\infty \le \delta_1$,

$$|G_k(u)| \le \frac{1}{4mC_1}|u|^2. \tag{3.3}$$

By (2.4), there exists a $0 < \delta < 1$ such that, for any $\|u\| \leq \delta$, the inequality $\|u\|_\infty \leq \delta_1$ holds. Then for $u \in H_T^1$ with $\|u\| = \delta_0$, $\delta_0$ small enough $(0 < \delta_0 < \min\{(2bC_1^{\gamma-2})^{1/2-\gamma}, \delta\})$ such that the following inequality holds

$$\begin{aligned} I(u) &\geq \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt + b\int_0^T |u(t)|^\gamma dt - \frac{1}{4C_1}\|u\|_\infty^2 \\ &\geq \min\left\{\frac{1}{2}\|u\|^2, bC_1^{\gamma-2}\|u\|^\gamma\right\} - \frac{1}{4}\|u\|^2 \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}\|u\|^2 \\ &\geq \frac{1}{4}\|\delta_0\|^2. \end{aligned} \tag{3.4}$$

By $(V_2')$, there exist $C_3, C_4 > 0$ such that

$$-V(t,u) \leq C_3|u|^\varrho + C_4, \qquad \forall (t,u) \in [0,T] \times \mathbb{R}^n. \tag{3.5}$$

By $(\mathrm{g}_1)$, there exist $C_5, C_6 > 0$ such that

$$G_k(u) \leq -C_5|u|^\theta + C_6, \quad \forall u \in \mathbb{R}^n, \ k \in \{1,2,\ldots,m\}. \tag{3.6}$$

To verify $(I_2)$ of Lemma 2.2, choose $u \in H_T^1$ such that $u(s_k) \neq 0$ for some $k \in \{1,2,\ldots,m\}$. Hence, we obtain

$$\begin{aligned} I(tu) \leq \frac{t^2}{2}\int_0^T |\dot{u}(t)|^2 dt + C_3|t|^\varrho \int_0^T |u(t)|^\varrho dt \\ - C_5|t|^\theta \sum_{k=1}^m |u(s_k)|^\theta + C_7, \end{aligned} \tag{3.7}$$

where $C_7$ is a positive constant. Since $\theta > 2$, $\varrho \in [2,\theta)$, (3.7) implies $I(tu) \to -\infty$ as $t \to \infty$. So, for $t$ large enough, $e = tu$ satisfies condition $(I_2)$. By the Mountain Pass Lemma (Lemma 2.2), $I$ possesses at least one non-zero critical point. Then, by Lemma 2.1, problem $(P_1)$ has at least one non-zero solution.

Finally, we verify that the solution is generated by impulses. Suppose $g_k \equiv 0$ and $u$ is a solution for problem $(P_1)$. Then, by $(V_1')$ and $(V_2')$,

$$
\begin{aligned}
0 &= \int_0^T \left[|\dot{u}|^2 - V_u(t,u)u\right] dt \\
&\geq \int_0^T \left[|\dot{u}|^2 + b|u|^{\gamma}\right] dt \\
&\geq \int_0^T |\dot{u}|^2 dt + b\|u\|_{\infty}^{\gamma-2} \int_0^T |u|^2 dt \\
&\geq \int_0^T |\dot{u}|^2 dt + bC_1^{\gamma-2}\|u\|^{\gamma-2} \int_0^T |u|^2 dt \\
&\geq \min\left\{1, bC_1^{\gamma-2}\|u\|^{\gamma-2}\right\}\|u\|^2 \\
&= \min\left\{\|u\|^2, bC_1^{\gamma-2}\|u\|^{\gamma}\right\}.
\end{aligned}
\tag{3.8}
$$

This implies that $u \equiv 0$, that is, problem $(P_1)$ does not possess any non-zero solutions when impulses are zero. Hence, by Definition 1.1, the non-zero solution obtained above is generated by impulses. □

*Proof of Theorem 1.4.* In order to apply Lemma 2.3, we let

$$
\begin{aligned}
\Phi(u) &= \int_0^T \left[\frac{1}{2}|\dot{u}(t)|^2 - V(t,u(t))\right] dt, \\
J(u) &= -\sum_{k=1}^{m} G_k(u(s_k)).
\end{aligned}
\tag{3.9}
$$

Obviously, $E = H_T^1$ is a separable and uniformly convex Banach space. By $(V_1)$, we have

$$
b_3\|u\|^2 \leq \Phi(u) \leq b_4\|u\|^2, \tag{3.10}
$$

where $b_3 = \min\{1/2, b_1\}$ and $b_4 = \min\{1/2, b_2\}$. Hence, by (3.10) and Lemma 2.1, we can obtain that $\Phi(u)$ is a coercive, $C^1$ functional, and bounded on each bounded subset of $E$. For a sequence $\{u_n\} \subset H_T^1$, if $u_n \rightharpoonup u \in H_T^1$ and $\liminf_{n\to\infty}\Phi(u_n) \leq \Phi(u)$, then $u_n \to u \in C[0,T]$. This means that $\int_0^T [(-1/2)|u(t)|^2 - V(t,u(t))]dt$ is weakly continuous. Hence, we have that $\Phi(u) = (1/2)\|u\|^2 + \int_0^T [(-1/2)|u(t)|^2 - V(t,u(t))]dt$ is sequentially weakly lower semicontinuous and $\liminf_{n\to\infty}\|u_n\|^2 \le \|u\|^2$. Therefore, $\{u_n\}$ has a subsequence converging strongly to $u$, that is, $\Phi(u)$ belongs to $\mathcal{W}_X$. For any $u \in H_T^1 \setminus \{0\}$, we have

$$\begin{aligned}\langle \Phi'(u), u\rangle &= \int_0^T [(\dot{u}(t), \dot{u}(t)) - (V_u(t,u(t)), u(t))]dt \\ &\ge \int_0^T \left[(\dot{u}(t), \dot{u}(t)) + b_1|u(t)|^2\right]dt \\ &\ge \min\{1, b_1\}\|u\|^2.\end{aligned} \tag{3.11}$$

So, $\lim_{\|u\|\to\infty}\langle\Phi'(u), u\rangle/\|u\| = +\infty$, that is, $\Phi'$ is coercive. For any $u, v \in H_T^1$, one has

$$\begin{aligned}\langle \Phi'(u) - \Phi'(v), u - v\rangle &= \int_0^T (\dot{u}(t) - \dot{v}(t), \dot{u}(t) - \dot{v}(t))dt \\ &\quad - \int_0^T (V_u(t,u(t)) - V_u(t,v(t)), u(t) - v(t))dt \\ &\ge \int_0^T (\dot{u}(t) - \dot{v}(t), \dot{u}(t) - \dot{v}(t))dt \\ &\quad + \int_0^T b_1|u(t) - v(t)|^2 dt \\ &\ge \min\{1, b_1\}\|u - v\|^2.\end{aligned} \tag{3.12}$$

So, $\Phi'$ is uniformly monotone. By [28, Theorem 26.A(d)], we have that $(\Phi')^{-1}$ exists and is continuous. For any $u, v \in H_T^1$,

$$\langle J'(u), u\rangle = -\sum_{k=1}^{m} g_k(u(s_k))u(s_k). \tag{3.13}$$

Let $u_n \rightharpoonup u \in H_T^1$, then $u_n \to u \in C[0,T]$. Hence, $J'(u_n) \to J'(u)$, $\Psi'(u_n) \to \Psi'(u)$ as $n \to \infty$. Therefore, we have that $J'$ is a compact operator by [28, Proposition 26.2]. In addition, $\Phi$ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. In view of $(H_1)$, there exist $\tau_1 > \tau_2 > 0$ such that

$$-\sum_{k=1}^{m} G_k(u) \le A|u|^2 \tag{3.14}$$

for any $|u| \in [0, \tau_2) \cup (\tau_1, +\infty)$. By the continuity of $G_k$, $\sum_{k=1}^{m} G_k(u)$ is bounded for any $|u| \in [\tau_2, \tau_1]$. We can choose $C_8 > 0$ and $\sigma > 2$ such that

$$-\sum_{k=1}^{m} G_k(u) \le A|u|^2 + C_8|u|^\sigma \tag{3.15}$$

for any $u \in \mathbb{R}^N$. Hence, by (2.4), we have

$$J(u) \leq AC_1^2\|u\|^2 + C_8C_1^{\sigma}\|u\|^{\sigma} \tag{3.16}$$

for all $u \in H_T^1$. By (3.16) and $(H_1)$, we obtain

$$\limsup_{u\to 0}\frac{J(u)}{\Phi(u)} \leq \limsup_{u\to 0}\left(\frac{AC_1^2}{b_3} + \frac{C_8C_1^{\sigma}}{b_3}\|u\|^{\sigma-2}\right) \leq \frac{AC_1^2}{b_3} < 1. \tag{3.17}$$

On the other hand, by $(H_1)$, $-\sum_{k=1}^m G_k(u) \geq 0$. Then, one has

$$\begin{aligned}\frac{-\sum_{k=1}^m G_k(u(s_k))}{\Phi(u)} &= \frac{-\sum_{k=1}^m G_k(u(s_k))|_{|u(s_k)|\leq\tau_1}}{\Phi(u)} + \frac{-\sum_{k=1}^m G_k(u(s_k))|_{|u(s_k)|>\tau_1}}{\Phi(u)} \\ &\leq \frac{C_9}{\Phi(u)} + \frac{-\sum_{k=1}^m G_k(u(s_k))|_{|u(s_k)|>\tau_1}}{\Phi(u)} \\ &\leq \frac{C_9}{b_3\|u\|^2} + \frac{A|u(s_k)|^2}{b_3\|u\|^2} \\ &\leq \frac{C_9}{b_3\|u\|^2} + \frac{AC_1^2\|u\|^2}{b_3\|u\|^2},\end{aligned} \tag{3.18}$$

where $C_9$ is a positive constant and $f|_A$ denotes the restriction of the functional $f$ to the set $A$. Hence, we have

$$\limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)} \leq \limsup_{\|u\|\to\infty}\left(\frac{C_9}{b_3\|u\|^2} + \frac{AC_1^2}{b_3}\right) \leq \frac{AC_1^2}{b_3} < 1. \tag{3.19}$$

Combining (3.18) with (3.19), one has $\alpha' < 1$.

By $\{H_2\}$, we obtain

$$\beta' = \sup_{u\in\Phi^{-1}(0,+\infty)}\frac{J(u)}{\Phi(u)} \geq \frac{-\sum_{k=1}^m G_k(\xi)}{-\int_0^T V(t,\xi)dt} > 1. \tag{3.20}$$

Then, for each compact interval $[a,b] \subset (1/\beta', 1/\alpha')$, $1 \in [a,b]$, there exists $B > 0$ with the following property: for every $f(t)$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$,

$$\Phi'(x) = J'(x) + \mu\Psi'(x). \tag{3.21}$$

Let $\mu = 0$, then we can obtain that problem $(P_1)$ has at least three solutions in $E$ whose norms are less than $B$. □

Finally, by Theorem 4 in [22], we can prove that problem $(P_1)$ does not possess any non-zero periodic solutions when impulses are zero. By Definition 1.1, problem $(P_1)$ has at least two solutions generated by impulses.

*Proof of Theorem 1.7.* Let

$$\Phi(u) = \int_0^T \left[\frac{1}{2}|\dot{u}(t)|^2 - V(t,u(t))\right] dt,$$
$$\Psi(u) = -\sum_{k=1}^{m} G_k(u(s_k)). \tag{3.22}$$

Similar to the proof of Theorems 3.1 and 3.3 in [27], it is easy to see that the functionals $\Phi, \Psi$ satisfy the regularity assumptions required in Lemma 2.4. Let us now verify that

$$\gamma < +\infty. \tag{3.23}$$

Let $\{\rho_n\}$ be a sequence of positive numbers such that $\rho_n \to +\infty$ and

$$\lim_{n\to+\infty} \frac{\max_{|u|\le\rho_n}\left[-\sum_{k=1}^m G_k(u)\right]}{\rho_n^2} = \liminf_{\rho\to+\infty} \frac{\max_{|u|\le\rho_n}\left[-\sum_{k=1}^m G_k(u)\right]}{\rho^2}. \tag{3.24}$$

If we put, for every $n \in \mathbb{N}$,

$$r_n = \frac{\rho_n^2 b_3}{C_1^2}, \tag{3.25}$$

then, by (2.4) and (3.10), $\Phi(u) \le r_n$ implies that $\|u\|_\infty \le \rho_n$. Since $0 \in \Phi^{-1}((-\infty, r_n))$, the following inequality holds:

$$\varphi(r_n) = \inf_{x\in\Phi^{-1}((-\infty,r_n))} \frac{\left(\sup_{x\in\Phi^{-1}((-\infty,r_n))}\Psi(x)\right) - \Psi(x)}{r_n - \Phi(x)}$$
$$\le \frac{C_1^2 \max_{|u|\le\rho_n}\left[-\sum_{k=1}^m G_k(u)\right]}{\rho_n^2 b_3}. \tag{3.26}$$

Hence,

$$0 < \gamma \le \frac{C_1^2 \alpha}{b_3} < +\infty. \tag{3.27}$$

From $\lambda \in \Lambda := (Tb_2/\beta, b_3/C_1^2\alpha)$, we have

$$\Lambda \subset \left(0, \frac{1}{\gamma}\right). \tag{3.28}$$

Fix $\lambda \in \Lambda$ and let us verify that the functional $I = \Phi - \lambda\Psi$ is unbounded from below. In fact, by the choice of $\lambda$ and the positivity of $\beta$, one has that there exists a sequence $\{y_n\}$ in $\mathbb{R}^n$ such that $\lim_{n\to\infty}|y_n| = +\infty$ and $-\sum_{k=1}^m G_k(y_n)/|y_n|^2 > Tb_2/\lambda$. Let $v_n(t) = y_n$. Then,

$$\begin{aligned} I(v_n) &= \int_0^T [-V(t, v_n)]dt - \lambda\left[-\sum_{k=1}^m G_k(v_n)\right] \\ &\leq \left(Tb_2 - \lambda\frac{-\sum_{k=1}^m G_k(y_n)}{|y_n|^2}\right)|y_n|^2. \end{aligned} \tag{3.29}$$

Note that $Tb_2 - \lambda(-\sum_{k=1}^m G_k(y_n)/|y_n|^2) < 0$, $I$ has no global minimum. Hence, by Lemma 2.4, there is a sequence $\{x_n\}$ of critical points (local minima) of $\Phi - \lambda\Psi$ such that $\lim_{n\to+\infty}\Phi(x_n) = +\infty$. By Theorem 4 in [22], we can prove that problem $(P_\lambda)$ does not possess any non-zero periodic solutions when impulses are zero. Therefore, by Definition 1.1, the solutions obtained above are all generated by impulses. □

## Acknowledgment

The paper was supported by the Foundation for Youth Key Teacher by the Hengyang Normal University. H. Chen was supported by NFSC (10871206).

## References

[1] R. K. George, A. K. Nandakumaran, and A. Arapostathis, "A note on controllability of impulsive systems," *Journal of Mathematical Analysis and Applications*, vol. 241, no. 2, pp. 276–283, 2000.

[2] T. E. Carter, "Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion," *Dynamics and Control*, vol. 10, no. 3, pp. 219–227, 2000.

[3] Y. Pei, C. Li, L. Chen, and C. Wang, "Complex dynamics of one-prey multi-predator system with defensive ability of prey and impulsive biological control on predators," *Advances in Complex Systems*, vol. 8, no. 4, pp. 483–495, 2005.

[4] X. Liu and A. R. Willms, "Impulsive controllability of linear dynamical systems with applications to maneuvers of spacecraft," *Mathematical Problems in Engineering*, vol. 2, no. 4, pp. 277–299, 1996.

[5] S. Gao, L. Chen, J. J. Nieto, and A. Torres, "Analysis of a delayed epidemic model with pulse vaccination and saturation incidence," *Vaccine*, vol. 24, no. 35-36, pp. 6037–6045, 2006.

[6] J. Chu and J. J. Nieto, "Impulsive periodic solutions of first-order singular differential equations," *Bulletin of the London Mathematical Society*, vol. 40, no. 1, pp. 143–150, 2008.

[7] S. T. Zavalishchin and A. N. Sesekin, *Dynamic Impulse Systems*, vol. 394 of *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.

[8] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific Publishing, Singapore, 1989.

[9] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Impulsive and Hybrid Dynamical Systems*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, NJ, USA, 2006.

[10] A. M. Samoĭlenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, World Scientific Publishing, River Edge, NJ, USA, 1995.

[11] R. P. Agarwal and D. O'Regan, "A multiplicity result for second order impulsive differential equations via the Leggett Williams fixed point theorem," *Applied Mathematics and Computation*, vol. 161, no. 2, pp. 433–439, 2005.

[12] H. Wang and H. Chen, "Boundary value problem for second-order impulsive functional differential equations," *Applied Mathematics and Computation*, vol. 191, no. 2, pp. 582–591, 2007.

[13] J. Chen, C. C. Tisdell, and R. Yuan, “On the solvability of periodic boundary value problems with impulse,” *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 902–912, 2007.

[14] J. Li, J. J. Nieto, and J. Shen, “Impulsive periodic boundary value problems of first-order fifferential equations,” *Journal of Mathematical Analysis and Applications*, vol. 303, pp. 288–303, 2005.

[15] M. E. Hernández, H. R. Henríquez, and M. A. McKibben, “Existence results for abstract impulsive second-order neutral functional differential equations,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2736–2751, 2009.

[16] J. J. Nieto and D. O’Regan, “Variational approach to impulsive differential equations,” *Nonlinear Analysis: Real World Applications*, vol. 10, no. 2, pp. 680–690, 2009.

[17] Y. Tian and W. Ge, “Applications of variational methods to boundary-value problem for impulsive differential equations,” *Proceedings of the Edinburgh Mathematical Society. Series II*, vol. 51, no. 2, pp. 509–527, 2008.

[18] Z. Zhang and R. Yuan, “An application of variational methods to Dirichlet boundary value problem with impulses,” *Nonlinear Analysis: Real World Applications*, vol. 11, no. 1, pp. 155–162, 2010.

[19] J. Zhou and Y. Li, “Existence and multiplicity of solutions for some Dirichlet problems with impulsive effects,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 2856–2865, 2009.

[20] J. Sun and H. Chen, “Variational method to the impulsive equation with Neumann boundary conditions,” *Boundary Value Problems*, vol. 2009, Article ID 316812, 17 pages, 2009.

[21] J. Sun, H. Chen, J. J. Nieto, and M. Otero-Novoa, “The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 12, pp. 4575–4586, 2010.

[22] H. Zhang and Z. Li, “Periodic and homoclinic solutions generated by impulses,” *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 39–51, 2011.

[23] G. Bonanno and R. Livrea, “Multiple periodic solutions for Hamiltonian systems with not coercive potential,” *Journal of Mathematical Analysis and Applications*, vol. 363, no. 2, pp. 627–638, 2010.

[24] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, vol. 74 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1989.

[25] P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, vol. 65 of *CBMS Regional Conference Series in Mathematics*, American Mathematical Society, Providence, RI, USA, 1986.

[26] B. Ricceri, “A further three critical points theorem,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 4151–4157, 2009.

[27] B. Ricceri, “A general variational principle and some of its applications,” *Journal of Computational and Applied Mathematics*, vol. 113, no. 1-2, pp. 401–410, 2000.

[28] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol. 2, Springer, Singapore, 1990.