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*Research Article*

# Some Identities on Bernstein Polynomials Associated with $q$-Euler Polynomials

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We investigate some interesting properties of the $q$-Euler polynomials. The purpose of this paper is to give some relationships between Bernstein and $q$-Euler polynomials, which are derived by the $p$-adic integral representation of the Bernstein polynomials associated with $q$-Euler polynomials.

## 1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ denote the ring of $p$-adic integers, the field of $p$-adic numbers, and the field of $p$-adic completion of the algebraic closure of $\mathbb{Q}_p$, respectively (see [1–15]). Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The normalized $p$-adic absolute value is defined by $|p|_p = 1/p$. As an indeterminate, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}\left(x+p^N\mathbb{Z}_p\right) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \end{aligned} \tag{1.1}$$

(see [7–10]). For $n \in \mathbb{N}$, we can derive the following integral equation from (1.1):

$$I_{-1}(f_n) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \tag{1.2}$$

where $f_n(x) = f(x+n)$ (see [7–11]). As well-known definition, the Euler polynomials are given by the generating function as follows:

$$\frac{2}{e^t+1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t}{n!}, \tag{1.3}$$

(see [3, 5, 7–15]), with usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case $x = 0$, $E_n(0) = E_n$ are called the $n$th Euler numbers. From (1.3), we can derive the following recurrence formula for Euler numbers:

$$E_0 = 1, \quad (E+1)^n + E_n = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{1.4}$$

(see [12]), with usual convention about replacing $E^n$ by $E_n$. By the definitions of Euler numbers and polynomials, we get

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l, \tag{1.5}$$

(see [3, 5, 7–15]). Let $C[0,1]$ denote the set of continuous functions on $[0,1]$. For $f \in C[0,1]$, Bernstein introduced the following well-known linear positive operator in the field of real numbers $\mathbb{R}$:

$$\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{k,n}(x), \tag{1.6}$$

where $\binom{n}{k} = n(n-1)\cdots(n-k+1)/k! = n!/k!(n-k)!$ (see [1, 2, 7, 11, 12, 14]). Here, $\mathbb{B}_n(f \mid x)$ is called the Bernstein operator of order $n$ for $f$. For $k, n \in \mathbb{Z}_+$, the Bernstein polynomials of degree $n$ are defined by

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad \text{for } x \in [0,1]. \tag{1.7}$$

In this paper, we study the properties of $q$-Euler numbers and polynomials. From these properties, we investigate some identities on the $q$-Euler numbers and polynomials. Finally, we give some relationships between Bernstein and $q$-Euler polynomials, which are derived by the $p$-adic integral representation of the Bernstein polynomials associated with $q$-Euler polynomials.

## 2. $q$-Euler Numbers and Polynomials

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Let $f(x) = q^x e^{xt}$. From (1.1) and (1.2), we have

$$\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \frac{2}{qe^t + 1}. \tag{2.1}$$

Now, we define the $q$-Euler numbers as follows:

$$\frac{2}{qe^t + 1} = e^{E_q t} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}, \tag{2.2}$$

with the usual convention about replacing $E_q^n$ by $E_{n,q}$.

By (2.2), we easily get

$$E_{0,q} = \frac{2}{q+1}, \qquad q\left(E_q + 1\right)^n + E_{n,q} = \begin{cases} 2 & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \tag{2.3}$$

with usual convention about replacing $E_q^n$ by $E_{n,q}$.

We note that

$$\frac{2}{qe^t + 1} = \frac{2}{e^t + q^{-1}} \cdot \frac{2}{1+q} = \frac{2}{1+q} \sum_{n=0}^{\infty} H_n\left(-q^{-1}\right) \frac{t^n}{n!}, \tag{2.4}$$

where $H_n(-q^{-1})$ is the $n$th Frobenius-Euler numbers.

From (2.1), (2.2), and (2.4), we have

$$\int_{\mathbb{Z}_p} q^x e^{xt} d\mu_{-1}(x) = E_{n,q} = \frac{2}{1+q} H_n\left(-q^{-1}\right), \quad \text{for } n \in \mathbb{Z}_+. \tag{2.5}$$

Now, we consider the $q$-Euler polynomials as follows:

$$\frac{2}{qe^t + 1} e^{xt} = e^{E_q(x)t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}, \tag{2.6}$$

with the usual convention $E_q^n(x)$ by $E_{n,q}(x)$.

From (1.2), (2.1), and (2.6), we get

$$\int_{\mathbb{Z}_p} q^x e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \tag{2.7}$$

By comparing the coefficients on the both sides of (2.6) and (2.7), we get the following Witt's formula for the $q$-Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} q^y (x+y)^n d\mu_{-1}(y) = E_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_{l,q}. \tag{2.8}$$

From (2.6) and (2.8), we can derive the following equation:

$$\frac{2q}{qe^t+1} e^{(1-x)t} = \frac{2}{1+q^{-1}e^{-t}} e^{-xt} = \sum_{n=0}^{\infty} E_{n,q^{-1}}(x)(-1)^n \frac{t}{n!}. \tag{2.9}$$

By (2.6) and (2.9), we obtain the following reflection symmetric property for the $q$-Euler polynomials.

**Theorem 2.1.** *For $n \in \mathbb{Z}_+$, one has*

$$(-1)^n E_{n,q^{-1}}(x) = qE_{n,q}(1-x). \tag{2.10}$$

From (2.5), (2.6), (2.7), and (2.8), we can derive the following equation: for $n \in \mathbb{N}$,

$$\begin{aligned} E_{n,q}(2) &= (E_q+1+1)^n = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}(1) \\ &= E_{0,q} + \frac{1}{q} \sum_{l=1}^{n} \binom{n}{l} qE_{l,q}(1) = \frac{2}{1+q} - \frac{1}{q} \sum_{l=1}^{n} \binom{n}{l} E_{l,q} \\ &= \frac{2}{1+q} + \frac{2}{q(1+q)} - \frac{1}{q} \sum_{l=0}^{n} \binom{n}{l} E_{l,q} \\ &= \frac{2}{q} - \frac{1}{q^2} qE_{n,q}(1) = \frac{2}{q} + \frac{1}{q^2} E_{n,q}, \end{aligned} \tag{2.11}$$

by using recurrence formula (2.3). Therefore, we obtain the following theorem.

**Theorem 2.2.** *For $n \in \mathbb{N}$, one has*

$$qE_{n,q}(2) = 2 + \frac{1}{q} E_{n,q}. \tag{2.12}$$

By using (2.5) and (2.8), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} q^{-x}(x-1)^n d\mu_{-1}(x) \\ &= (-1)^n E_{n,q^{-1}}(-1) = q\int_{\mathbb{Z}_p}(x+2)^n d\mu_{-1}(x) = q\left(\frac{2}{q}+\frac{1}{q^2}E_{n,q}\right) \\ &= 2+\frac{1}{q}E_{n,q} = 2+\frac{1}{q}\int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x), \quad \text{for } n>0.\end{aligned} \tag{2.13}$$

Therefore, we obtain the following theorem.

**Theorem 2.3.** *For $n \in \mathbb{N}$, one has*

$$\int_{\mathbb{Z}_p} q^{-x}(1-x)^n d\mu_{-1}(x) = 2+\frac{1}{q}\int_{\mathbb{Z}_p} x^n q^x d\mu_{-1}(x). \tag{2.14}$$

By using Theorem 2.3, we will study for the $p$-adic integral representation on $\mathbb{Z}_p$ of the Bernstein polynomials associated with $q$-Euler polynomials in Section 3.

## 3. Bernstein Polynomials Associated with $q$-Euler Numbers and Polynomials

Now, we take the $p$-adic integral on $\mathbb{Z}_p$ for the Bernstein polynomials in (1.7) as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} B_{k,n}(x) q^x d\mu_{-1}(x) &= \int_{\mathbb{Z}_p}\binom{n}{k} x^k (1-x)^{n-k} q^x d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}\int_{\mathbb{Z}_p} x^{n-j} q^x d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{n-k-j}E_{n-j,q} \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}E_{k+j,q}, \quad \text{where } n,k\in\mathbb{Z}_+.\end{aligned} \tag{3.1}$$

By the definition of Bernstein polynomials, we see that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \quad \text{where } n,k\in\mathbb{Z}_+. \tag{3.2}$$

Let $n, k \in \mathbb{Z}_+$ with $n > k$. Then, by (3.2), we get

$$\begin{aligned}
\int_{\mathbb{Z}_p} q^x B_{k,n}(x)\,d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} q^x B_{n-k,n}(1-x)\,d\mu_{-1}(x) \\
&= \binom{n}{n-k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\int_{\mathbb{Z}_p}(1-x)^{n-j}q^x\,d\mu_{-1}(x) \\
&= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(2+q\int_{\mathbb{Z}_p}x^{n-j}q^x\,d\mu_{-1}(x)\right) \\
&= \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\left(2+qE_{n-j,q^{-1}}\right) \\
&= \begin{cases} 2+qE_{n,q^{-1}} & \text{if } k=0, \\ \binom{n}{k}q\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E_{n-j,q^{-1}} & \text{if } k>0. \end{cases}
\end{aligned} \tag{3.3}$$

Thus, we obtain the following theorem.

**Theorem 3.1.** *For $n, k \in \mathbb{Z}_+$ with $n > k$, one has*

$$\int_{\mathbb{Z}_p} q^{1-x} B_{k,n}(x)\,d\mu_{-1}(x) = \begin{cases} 2q+E_{n,q} & \text{if } k=0, \\ \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}E_{n-j,q} & \text{if } k>0. \end{cases} \tag{3.4}$$

By (3.1) and Theorem 3.1, we get the following corollary.

**Corollary 3.2.** *For $n, k \in \mathbb{Z}_+$ with $n > k$, one has*

$$\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^{j}E_{k+j,q^{-1}} = \begin{cases} 2+\dfrac{1}{q}E_{n,q} & \text{if } k=0, \\ \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\dfrac{1}{q}E_{n-j,q} & \text{if } k>0. \end{cases} \tag{3.5}$$

For $m, n, k \in \mathbb{Z}_+$ with $m+n > 2k$. Then, we get

$$
\begin{aligned}
&\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{-x} d\mu_{-1}(x) \\
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j+2k}\int_{\mathbb{Z}_p} q^{-x}(1-x)^{n+m-j} d\mu_{-1}(x) \\
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j+2k} q \int_{\mathbb{Z}_p} (x+2)^{n+m-j} q^{x} d\mu_{-1}(x) \\
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{j+2k} q \left(\frac{2}{q} + \frac{1}{q^2} E_{n+m-j,q}\right) \\
&= \begin{cases} 2 + \dfrac{1}{q} E_{n+m,q} & \text{if } k = 0, \\ \dbinom{n}{k}\dbinom{m}{k}\displaystyle\sum_{j=0}^{2k}\dbinom{2k}{j}(-1)^{j+2k}\dfrac{1}{q} E_{n+m-j,q} & \text{if } k > 0. \end{cases}
\end{aligned} \tag{3.6}
$$

Therefore, we obtain the following theorem.

**Theorem 3.3.** *For $m, n, k \in \mathbb{Z}_+$ with $m+n > 2k$, one has*

$$
\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{1-x} d\mu_{-1}(x) = \begin{cases} 2q + E_{n+m,q} & \textit{if } k = 0, \\ \dbinom{n}{k}\dbinom{m}{k}\displaystyle\sum_{j=0}^{2k}\dbinom{2k}{j}(-1)^{j+2k} E_{n+m-j,q} & \textit{if } k > 0. \end{cases} \tag{3.7}
$$

By using binomial theorem, for $m, n, k \in \mathbb{Z}_+$, we get

$$
\int_{\mathbb{Z}_p} B_{k,n}(x) B_{k,m}(x) q^{1-x} d\mu_{-1}(x) \tag{3.8}
$$

$$
\begin{aligned}
&= \binom{n}{k}\binom{m}{k}\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^{j}\int_{\mathbb{Z}_p} x^{j+2k} q^{1-x} d\mu_{-1}(x) \\
&= q\binom{n}{k}\binom{m}{k}\sum_{j=0}^{n+m-2k}\binom{n+m-2k}{j}(-1)^{j} E_{j+2k,q^{-1}}.
\end{aligned} \tag{3.9}
$$

By comparing the coefficients on the both sides of (3.8) and Theorem 3.3, we obtain the following corollary.

**Corollary 3.4.** *Let $m, n, k \in \mathbb{Z}_+$ with $m + n > 2k$. Then, we get*

$$\sum_{j=0}^{n+m-2k} \binom{n+m-2k}{j} (-1)^j E_{j+2k,q^{-1}} = \begin{cases} 2 + \dfrac{1}{q} E_{n+m,q} & \text{if } k = 0, \\ \dfrac{1}{q} \displaystyle\sum_{j=0}^{2k} \binom{2k}{j} (-1)^{j+2k} E_{n+m-j,q} & \text{if } k > 0. \end{cases} \tag{3.10}$$

For $s \in \mathbb{N}$, let $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$. By induction, we get

$$\begin{aligned} &\int_{\mathbb{Z}_p} B_{k,n_1}(x) \cdots B_{k,n_s}(x) q^{-x} d\mu_{-1}(x) \\ &= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \int_{\mathbb{Z}_p} x^{sk} (1-x)^{n_1+\cdots+n_s-sk} q^{-x} d\mu_{-1}(x) \\ &= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} \int_{\mathbb{Z}_p} (1-x)^{n_1+\cdots+n_s-j} q^{-x} d\mu_{-1}(x) \\ &= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} q \int_{\mathbb{Z}_p} (x+2)^{n_1+\cdots+n_s-j} q^{x} d\mu_{-1}(x) \\ &= \left( \prod_{i=1}^{s} \binom{n_i}{k} \right) \sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} q \left( \frac{2}{q} + \frac{1}{q^2} E_{n_1+\cdots+n_s-j,q} \right) \\ &= \begin{cases} 2 + \dfrac{1}{q} E_{n_1+\cdots+n_s,q} & \text{if } k = 0, \\ \left( \displaystyle\prod_{i=1}^{s} \binom{n_i}{k} \right) \dfrac{1}{q} \displaystyle\sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+\cdots+n_s-j,q} & \text{if } k > 0. \end{cases} \end{aligned} \tag{3.11}$$

Therefore, we obtain the following theorem.

**Theorem 3.5.** *Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, one has*

$$\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{1-x} d\mu_{-1}(x) = \begin{cases} 2q + E_{n_1+n_2+\cdots+n_s,q} & \text{if } k = 0, \\ \left( \displaystyle\prod_{i=1}^{s} \binom{n_i}{k} \right) \displaystyle\sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s-j,q} & \text{if } k > 0. \end{cases} \tag{3.12}$$

For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ by binomial theorem, we get

$$\int_{\mathbb{Z}_p} \left( \prod_{i=1}^{s} B_{k,n_i}(x) \right) q^{-x} d\mu_{-1}(x)$$
$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j \int_{\mathbb{Z}_p} x^{j+sk} q^{-x} d\mu_{-1}(x)$$
$$= \binom{n_1}{k} \cdots \binom{n_s}{k} \sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk,q^{-1}}. \tag{3.13}$$

By using (3.13) and Theorem 3.5, we obtain the following corollary.

**Corollary 3.6.** *Let $s \in \mathbb{N}$. For $n_1, n_2, \ldots, n_s, k \in \mathbb{Z}_+$ with $n_1 + n_2 + \cdots + n_s > sk$, one has*

$$\sum_{j=0}^{n_1+\cdots+n_s-sk} \binom{n_1+\cdots+n_s-sk}{j} (-1)^j E_{j+sk,q^{-1}} = \begin{cases} 2 + \dfrac{1}{q} E_{n_1+n_2+\cdots+n_s,q} & \text{if } k = 0, \\ \dfrac{1}{q} \displaystyle\sum_{j=0}^{sk} \binom{sk}{j} (-1)^{sk+j} E_{n_1+n_2+\cdots+n_s-j,q} & \text{if } k > 0. \end{cases} \tag{3.14}$$

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