Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2011, Article ID 264795, 18 pages
doi:10.1155/2011/264795

Research Article

Global Existence of Cylinder Symmetric Solutions for the Nonlinear Compressible Navier-Stokes Equations

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Received 4 September 2011; Accepted 6 October 2011

Academic Editor: Meng Fan

We prove the global existence of cylinder symmetric solutions to the compressible Navier-Stokes equations with external forces and heat source in R^3 for any large initial data. Some new ideas and more delicate estimates are used to prove this result.

1. Introduction

In this paper, we study the global existence of cylinder symmetric solutions to the nonlinear compressible Navier-Stokes equations with external forces and heat source in a bounded domain $G = \{r \in R^+,\ 0 < a < r < b < +\infty\}$ of R^3, where r is the radial variable. In the Eulerian coordinates, the system under consideration are expressed as

$$\rho_t + (\rho u)_r + \frac{\rho u}{r} = 0, \tag{1.1}$$

$$\rho\left(u_t + uu_r - \frac{v^2}{r}\right) + P_r - \nu\left(u_r + \frac{u}{r}\right)_r = f_1(r,t), \tag{1.2}$$

$$\rho\left(v_t + uv_r + \frac{uv}{r}\right) - \mu\left(v_r + \frac{v}{r}\right)_r = f_2(r,t), \tag{1.3}$$

$$\rho(w_t + uw_r) - \mu\left(w_{rr} + \frac{w_r}{r}\right) = f_3(r,t), \tag{1.4}$$

$$C_V\rho(\theta_t + u\theta_r) - \kappa\left(\theta_{rr} + \frac{\theta_r}{r}\right) + P\left(u_r + \frac{u}{r}\right) - Q = g(r,t), \tag{1.5}$$

where

$$P = \gamma\rho\theta, \qquad Q = \lambda\left(u_r + \frac{u}{r}\right)^2 + \mu\left[\left(v_r - \frac{v}{r}\right)^2 + w_r^2 + 2u_r^2 + 2\left(\frac{u}{r}\right)^2\right], \tag{1.6}$$

and ρ is the mass density, θ is the absolute temperature, u, v, w are the radial velocity, angular velocity, and axial velocity, respectively, and λ, μ, ν, γ, C_V, κ, λ are the constants satisfying $\gamma, C_V, \kappa, \mu > 0$, $3\lambda + 2\mu \geq 0$ $(\nu = \lambda + 2\mu)$. f_1, f_2, f_3, and g represent external forces and heat source, respectively. For system (1.1)–(1.5), we consider the following initial boundary value problem:

$$\rho(r,0) = \rho_0(r), \qquad (u,v,w)(r,0) = (u_0,v_0,w_0)(r), \qquad \theta(r,0) = \theta_0(r), \quad r \in G, \tag{1.7}$$

$$(u,v,w)(a,t) = (u,v,w)(b,t) = 0, \qquad \theta_r(a,t) = \theta_r(b,t) = 0, \quad t \geq 0. \tag{1.8}$$

To show the global existence, it is convenient to transform the system (1.1)–(1.5) to that in the Lagrangian coordinates. The Eulerian coordinates (r,t) are connected to the Lagrangian coordinates (ξ,t) by the relation

$$r(\xi,t) = r_0(\xi) + \int_0^t \tilde{u}(\xi,\tau)d\tau, \tag{1.9}$$

where $\tilde{u}(\xi,t) = u(r(\xi,t),t)$ and

$$r_0(\xi) = \eta^{-1}(\xi), \qquad \eta(r) = \int_a^r s\rho_0(s)ds, \quad r \in G. \tag{1.10}$$

It should be noted that if $\inf\{\rho_0(s) : s \in (a,b)\} > 0$, then η is invertible. It follows from (1.1), (1.8), and (1.10) that

$$\int_a^r s\rho(s,t)ds = \int_a^{r_0} s\rho_0(s)ds = \xi, \tag{1.11}$$

and G is transformed into $\Omega = (0,L)$ with

$$L = \int_a^b s\rho(s,t)ds = \int_a^b s\rho_0(s), \quad \forall t \geq 0. \tag{1.12}$$

Differentiating (1.11) with respect to ξ yields

$$\partial_\xi r(\xi,t) = r(\xi,t)^{-1}\rho^{-1}(r(\xi,t),t). \tag{1.13}$$

In general, for a function $\phi(r,t) = \widetilde{\phi}(\xi,t) = \phi(r(\xi,t),t)$, we easily get

$$\partial_t\widetilde{\phi}(\xi,t) = \partial_t\phi(r,t) + u\partial_r\phi(r,t), \tag{1.14}$$

$$\partial_\xi\widetilde{\phi}(\xi,t) = \partial_r\phi(r,t)\partial_\xi r(\xi,t) = \frac{\partial_r\phi(r,t)}{r}\rho^{-1}(r,t). \tag{1.15}$$

Without danger of confusion, we denote $(\widetilde{\rho},\widetilde{u},\widetilde{v},\widetilde{w},\widetilde{\theta})$ still by (ρ,u,v,w,θ) and (ξ,t) by (x,t). We set $\tau := 1/\rho$ to denote the specific volume. Therefore, by virtue of (1.13)–(1.15), system (1.1)–(1.8) in the new variables (x,t) read

$$\tau_t = (ru)_x, \tag{1.16}$$

$$u_t = r\left[\frac{\nu(ru)_x - \gamma\theta}{\tau}\right]_x + \frac{v^2}{r} + f_1(r(x,t),t), \tag{1.17}$$

$$v_t = \mu r\left[\frac{(rv)_x}{\tau}\right]_x - \frac{uv}{r} + f_2(r(x,t),t), \tag{1.18}$$

$$w_t = \mu r\left[\frac{(rw)_x}{\tau}\right]_x + \mu\frac{\tau w}{r^2} + f_3(r(x,t),t), \tag{1.19}$$

$$C_V\theta_t = \kappa\left[\frac{r^2\theta_x}{\tau}\right]_x + \frac{1}{\tau}\left[\nu(ru)_x - \gamma\theta\right](ru)_x + \mu\frac{[(rv)_x]^2}{\tau} + \mu\frac{r^2w_x^2}{\tau} - 2\mu\left(u^2+v^2\right)_x + g(r(x,t),t), \tag{1.20}$$

together with

$$\tau(x,0) = \tau_0(x), \qquad (u,v,w)(x,0) = (u_0,v_0,w_0)(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in [0,L], \tag{1.21}$$

$$(u,v,w)(0,t) = (u,v,w)(L,t) = 0, \qquad \theta_x(0,t) = \theta_x(L,t) = 0, \quad t \geq 0. \tag{1.22}$$

By (1.9) and (1.13), we have

$$\begin{gathered} r(x,t) = r_0(x) + \int_0^t u(x,s)ds, \quad r_0(x) = \left[a^2 + 2\int_0^x \tau_0(y)dy\right]^{1/2}, \\ r_t(x,t) = u(x,t), \qquad r(x,t)r_x(x,t) = \tau(x,t). \end{gathered} \tag{1.23}$$

Now let us first recall the related results in the literature. When there were no external forces and heat source, in two or three dimensions, the global existence and large time behavior of smooth solutions to the equations of a viscous polytropic ideal gas have been investigated for general domains only in the case of sufficiently small initial data, see, for example, [1–3]. For any large initial data, the global existence of generalized solutions was shown in [4–7]. Recently, Qin [8] proved the exponential stability in H^1 and H^2, and Qin and Jiang [9] studied the global existence and exponential stability in H^4 with smallness of initial total energy.

When there exist external forces and heat forces, for one-dimensional case, the system is isentropic compressible Navier-Stokes equations. Mucha [10] obtained the exponential stability under various boundary conditions, Yanagi [11] established the existence of classical solutions, and Qin and Zhao [12] proved the global existence and asymptotic behavior of solutions for pressure $P = \rho^\gamma$ with $\gamma = 1$. Later on, Zhang and Fang [13] studied the global existence and uniqueness for $\gamma > 1$. For nonisentropic compressible Navier-Stokes equations, Qin and Yu [14] proved the global existence and asymptotic behavior for perfect gas. In two- or three-dimensional case and the external force and heat source $f \neq 0$, $g \neq 0$, Qin and Wen [15] proved the global existence of spherically symmetric solutions. In this paper, we will prove the global existence of cylinder symmetric solutions with external forces and heat source in a bounded domain in R^3.

The notation in this paper will be as follows: $L^{\overline{p}}, 1 \leq \overline{p} \leq +\infty$, $W^{m,\overline{p}}$, $m \in N$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on $(0, L)$. In addition, $\|\cdot\|_B$ denotes the norm in the space B; we also put $\|\cdot\| = \|\cdot\|_{L^2}$. We denote by $C^k(I, B), k \in N_0$, the space of k-times continuously differentiable functions from $I \subseteq R$ into a Banach space B, and likewise by $L^{\overline{p}}(I, B)$, $1 \leq \overline{p} \leq +\infty$ the corresponding Lebesgue spaces. Subscripts t and x denote the (partial) derivatives with respect to t and x, respectively. We use C_1 to denote the generic positive constant depending on the H^1-norm of the initial data and time T.

We suppose that $f_i(r(x,t),t)(i = 1, 2, 3)$, $g(r,t)$ satisfy, for any $T > 0$,

$$f_i \in L^1([0,T], L^\infty[0,L]) \cap L^2\Big([0,T], L^2[0,L]\Big), \tag{1.24}$$

$$g(r,t) > 0, \qquad g \in L^1([0,T], L^\infty[0,L]) \cap L^2\Big([0,T], L^2[0,L]\Big). \tag{1.25}$$

We are now in a position to state our main theorems.

Theorem 1.1. *Assume that* (1.24)-(1.25) *hold; if* $(\tau_0, u_0, v_0, w_0, \theta_0) \in H^1[0,L] \times H_0^1[0,L] \times H_0^1[0,L] \times H_0^1[0,L] \times H^1[0,L]$, $\tau_0(x) > 0, \theta_0(x) > 0$ *on* $[0,L]$ *and the initial data are compatible with the boundary conditions* (1.22)*, then for problem* (1.16)–(1.22) *there exists a unique global solution* $(\tau, u, v, w, \theta) \in C([0,T], H^1[0,L] \times H_0^1[0,L] \times H_0^1[0,L] \times H_0^1[0,L] \times H^1[0,L])$ *such that, for any* $T > 0$,

$$\begin{gathered}
0 < a \leq r(x,t) \leq b, \quad (x,t) \in [0,L] \times [0,T], \\
0 < C_1^{-1} \leq \tau(x,t) \leq C_1, \quad (x,t) \in [0,L] \times [0,T], \\
\|\tau(t)\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 + \|\theta(t)\|_{H^1}^2 + \|r(t)\|_{H^2}^2 \\
+ \int_0^t \Big(\|\tau\|_{H^1}^2 + \|u\|_{H^2}^2 + \|v\|_{H^2}^2 + \|w\|_{H^2}^2 + \|\theta\|_{H^2}^2 + \|u_t\|^2 \\
+ \|v_t\|^2 + \|w_t\|^2 + \|\theta_t\|^2 \Big)(\tau) d\tau \leq C_1, \quad \forall t \in [0,T].
\end{gathered} \tag{1.26}$$

2. Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1. To this end, we assume that in this section all assumptions in Theorem 1.1 hold. The proof of Theorem can be divided into the following several lemmas.

Lemma 2.1. *One has*

$$a = r(0,t) \le r(x,t) \le r(L,t) = b, \quad \forall (x,t) \in [0,L] \times [0,+\infty). \tag{2.1}$$

Proof. The proof of (2.1) is borrowed from [6, 8]; please refer to (2.1) in [6] or Lemma 2.1 in [8] for detail. □

Lemma 2.2. *The global solution* $(\tau(t), u(t), v(t), w(t), \theta(t))$ *to problems* (1.16)–(1.22) *satisfies the following estimates:*

$$\int_0^L \left[\frac{1}{2}\left(u^2+v^2+w^2\right) + C_V\theta\right](x,t)\,dx \le C_1, \tag{2.2}$$

$$\int_0^L U(x,t)\,dx + \int_0^t\int_0^L \left(\frac{\kappa r^2\theta_x^2}{\tau\theta^2} + \frac{\tau u^2}{\theta} + \frac{u_x^2}{\tau\theta} + \frac{(ru)_x^2}{\tau\theta} + \frac{w_x^2}{\tau\theta} + \frac{\left(\tau r^{-1}v - rv_x\right)^2}{\tau\theta} + \frac{g}{\theta}\right)(x,s)\,dx\,ds \le C_1, \tag{2.3}$$

where

$$U(x,t) = \frac{1}{2}\left(u^2+v^2+w^2\right) + \gamma(\tau - \log\tau - 1) + C_V(\theta - \log\theta - 1). \tag{2.4}$$

Proof. Multiplying (1.17)–(1.19) by u, v, and w, respectively, adding up the results, and using (1.16), we have

$$\begin{aligned}\frac{d}{dt}&\left[\frac{1}{2}\left(u^2+v^2+w^2\right) + C_V\theta\right]\\ &= \left[\frac{\kappa r^2\theta_x}{\tau} + \frac{ru(\nu(ru)_x - \gamma\theta)}{\tau} + \frac{\mu rv(rv)_x}{\tau} + \frac{\mu r^2ww_x}{\tau} - 2\mu\left(u^2+v^2\right)\right]_x\\ &\quad + f_1u + f_2v + f_3w + g.\end{aligned} \tag{2.5}$$

Integrating (2.5) with respect to x and t over $Q_T = [0, L] \times [0, t]$ ($t \in [0, T]$, $\forall T > 0$), using boundary condition (1.22), we obtain

$$\begin{aligned}\int_0^L \left[\frac{1}{2}\left(u^2+v^2+w^2\right)+C_V\theta\right]dx &= \int_0^L \left[\frac{1}{2}\left(u_0^2+v_0^2+w_0^2\right)+C_V\theta_0\right]dx \\ &\quad + \int_0^t\int_0^L (f_1u+f_2v+f_3w+g)(x,s)\,dx\,ds \\ &\le C_1 + C_1\int_0^t\int_0^L \left(u^2+v^2+w^2\right)(x,s)\,dx\,ds \\ &\quad + C_1\int_0^t\int_0^L \left(f_1^2+f_2^2+f_3^2\right)(x,s)\,dx\,ds + C_1\int_0^t \|g\|_{L^\infty}ds\end{aligned} \tag{2.6}$$

which, by using Gronwall's inequality and (1.24)-(1.25), gives (2.2).

By (1.16)–(1.20), we can easily obtain

$$\begin{aligned}&U_t + \frac{\kappa r^2\theta_x^2}{\tau\theta^2} + \frac{\nu(ru)_x^2+\mu(rv)_x^2+\mu r^2w_x^2}{\tau\theta} - \frac{2\mu\left(u^2+v^2\right)_x}{\theta} + \frac{g}{\theta} \\ &\quad = \left[\frac{\kappa(\theta-1)r^2\theta_x}{\tau\theta} + \frac{ru(\nu(ru)_x-\gamma\theta)}{\tau} + \frac{\mu rv(rv)_x}{\tau} + \frac{\mu r^2ww_x}{\tau} - 2\mu\left(u^2+v^2\right) + \gamma ru\right]_x \\ &\qquad + f_1u+f_2v+f_3w+g.\end{aligned} \tag{2.7}$$

Note that constants $\nu = \lambda + 2\mu$ and

$$\begin{aligned}\frac{\nu(ru)_x^2+\mu(rv)_x^2}{\tau\theta} - \frac{2\mu\left(u^2+v^2\right)_x}{\theta} &= \frac{2\mu\left(\tau^2r^{-2}u^2+r^2u_x^2\right)+\lambda(ru)_x^2}{\tau\theta} + \frac{\mu\left(\tau r^{-1}v-rv_x\right)^2}{\tau\theta} \\ &\ge C_1^{-1}\left(\frac{\tau u^2}{\theta} + \frac{u_x^2+\lambda(ru)_x^2}{\tau\theta} + \frac{\mu\left(\tau r^{-1}v-rv_x\right)^2}{\tau\theta}\right).\end{aligned} \tag{2.8}$$

Integrating (2.7) with respect to x and t over Q_T, using (1.22), (1.24)-(1.25), and (2.8), we conclude

$$\begin{aligned}&\int_0^L U(x,t)\,dx + \int_0^t\int_0^L \left(\frac{\kappa r^2\theta_x^2}{\tau\theta^2} + \frac{\tau u^2}{\theta} + \frac{u_x^2+(ru)_x^2+w_x^2+\left(\tau r^{-1}v-rv_x\right)^2}{\tau\theta} + \frac{g}{\theta}\right)dx\,ds \\ &\quad \le C_1 + \int_0^t\int_0^L (f_1u+f_2v+f_3w+g)(x,s)\,dx\,ds \\ &\quad \le C_1 + C_1\int_0^t\int_0^L \left(u^2+v^2+w^2+f_1^2+f_2^2+f_3^2\right)(x,s)\,dx\,ds + C_1\int_0^t \|g\|_{L^\infty}ds \\ &\quad \le C_1.\end{aligned} \tag{2.9}$$

The proof is complete. □

Next we adapt and modify an idea of Qin and Wen [15] for one-dimensional case to give a representation for τ.

Let

$$\sigma(x,t) := \frac{\nu(ru)_x - \gamma\theta}{\tau} + \int_0^x \left(\frac{v^2 - u^2}{r^2} + \frac{f_1(r(y,t),t)}{r} \right) dy, \tag{2.10}$$

$$h(x,t) := \int_0^x \frac{u_0}{r_0} dy + \int_0^t \sigma(x,s) dx. \tag{2.11}$$

Then, we infer from (1.16) and (1.17) that

$$h_x = \frac{u}{r}, \qquad h_t = \sigma. \tag{2.12}$$

By (1.16) and (2.12), we have

$$(h\tau)_t = (ruh)_x - u^2 + \nu(ru)_x - \gamma\theta + \tau \int_0^x \left(\frac{v^2 - u^2}{r^2} + \frac{f_1}{r} \right) dx. \tag{2.13}$$

Integrating (2.13) with respect to x and t over Q_T, we obtain

$$\begin{aligned}
\int_0^L h\tau dx &= \int_0^L h_0\tau_0 dx - \int_0^t\int_0^L \left(u^2 + \gamma\theta\right) dx\, ds + \int_0^t\int_0^L \tau \int_0^x \left(\frac{v^2 - u^2}{r^2} + \frac{f_1}{r} \right) dy\, dx\, ds \\
&= \int_0^L h_0\tau_0 dx - \int_0^t\int_0^L \left(u^2 + \gamma\theta\right) dx\, ds + \int_0^t\int_0^L (rr_x) \int_0^x \left(\frac{v^2 - u^2}{r^2} + \frac{f_1}{r} \right) dy\, dx\, ds \\
&= \int_0^L h_0\tau_0 dx - \int_0^t\int_0^L \left(u^2 + \gamma\theta\right) dx\, ds + \frac{b^2}{2}\int_0^t\int_0^L \left(\frac{v^2 - u^2}{r^2} + \frac{f_1}{r} \right) dx\, ds \\
&\quad - \int_0^t\int_0^L \left(\frac{v^2 - u^2}{2} + \frac{rf_1}{2} \right) dx\, ds,
\end{aligned} \tag{2.14}$$

where $h_0(x) := h(x,0)$. It follows from integration of (1.16) over Q_T and use of (1.22) that

$$\int_0^L \tau(x,t) dx = \int_0^L \tau_0(x) dx = \tau^*. \tag{2.15}$$

If we apply the mean value theorem to (2.14) and use (2.15), we conclude there is an $x_0(t) \in [0,L]$ such that

$$h(x_0(t),t) = \frac{1}{\tau^*}\int_0^L h(x,t)\tau(x,t) dx. \tag{2.16}$$

Therefore, we derive from (2.11), (2.14), and (2.16) that

$$\int_0^t \sigma(x_0(t),s)ds = h(x_0(t),t) - \int_0^{x_0(t)} \frac{u_0}{r_0}dx$$
$$= -\frac{1}{\tau^*}\int_0^t\int_0^L \left(\frac{v^2+u^2}{2} + \gamma\theta + \frac{rf_1}{2}\right)dx\,ds + \frac{b^2}{2\tau^*}\int_0^t\int_0^L \left(\frac{v^2-u^2}{r^2} + \frac{f_1}{r}\right)dx\,ds$$
$$+ \frac{1}{\tau^*}\int_0^L h_0(x)\tau_0(x)dx - \int_0^{x_0(t)} \frac{u_0}{r_0}dx. \tag{2.17}$$

Using (2.17), we will show the representation of specific volume τ.

Lemma 2.3. *One has the following representation:*

$$\tau(x,t) = \frac{D(x,t)}{B(x,t)}\left[1 + \frac{\gamma}{\nu}\int_0^t \frac{\theta(x,s)B(x,s)}{D(x,s)}ds\right], \quad x \in [0,L], \tag{2.18}$$

where

$$D(x,t) = \tau_0(x)\exp\left\{\frac{1}{\nu}\left[\frac{1}{\tau^*}\int_0^L \tau_0 h_0 dx - \int_0^x \frac{u_0}{r_0}dy + \int_{x_0(t)}^x \frac{u}{r}dy\right.\right.$$
$$\left.\left. - \int_0^t\int_0^x \frac{f_1}{r}dy\,ds - \frac{1}{\tau^*}\int_0^t\int_0^L \frac{rf_1}{2}dx\,ds + \frac{b^2}{2\tau^*}\int_0^t\int_0^L \frac{f_1}{r}dx\,ds\right]\right\},$$
$$B(x,t) = \exp\left\{\frac{1}{\nu}\left[\frac{1}{\tau^*}\int_0^t\int_0^L \left(\frac{u^2+v^2}{2} + \gamma\theta\right)dx\,ds - \frac{b^2}{2\tau^*}\int_0^t\int_0^L \frac{v^2-u^2}{r^2}dx\,ds\right.\right.$$
$$\left.\left. + \int_0^t\int_0^x \frac{v^2-u^2}{r^2}dx\,ds\right]\right\}. \tag{2.19}$$

Proof. By (1.16) and (1.17), we have

$$\left(\frac{u}{r}\right)_t = \sigma_x = \nu(\log\tau)_{tx} - \gamma\left(\frac{\theta}{\tau}\right)_x + \frac{v^2-u^2+rf_1}{r^2}. \tag{2.20}$$

Integrating (2.20) over $[x_0(t), x] \times [0, t]$ and using (2.17), we derive

$$
\begin{aligned}
&\nu \log \tau(x,t) - \gamma \int_0^t \frac{\theta(x,s)}{\tau(x,s)} ds \\
&\quad = \nu \log \tau_0(x) + \int_0^t \sigma(x_0(t), s) ds - \int_0^t \int_0^x \frac{v^2 - u^2 + r f_1}{r^2} + \int_{x_0(t)}^x \left(\frac{u}{r} - \frac{u_0}{r_0} \right) dy \\
&\quad = \nu \log \tau_0(x) - \frac{1}{\tau^*} \int_0^t \int_0^L \left(\frac{v^2 + u^2}{2} + \gamma\theta + \frac{r f_1}{2} \right) dx\, ds + \frac{b^2}{2\tau^*} \int_0^t \int_0^L \left(\frac{v^2 - u^2}{r^2} + \frac{f_1}{r} \right) dx\, ds \\
&\qquad + \frac{1}{\tau^*} \int_0^L h_0(x) \tau_0(x) dx - \int_0^x \frac{u_0}{r_0} dx - \int_0^t \int_0^x \frac{v^2 - u^2 + r f_1}{r^2} dy\, ds + \int_{x_0(t)}^x \frac{u}{r} dy
\end{aligned}
\tag{2.21}
$$

which, when the exponentials are taken, turns into

$$
\frac{B(x,t)}{D(x,t)} = \frac{1}{\tau(x,t)} \exp\left(\frac{\gamma}{\nu} \int_0^t \frac{\theta(x,s)}{\tau(x,s)} ds \right). \tag{2.22}
$$

Multiplying (2.22) by $\gamma\theta/\nu$ and integrating the resulting equation with respect to t, we arrive at

$$
\exp\left(\frac{\gamma}{\nu} \int_0^t \frac{\theta(x,s)}{\tau(x,s)} ds \right) = 1 + \frac{\gamma}{\nu} \int_0^t \frac{\theta(x,s) B(x,s)}{D(x,s)} ds. \tag{2.23}
$$

Substituting this into (2.22), we obtain (2.18). The proof is complete. □

Lemma 2.4. *There are positive constants $\overline{\tau}$ and $\underline{\tau}$, such that, for any $T > 0$,*

$$
\underline{\tau} \le \tau(x,t) \le \overline{\tau}, \quad (x,t) \in [0, L] \times [0, T]. \tag{2.24}
$$

Proof. Recalling the definition $D(x,t)$, we have by (1.24), Cauchy-Schwarz's inequality, and Lemma 2.1 that

$$
\left| \frac{1}{\tau^*} \int_0^t \int_0^L \frac{r f_1}{2} dx\, ds - \frac{b^2}{2\tau^*} \int_0^t \int_0^L \frac{f_1}{r} dx\, ds - \int_0^t \int_0^x \frac{f_1}{r} dy\, ds \right| \le C_1 \int_0^t \|f_1\|_{L^\infty} \le C_1 \tag{2.25}
$$

which, along with Lemma 2.2, gives

$$
0 < C_1^{-1} \le D(x,t) \le C_1, \quad (x,t) \in [0, L] \times [0, T]. \tag{2.26}
$$

By Lemmas 2.1 and 2.2, we easily obtain, for any $0 \le s \le t$,

$$
B(x,t) \le C_1, \quad \frac{B(x,s)}{B(x,t)} \le \exp\{-C_1(t - s)\}. \tag{2.27}
$$

Therefore, we derive from (2.2), (2.18), and (2.26)-(2.27) that

$$\tau(x,t) \geq \frac{D(x,t)}{B(x,t)} \geq \underline{\tau}, \tag{2.28}$$

$$\begin{aligned}
\tau(x,t) &\leq C_1\left[1+\int_0^t \theta e^{-C_1(t-s)}ds\right] \\
&\leq C_1+\int_0^t\left[\left(\int_0^L \theta dx\right)^{1/2}+\left(\int_0^L \frac{r^2\theta_x^2}{\tau\theta^2}dx\right)^{1/2}\left(\int_0^L \frac{\tau\theta}{r^2}dx\right)^{1/2}\right]^2 e^{-C_1(t-s)}ds \\
&\leq C_1+C_1\left(1+\int_0^t \max_{[0,L]}\tau(\cdot,s)\int_0^L \frac{r^2\theta_x^2}{\tau\theta^2}dx\,ds\right)ds
\end{aligned} \tag{2.29}$$

which, by using Gronwall's inequality and (2.28), gives (2.24). The proof is complete. □

Remark 1. If the initial data or initial energy are small enough, we can obtain the uniform estimate independent of time t about specific volume τ under assumptions of external forces. Moreover, we can prove the large-time behavior of solutions.

Lemma 2.5. *Under the assumptions of Theorem 1.1, one has, for any $T>0$ and for all $t\in[0,T]$,*

$$\int_0^L\left(\theta^2+u^4+v^4+w^4\right)dx+\int_0^t\int_0^L\left(\theta_x^2+u^2u_x^2+v^2v_x^2+w^2w_x^2\right)(x,s)dx\,ds \leq C_1. \tag{2.30}$$

Proof. Multiplying (2.5) by $(1/2)(u^2+v^2+w^2)+C_V\theta$ and then integrating the result over Q_T, we have

$$\begin{aligned}
&\frac{1}{2}\int_0^L\left(\frac{1}{2}(u^2+v^2+w^2)+C_V\theta\right)^2dx \\
&\quad\leq C_1-\frac{C_V\kappa}{2}\int_0^t\int_0^L \frac{r^2\theta_x^2}{\tau}dx\,ds \\
&\quad\quad+C_1\int_0^t\int_0^L\left\{\frac{r^2u^2u_x^2+r^2v^2v_x^2+r^2w^2w_x^2}{\tau}+u^4+v^4+\theta^2+\theta^2u^2\right\}dx\,ds \\
&\quad\quad+\int_0^t\int_0^L(f_1u+f_2v+f_3w+g)\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+C_V\theta\right)dx\,ds,
\end{aligned} \tag{2.31}$$

where

$$\begin{aligned}
&\int_0^t\int_0^L g\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+C_V\theta\right)dx\,ds \\
&\qquad \le \int_0^t \|g\|_{L^\infty}\int_0^L\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+C_V\theta\right)dx\,ds \le C_1, \\
&\int_0^t\int_0^L (f_1u+f_2v+f_3w)\left(\frac{1}{2}\left(u^2+v^2+w^2\right)+C_V\theta\right)dx\,ds \\
&\qquad \le C_1\int_0^t(\|f_1\|_{L^\infty}+\|f_2\|_{L^\infty}+\|f_3\|_{L^\infty})\int_0^L\left(u^2+u^4+v^2+v^4+w^2+w^4\right)dx\,ds \\
&\qquad\quad +\int_0^t\int_0^L \theta^2 dx\,ds.
\end{aligned} \tag{2.32}$$

Multiplying (1.17) by u^3 and then integrating the result over Q_T, we get

$$\begin{aligned}
\frac{1}{4}\int_0^L u^4dx &\le C_1-\frac{\nu}{\overline{\overline{\tau}}}\int_0^t\int_0^1 r^2u^2u_x^2dx\,ds+C_1\int_0^t\int_0^L\left(u^4+\theta^2u^2\right)(x,s)dx\,ds \\
&\quad +\int_0^t\int_0^L\left(\frac{v^2}{r}+f_1\right)u^3dx\,ds \\
&\le C_1-\frac{\nu}{\overline{\overline{\tau}}}\int_0^t\int_0^1 r^2u^2u_x^2dx\,ds+C_1\int_0^t\int_0^L\left(u^4+\theta^2u^2\right)(x,s)dx\,ds \\
&\quad +C_1\int_0^t\left(\|f_1\|_{L^\infty}+\|u\|_{L^\infty}^2\right)\int_0^L\left(v^4+u^2+u^4\right)dx\,ds.
\end{aligned} \tag{2.33}$$

Similarly, multiplying (1.18) and (1.19) by v^3 and w^3, respectively, and then integrating over Q_T, we have

$$\begin{aligned}
\frac{1}{4}\int_0^L v^4dx &\le C_1-\frac{\mu}{\overline{\overline{\tau}}}\int_0^t\int_0^1 r^2v^2v_x^2dx\,ds+C_1\int_0^t\int_0^L\left(v^4+u^2v^4+f_2\left(v^2+v^4\right)\right)dx\,ds \\
&\le C_1-\frac{\mu}{\overline{\overline{\tau}}}\int_0^t\int_0^1 r^2v^2v_x^2dx\,ds+C_1\int_0^t\left(\|f_2\|_{L^\infty}+\|u\|_{L^\infty}^2+1\right)\int_0^L\left(v^2+v^4\right)dx\,ds, \\
\frac{1}{4}\int_0^L w^4dx &\le C_1-\frac{\mu}{\overline{\overline{\tau}}}\int_0^t\int_0^1 r^2w^2w_x^2dx\,ds+C_1\int_0^t\int_0^L\left(w^4+f_3\left(w^2+w^4\right)\right)dx\,ds \\
&\le C_1-\frac{\mu}{\overline{\overline{\tau}}}\int_0^t\int_0^1 r^2w^2w_x^2dx\,ds+C_1\int_0^t(\|f_3\|_{L^\infty}+1)\int_0^L\left(w^2+w^4\right)dx\,ds.
\end{aligned} \tag{2.34}$$

Multiplying (2.31) and (2.33) by $\mu/(2\overline{\tau}C_1)$ and μ/ν, respectively, adding up the resulting inequalities, and using (2.34) to obtain, with the help of (2.32), the following result:

$$\begin{aligned}
&\int_0^L \left(u^4+v^4+w^4+\theta^2\right)dx+\int_0^t\int_0^L\left(\theta_x^2+u^2u_x^2+v^2v_x^2+w^2w_x^2\right)(x,s)dx\,ds\\
&\quad\le C_1+\int_0^t(\|f_1\|_{L^\infty}+\|f_2\|_{L^\infty}+\|f_3\|_{L^\infty}+1)\int_0^L\left(\theta^2+u^4+v^4+w^4\right)dx\,ds\\
&\qquad+\int_0^t\|u\|_{L^\infty}^2\int_0^L\left(\theta^2+u^4+v^4+w^4\right)dx\,ds.
\end{aligned}\tag{2.35}$$

On the other hand, by (2.3) and (2.20),

$$\int_0^t\|u\|_{L^\infty}^2ds\le\int_0^t\left(\int_0^L|u_x|^2dx\right)^2ds\le\int_0^t\int_0^L\frac{u_x^2}{\tau\theta}dx\int_0^L\tau\theta dx\,ds\le C_1.\tag{2.36}$$

In view of (1.24)-(1.25) and (2.36), we apply Gronwall's inequality to (2.35) to obtain (2.30). The proof is complete. □

Lemma 2.6. *Under the assumptions of Theorem 1.1, one has, for any $T>0$,*

$$\int_0^L\tau_x^2(x,t)dx+\int_0^t\int_0^L\theta\tau_x^2(x,s)dx\,ds\le C_1,\quad\forall t\in[0,T].\tag{2.37}$$

Proof. By means of (1.16), we rewrite (1.17) as

$$\left(\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right)_t=\frac{\gamma(\theta\tau_x-\tau\theta_x)}{\tau^2}+\frac{v^2-u^2+rf_1}{r^2}.\tag{2.38}$$

Multiplying (2.38) by $(u/r)-(\nu\tau_x/\tau)$ in $L^2[0,L]$ and using Lemmas 2.1–2.5, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right\|^2+\nu\gamma\int_0^L\frac{\theta\tau_x^2}{\tau^2r}dx\\
&\quad=\int_0^L\left[\frac{\gamma(\theta\tau_x-\tau\theta_x)u}{r\tau^2}+\frac{\gamma\nu\tau_x\theta_x}{r\tau^2}+\frac{v^2-u^2+rf_1}{r^2}\left(\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right)\right]dx\\
&\quad\le\frac{1}{2}\nu\gamma\int_0^L\frac{\theta\tau_x^2}{\tau^2r}dx+C_1\int_0^L\left(\theta u^2+\theta_x^2+u^2+\frac{\theta_x^2}{\theta}+u^4+v^4+f_1^2+\left(\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right)^2\right)dx.
\end{aligned}\tag{2.39}$$

Integrating it with respect to t, using Lemmas 2.1-2.5, (1.24), and (2.36), we get

$$\begin{aligned}
&\left\|\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right\|^2+\int_0^t\int_0^L \theta\tau_x^2 dx\,ds \\
&\le C_1+\int_0^t\left\|\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right\|^2 ds+C_1\int_0^t\int_0^L\left(\theta u^2+f_1^2+\theta_x^2+\frac{\theta_x^2}{\theta}\right)dx\,ds \\
&\le C_1+\int_0^t\left\|\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right\|^2 ds+\int_0^t\|u\|_{L^\infty}^2\int_0^L\theta dx\,ds+\int_0^t\int_0^L\theta_x^2\left(1+\frac{1}{\theta^2}\right)dx\,ds \\
&\le C_1+\int_0^t\left\|\frac{u}{r}-\frac{\nu\tau_x}{\tau}\right\|^2 ds.
\end{aligned} \tag{2.40}$$

We exploit the Gronwall inequality to (2.40) to obtain (2.37). The proof is complete. □

Lemma 2.7. *Under the assumptions of Theorem 1.1, one has, for any $T>0$,*

$$\int_0^L u_x^2(x,t)dx+\int_0^t\int_0^L\left(u_t^2+u_{xx}^2\right)(x,s)dx\,ds\le C_1,\quad \forall t\in[0,T], \tag{2.41}$$

$$\int_0^L v_x^2(x,t)dx+\int_0^t\int_0^L\left(v_t^2+v_{xx}^2\right)(x,s)dx\,ds\le C_1,\quad \forall t\in[0,T], \tag{2.42}$$

$$\int_0^L w_x^2(x,t)dx+\int_0^t\int_0^L\left(w_t^2+w_{xx}^2\right)(x,s)dx\,ds\le C_1,\quad \forall t\in[0,T]. \tag{2.43}$$

Proof. Multiplying (1.17) by u_t, integrating the result over Q_T, using Lemmas 2.1–2.6, and taking into account that $(ru)_x(ru_t)_x/\tau=(1/2)((ru)_x^2/\tau)_t-((ru)_x(u^2)_x/\tau)+((ru)_x^3/2\tau^2)$, we obtain

$$\begin{aligned}
&\int_0^t\int_0^L u_t^2 dx\,ds \\
&\quad=-\int_0^t\int_0^L(ru_t)_x\frac{\nu(ru)_x}{\tau}-\int_0^t\int_0^L\left(\frac{\gamma\theta}{\tau}\right)_x ru_t dx\,ds+\int_0^t\int_0^L\left(\frac{v^2}{r}+f_1\right)u_t dx\,ds \\
&\quad=\frac{\nu}{2}\int_0^L\frac{(r_0u_0)_x^2}{\tau_0}dx-\frac{\nu}{2}\int_0^L\frac{(ru)_x^2}{\tau}dx+\gamma\int_0^t\int_0^L r\frac{\tau\theta_x-\theta\tau_x}{\tau^2}u_t dx\,ds \\
&\qquad-\nu\int_0^t\int_0^L u^2\left(\frac{(ru)_x}{\tau}\right)_x dx\,ds+\frac{\nu}{2}\int_0^t\int_0^L ru\left(\frac{(ru)_x^2}{\tau^2}\right)_x dx\,ds+\int_0^t\int_0^L\left(\frac{v^2}{r}+f_1\right)u_t dx\,ds \\
&\quad\le C_1-\frac{\nu}{2}\int_0^L\frac{(ru)_x^2}{\tau}dx+\frac{1}{4}\int_0^t\int_0^L u_t^2 dx\,ds+C_1\int_0^t\int_0^L\left(\theta^2\tau_x^2+\theta_x^2+f_1^2+v^4\right)dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&-\nu\int_0^t\int_0^L u^2\left[\frac{u_t}{r}+\frac{1}{r}\left(\frac{\gamma\theta}{\tau}\right)_x-\frac{v^2+f_1r}{r^2}\right]dx\,ds\\
&+\frac{\nu}{2}\int_0^t\int_0^L 2ru\frac{(ru)_x}{\tau}\left[\frac{u_t}{r}+\frac{1}{r}\left(\frac{\gamma\theta}{\tau}\right)_x-\frac{v^2+f_1r}{r^2}\right]dx\,ds\\
&\leq C_1-\frac{\nu}{2}\int_0^L\frac{(ru)_x^2}{\tau}dx+\frac{1}{2}\int_0^t\int_0^L u_t^2dx\,ds+C_1\int_0^t\int_0^L\left(\theta^2\tau_x^2+\theta_x^2+f_1^2+v^4+u^4\right)dx\,ds\\
&+C_1\int_0^t\int_0^L u^2\frac{(ru)_x^2}{\tau}(x,s)dx\,ds.
\end{aligned}
\tag{2.44}
$$

We derive from (2.30) that

$$
\|\theta\|_{L^\infty}^2\leq\int_0^L\theta^2dx+2\int_0^L|\theta\theta_x|dx\leq C_1\int_0^L\left(\theta_x^2+\theta^2\right)dx\leq C_1+C_1\int_0^L\theta_x^2dx.
\tag{2.45}
$$

Therefore, using (1.24), (2.30), (2.36), (2.37), and (2.44)-(2.45), we conclude

$$
\int_0^L\frac{(ru)_x^2}{\tau}dx+\int_0^t\int_0^L u_t^2dx\,ds\leq C_1+C_1\int_0^t\|\theta\|_{L^\infty}^2\int_0^L\tau_x^2dx\,ds+C_1\int_0^t\|u\|_{L^\infty}^2\int_0^L\frac{(ru)_x^2}{\tau}(x,s)dx\,ds
\tag{2.46}
$$

which, by applying the Gronwall inequality, implies

$$
\int_0^L u_x^2(x,t)dx+\int_0^t\int_0^L u_t^2(x,s)dx\,ds\leq C_1.
\tag{2.47}
$$

By (1.17), we have

$$
\begin{aligned}
\int_0^t\int_0^L u_{xx}^2(x,s)dxds&\leq C_1\int_0^t\int_0^L\left(u_t^2+u_x^2+u^2+\tau_x^2u_x^2+\theta^2\tau_x^2+\theta_x^2+v^4+f_1^2\right)(x,s)dx\,ds\\
&\leq C_1+C_1\int_0^t\left(\|\theta\|_{L^\infty}^2+\|u_x\|_{L^\infty}^2\right)\int_0^L\tau_x^2dx\,ds\\
&\leq C_1+C_1\int_0^t\left(1+\|\theta_x\|^2+\|u_x\|\|u_{xx}\|+\|u_x\|\right)(s)ds\\
&\leq C_1+\frac{1}{2}\int_0^t\int_0^L u_{xx}^2(x,s)dx\,ds.
\end{aligned}
\tag{2.48}
$$

Therefore,

$$\int_0^t \|u_{xx}(s)\|^2 ds \le C_1 \tag{2.49}$$

which, along with (2.47), gives (2.41).

Analogously, multiplying (1.18) by v_t, integrating the result over Q_T, and using assumptions (1.24) and Lemmas 2.1–2.6, we deduce

$$\begin{aligned}
&\frac{\mu}{2}\int_0^L \frac{(rv)_x^2}{\tau}dx + \int_0^t \|v_t(s)\|^2 ds \\
&\quad = \frac{\mu}{2}\int_0^L \frac{(r_0v_0)_x^2}{\tau_0}dx + \frac{\mu}{2}\int_0^t\int_0^L ru\left[\frac{(rv)_x^2}{\tau}\right]_x dx\,ds \\
&\qquad - \mu\int_0^t\int_0^L \left[\frac{(rv)_x}{\tau}\right]_x uv dx\,ds - \int_0^t\int_0^L \frac{uvv_t}{r}dx\,ds + \int_0^t\int_0^L f_2 v_t dx\,ds \\
&\quad \le C_1 + \frac{1}{4}\int_0^t \|v_t(s)\|^2 ds - \frac{1}{2}\int_0^t\int_0^L u\frac{(rv)_x}{\tau}\left[v_t + \frac{uv}{r} - f_2\right]dx\,ds \\
&\qquad - \int_0^t\int_0^L uv\left[\frac{v_t - f_2}{r} + \frac{uv}{r^2}\right]dx\,ds \\
&\quad \le C_1 + \frac{1}{2}\int_0^t \|v_t(s)\|^2 ds + \int_0^t \|u\|_{L^\infty}^2 \int_0^L \frac{(rv)_x^2}{\tau}dx\,ds + C_1\int_0^t\int_0^L \left(u^2v^2 + f_2^2\right)dx\,ds \\
&\quad \le C_1 + \frac{1}{2}\int_0^t \|v_t(s)\|^2 ds + \int_0^t \|u\|_{L^\infty}^2 \int_0^L \left(v^2 + \frac{(rv)_x^2}{\tau}\right)dx\,ds.
\end{aligned} \tag{2.50}$$

In view of (2.36), we apply Gronwall's inequality to (2.50) to obtain

$$\int_0^L \frac{(rv)_x^2}{\tau}dx + \int_0^t\int_0^L v_t^2(x,s)dx\,ds \le C_1. \tag{2.51}$$

By (1.18) and (2.51), we easily deduce

$$\int_0^t\int_0^L v_{xx}^2(x,s)dx\,ds \le C_1 \tag{2.52}$$

which, along with (2.51) and Lemmas 2.1–2.4, implies (2.42). The proof of (2.43) iS similar to that of (2.41) and (2.42). The proof is now complete. □

Lemma 2.8. *Under the assumptions of Theorem 1.1, one has, for any $T > 0$,*

$$\int_0^L \theta_x^2(x,t)dx + \int_0^t\int_0^L \left(\theta_t^2 + \theta_{xx}^2\right)(x,s)dx\,ds \le C_1, \quad \forall t \in [0,T]. \tag{2.53}$$

Proof. Multiplying (1.20) by θ_t over Q_T, we have

$$\begin{aligned}
&\frac{\kappa}{2}\int_0^L \frac{r^2\theta_x^2}{\tau}dx + C_V\int_0^t\int_0^L \theta_t^2(x,s)dx\,ds \\
&\quad= \frac{\kappa}{2}\int_0^L \frac{r_0^2\theta_{0x}^2}{\tau_0}dx + \frac{\kappa}{2}\int_0^t\int_0^L \left(\frac{r^2}{\tau}\right)_t \theta_x^2 dx\,ds \\
&\qquad+ \int_0^t\int_0^L \bigg[\frac{1}{\tau}\left[\nu(ru)_x - \gamma\theta\right](ru)_x + \mu\frac{[(rv)_x]^2}{\tau} + \mu\frac{r^2 w_x^2}{\tau} - 2\mu\left(u^2+v^2\right)_x \\
&\qquad\qquad + g(r(x,t),t)\bigg]\theta_t(x,s)dx\,ds.
\end{aligned} \tag{2.54}$$

Using Lemmas 2.1–2.7, the Cauchy-Schwarz inequality, and the interpolation inequality, we have

$$\begin{aligned}
&\left|\frac{\kappa}{2}\int_0^t\int_0^L \left(\frac{r^2}{\tau}\right)_t \theta_x^2 dx\,ds\right| \\
&\quad\le C_1\int_0^t \left(\|u\|_{L^\infty} + \|(ru)_x\|_{L^\infty}\right)\int_0^L \theta_x^2 dx\,ds \\
&\quad\le C_1\int_0^t \left(\|u_x\| + \|(ru)_x\|^{1/2}\|(ru)_{xx}\|^{1/2}\right)\int_0^L \theta_x^2 dx\,ds \\
&\quad\le C_1\int_0^t \left(\|u_x\| + \|(ru)_{xx}\|^2 + \|(ru)_x\|^{2/3}\right)\int_0^L \theta_x^2 dx\,ds \\
&\quad\le C_1\int_0^t\int_0^L \theta_x^2 dx\,ds + C_1\int_0^t \left(\int_0^L \left(u^2\tau_x^2 + u_x^2 + u^2 + u_{xx}^2\right)dx\right)\int_0^L \theta_x^2 dx\,ds \\
&\quad\le C_1 + C_1\int_0^t \left(\|u\|_{L^\infty}^2\|\tau_x\|^2 + \|u\|^2 + \|u_{xx}\|^2\right)\int_0^L \theta_x^2 dx\,ds \\
&\quad\le C_1 + C_1\int_0^t \left(\|u\|_{L^\infty}^2 + \|u_{xx}\|^2\right)\int_0^L \theta_x^2 dx\,ds,
\end{aligned} \tag{2.55}$$

$$\begin{aligned}
&\left|\int_0^t\int_0^L \left[\frac{1}{\tau}\left[\nu(ru)_x - \gamma\theta\right](ru)_x + \mu\frac{[(rv)_x]^2}{\tau} + \mu\frac{r^2w_x^2}{\tau} - 2\mu\left(u^2+v^2\right)_x + g(r,t)\right]\theta_t dx\,ds\right| \\
&\quad\le \frac{1}{4}\int_0^t \|\theta_t\|^2 ds + C_1\int_0^t\int_0^L \left[(ru)_x^4 + (rv)_x^4 + \theta^2(ru)_x^2 + w_x^4 + u^2u_x^2 + v^2v_x^2 + g^2\right]dx\,ds
\end{aligned}$$

$$\leq \frac{1}{4}\int_0^t \|\theta_t\|^2 ds + C_1 \int_0^t \Big[\|(ru)_x\|_{L^\infty}^2 \left(\|(ru)_x\|^2 + \|\theta\|^2 \right) + \|(rv)_x\|_{L^\infty}^2 \|(rv)_x\|^2$$
$$+ \|w_x\|_{L^\infty}^2 \|w_x\|^2 \Big](s) ds + C_1 \int_0^t \int_0^L \left(u^2 u_x^2 + v^2 v_x^2 + g^2 \right)(x,s) dx\, ds. \tag{2.56}$$

Similarly to (2.55), by virtue of (1.25), (2.30), and (2.41)–(2.43), we arrive at

$$\left| \int_0^t \int_0^L \left[\frac{1}{\tau} [\nu (ru)_x - \gamma\theta] (ru)_x + \mu \frac{[(rv)_x]^2}{\tau} + \mu \frac{r^2 w_x^2}{\tau} - 2\mu \left(u^2 + v^2 \right)_x + g(r,t) \right] \theta_t dx\, ds \right|$$
$$\leq C_1 + \frac{1}{4}\int_0^t \|\theta_t\|^2 ds + C_1 \int_0^t \left(\|(ru)_x\|_{L^\infty}^2 + \|(rv)_x\|_{L^\infty}^2 + \|w_x\|_{L^\infty}^2 \right)$$
$$\leq C_1 + \frac{1}{4}\int_0^t \|\theta_t\|^2 ds + C_1 \int_0^t \left(\|\tau_x\|^2 + \|u_x\|_{H^1}^2 + \|v_x\|_{H^1}^2 + \|w_x\|_{H^1}^2 \right)$$
$$\leq C_1 + \frac{1}{4}\int_0^t \|\theta_t\|^2 ds. \tag{2.57}$$

Combining (2.54)–(2.57), we conclude

$$\|\theta_x\|^2 + \int_0^t \|\theta_t(s)\|^2 ds \leq C_1 + C_1 \int_0^t \left(\|u\|_{L^\infty}^2 + \|u_{xx}\|^2 \right) \|\theta_x\|^2 ds. \tag{2.58}$$

In view of (2.36) and (2.41), we apply Gronwall's inequality to (2.58) to obtain

$$\|\theta_x(t)\|^2 + \int_0^t \|\theta_t(s)\|^2 ds \leq C_1, \quad \forall t \in [0,T]. \tag{2.59}$$

Similarly to proof of (2.41), by Lemmas 2.1–2.7, (1.20), (1.25), and (2.59), we obtain

$$\int_0^t \|\theta_{xx}(s)\|^2 ds \leq C_1 \tag{2.60}$$

which, together with (2.59), implies (2.53). The proof is complete. □

Proof of Theorem 1.1. By Lemmas 2.1–2.8, we complete the proof of Theorem 1.1. □

Acknowledgments

The work is in part supported by Doctoral Foundation of North China University of Water Sources and Electric Power (no. 201087) and the Natural Science Foundation of Henan Province of China (no. 112300410040).

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