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Research Article

On a Constructive Approach for Derivative-Dependent Singular Boundary Value Problems

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We present a constructive approach to establish existence and uniqueness of solution of singular boundary value problem $-(p(x)y'(x))' = q(x)f(x, y, py')$ for $0 < x \leq b$, $y(0) = a$, $\alpha_1 y(b) + \beta_1 p(b) y'(b) = \gamma_1$. Here $p(x) > 0$ on $(0, b)$ allowing $p(0) = 0$. Further $q(x)$ may be allowed to have integrable discontinuity at $x = 0$, so the problem may be doubly singular.

1. Introduction

Consider the following singular boundary value problem:

$$My \equiv -(p(x)y'(x))' = q(x)f(x, y(x), p(x)y'(x)), \quad 0 < x \leq b, \tag{1.1}$$

$$y(0) = a, \quad \alpha_1 y(b) + \beta_1 p(b) y'(b) = \gamma_1, \tag{1.2}$$

where $\alpha_1 > 0$, $\beta_1 \geq 0$, γ_1 is any finite constant. We assume that $p(x)$ and $q(x)$ satisfy the following conditions:

(A1) $p(x) > 0$ in $(0, b]$, $p \in C[0, b] \cap C^1(0, b)$ and $\int_0^b (dt/p(t)) < \infty$,

(A2) $q(x) > 0$ in $(0, b]$ and $\int_0^b q(t)dt < \infty$.

In this work we establish existence and uniqueness of solution of the singular problem (1.1)-(1.2). We use monotone iterative method. For this we require an appropriate iterative scheme. In this regard Cherpion et al. [1] suggest the following approximation scheme:

$$-\alpha''_{n+1} + \tilde{k}(x)\alpha'_{n+1} + \tilde{l}(x)\alpha_{n+1} = f(x, \alpha_n, \alpha'_n) + \tilde{k}(x)\alpha'_n + \tilde{l}(x)\alpha_n,$$
$$\alpha_{n+1}(0) = \alpha_{n+1}(1) = 0, \tag{1.3}$$

for the regular boundary value problem $-y'' = f(x, y, y'), y(0) = y(1) = 0$. They also suggest that (1.3) with $\tilde{l}(x) = 0$ or with constant $\tilde{k}$ and $\tilde{l}$ does not work for the Dirichlet boundary condition.

Thus, for our problem we consider the following iterative scheme:

$$Ly_{n+1} = F(x, y_n, py'_n), \quad 0 < x \le b,$$
$$y_{n+1}(0) = 0, \quad \alpha_1 y_{n+1}(b) + \beta_1 p(b) y'_{n+1}(b) = \gamma_1, \tag{1.4}$$

where

$$Ly = -(p(x)y'(x))' - \mu(x)q(x)p(x)y'(x) - \lambda k(x)q(x)y(x),$$
$$F(x, y, py') = q(x)f(x, y, py') - \mu(x)q(x)p(x)y'(x) - \lambda k(x)q(x)y(x). \tag{1.5}$$

We assume that $k(x)$ and $\mu(x)$ satisfy the following conditions.

(A3) $k(x) \in C[0, b]$, $\exists\ m_k, M_k \in R$ such that $0 < m_k \le k(x) \le M_k$.

(A4) $\mu(x) \in L^1_q(0, b)$, that is, $\int_0^b q(x)\mu(x)dx < \infty$.

(A5) Further, we assume that the homogeneous boundary value problem $Ly = 0$, $y(0) = 0$, and $\alpha_1 y(b) + \beta_1 p(b) y'(b) = 0$ has only trivial solution.

For $\int_0^b (dt/p(t)) < \infty$, several researchers ([2–5]) suggest to reduce the singular problem to regular problem by a change of variable. But in [6] it is suggested that a direct consideration of singular problems provide better results.

Further, the following sign restrictions are imposed by several researchers ([4, 5, 7–9]):

(i) $yf(x, y, 0) > 0$, $|y| > M_0$, where M_0 is a constant ([7–9]) or

(ii) $f(x, M_1, 0) \ge 0 \ge f(x, -M_2, 0)$, $M_1, M_2 \ge 0$ ([4, 5]).

But such sign restrictions are quite restrictive as the simple differential equation $y'' = 2$ fails to satisfy the sign restrictions (i) and (ii) ([7]).

In the present work we consider the singular boundary value problem (SBVP) directly and do not impose any sign restriction. Further, we do not assume that the point $x = 0$ is a regular singular point as assumed in [6, 9]. We use iterative scheme (1.4) to establish existence and uniqueness of the solution of the problem. With the help of nonnegativity of Green's function, existence uniqueness of linear singular boundary value problem (LSBVP) is established.

This paper is divided in four sections. In Section 2, we show that singular point $x = 0$ is of limit circle type; hence, spectrum is pure point spectrum with complete set of orthonormal eigenfunctions. In Section 3, we prove the existence uniqueness of the corresponding LSBVP. Finally, in Section 4, using the results of Section 3, we establish the existence uniqueness of solutions of the nonlinear problem (1.1)-(1.2).

2. Eigenfunction Expansion

Let $L_s^2(0,b)$ be a Hilbert space with weight $s(x)=k(x)q(x)$ and the inner product defined as

$$\langle u,v\rangle=\int_0^b s(t)u(t)\overline{v(t)}dt. \tag{2.1}$$

Conditions on p, q, μ, and k guarantee that the singular point $x=0$ is of limit circle type (Weyl's Theorem, [10, page 438]. Thus, we have pure point spectrum ([11, page 125]. Next, from the Lagrange's identity, it is easy to see that all the eigenvalues are real, simple, positive, and eigenfunctions are orthogonal. Let the eigenvalues be $0<\lambda_0<\lambda_1<\lambda_2<\cdots$, and let the corresponding eigenfunctions be $\psi_0,\psi_1,\psi_2,\ldots$, respectively. Next, we transform $Ly=0$ by changing variable $z=\sqrt{g(x)}y$, where $g(x)=e^{\int_0^x \mu(t)q(t)dt}$, to

$$-(p(x)z'(x))'+r(x)z(x)=\lambda s(x)z(x),\quad 0<x\leq b, \tag{2.2}$$

where $r(x)=(1/2)\{\mu(x)q(x)p(x)\}'+(1/4)\{\mu(x)q(x)\}^2p(x)$. Now following the analysis of Theorem 2.7, (i), (ii) and Theorem 2.17 of [11] for the operator $(1/s)(r+M)$, where M is defined by (1.1), the following results can be established.

Theorem 2.1. *Let $f(x)$ be the primitive of an absolutely continuous function, and let*

$$\begin{gathered}\frac{1}{s}(r+M)f\in L_s^2(0,b),\\ f(b)\sin\alpha-p(b)f'(b)\cos\alpha=0,\quad \text{where } \alpha \text{ is real},\\ \lim_{x\to 0}p(x)W_x[f,\psi]=0,\end{gathered} \tag{2.3}$$

for every nonreal λ, where $\psi(x,\lambda)\in L_s^2(0,b)$ is a solution of (2.2) *and $W[f,\psi]$ is the wronskian of f and ψ. Then*

$$f(x)=\sum_{n=0}^{\infty}c_n\psi_n(x),\quad (0\leq x\leq b), \tag{2.4}$$

being the series absolutely and uniformly convergent on $[0,b]$.

Theorem 2.2. *Let $f\in L_s^2(0,b)$. Then*

$$\int_0^b s(x)\{f(x)\}^2dx=\sum_{n=0}^{\infty}c_n^2. \tag{2.5}$$

Theorem 2.3. *Let $f\in L_s^2(0,b)$, and let $\Phi(x,\lambda)$ be the solution of*

$$-(p(x)z'(x))'+r(x)z(x)-\lambda s(x)z(x)=s(x)f(x),\quad 0<x\leq b, \tag{2.6}$$

satisfying $\alpha_{11}z(b) + \beta_1 p(b)z'(b) = 0$, *where* $\alpha_{11} = \alpha_1 - (1/2)\beta_1\mu(b)p(b)q(b)$. *Then for* λ *not equal to any of the values of* λ_n, *one has*

$$\Phi(x,\lambda) = \sum_{n=0}^{\infty} \frac{c_n \varphi_n}{\lambda - \lambda_n}, \tag{2.7}$$

where the series is absolutely convergent.

Remark 2.4. Since $\|\cdot\|_s$ on $L_s^2(0,b)$ is equivalent to $\|\cdot\|_q$ on $L_q^2(0,b)$, we can apply Theorems 2.1–2.3 in $L_q^2(0,b)$ also.

3. Linear Singular Sturm-Liouville's Problem

In this section we apply Theorem 1.1 of [12] to the differential operator L and generate two linearly independent solutions of the linear problem. Further, with the help of these solutions, Green's function is constructed, and nonnegativity of the Green's function is established.

Theorem 3.1. *Let* $p(x)$, $q(x)$, $k(x)$, *and* $\mu(x)$ *satisfy (A1), (A2), (A3), and (A4), respectively. Then the initial value problems (IVPs)*

$$Ly = 0,\ 0 < x \le b,\ y(0) = a_0, \quad \lim_{x\to 0^+} p(x)y'(x) = b_0, \tag{3.1}$$

$$Ly = 0,\ 0 < x \le b,\ y(b) = c_0, \quad p(b)y'(b) = d_0 \tag{3.2}$$

have a solution in $L_s^2(0,b)$ *or equivalently in* $L_q^2(0,b)$ *(Remark 2.4).*

3.1. Green's Function

Green's function $G(x,t,\lambda)$ for the differential operator L can be defined as

$$G(x,t,\lambda) = \frac{1}{p(x)W(\varphi,\phi)} \begin{cases} -\varphi(t)\phi(x), & \text{if } 0 < t \le x, \\ -\varphi(x)\phi(t), & \text{if } x \le t \le b, \end{cases} \tag{3.3}$$

where $\phi = S\{\alpha_1 y_1(x) + \beta_1 y_2(x)\}$, $S = 1/\sqrt{\alpha_1^2 + \beta_1^2}$, $y_1(b) = 0$, $p(b)y_1'(b) = -1$, $y_2(b) = 1$, $p(b)y_2'(b) = 0$, and φ is a nontrivial solution of IVP (3.1) with $a_0 = 0$, $b_0 = 1$. From (A5), it is easy to conclude that $p(x)W(\varphi,\phi)|_{x=b} \neq 0$; thus, ϕ and φ are linearly independent.

Next we establish nonnegativity of Green's function. For this we need to establish following results.

Lemma 3.2. *If* $y(x)$ *satisfies* $Ly(x) = q(x)f(x) \ge 0$, *for* $0 < x \le b$, $y(0) = 0$, *and* $\alpha_1 y(b) + \beta_1 p(b)y'(b) = \gamma_1 \ge 0$, *where* $p(x)$, $q(x)$, $\mu(x)$, *and* $k(x)$ *satisfy (A1), (A2), (A3), and (A4), respectively, then* $y(x) \ge 0$ *provided that* $\lambda \le 0$.

Proof. We divide the proof in two cases as follows.

Case i. When $\lambda < 0$. On contrary assume that there exists a point $c \in (0, b)$ such that $y(c) < 0$. Then from the continuity of the solutions there exists a point $d \in (0, b)$ such that $y(d) < 0$, $y'(0) = 0$, and $y''(d) \geq 0$. Now at the point d, we have

$$-p(d)y''(d) - p'(d)y'(d) - \mu(d)q(d)p(d)y'(d) - \lambda k(d)q(d)y(d) < 0, \tag{3.4}$$

which is a contradiction. Hence, $y(x) \geq 0$ when $\lambda < 0$.

Case ii. When $\lambda = 0$. Using the same notations as in the previous case, we have $y'(x) > 0$ for $x > d$ and $y'(x) < 0$ for $x < d$.

Now, we consider the interval $[d, x_0] \subset (0, b)$ where $y'(d) = 0$, $y < 0$ in $[d, x_0]$, and $y' > 0$ in $(d, x_0]$. Then

$$P = \int_d^{x_0} \left\{ p(t)g(t)\left(y'(t)\right)^2 - s(t)g(t)f(t)y(t) \right\} dt > 0. \tag{3.5}$$

Integrating the first term by parts, we get

$$P = p(x_0)g(x_0)y'(x_0)y(x_0) < 0, \tag{3.6}$$

which is again a contradiction. Thus, $y(x) \geq 0$ when $\lambda \leq 0$. □

Lemma 3.3. *Consider the following differential equation:*

$$Ly(x) = 0, \quad 0 < x \leq b, \tag{3.7}$$

where $p(x)$, $q(x)$, $\mu(x)$, and $k(x)$ satisfy (A1), (A2), (A3), and (A4), respectively, with the boundary conditions:

$$y(0) = 0, \quad \alpha_1 p(b) + \beta_1 p(b) y'(b) = \gamma_1. \tag{3.8}$$

Then LSBVP (3.7)-(3.8) *has a unique solution given by*

$$y(x) = \frac{\gamma_1 \varphi(x)}{\alpha_1 \varphi(b) + \beta_1 p(b) \varphi'(b)}, \tag{3.9}$$

provided that λ is none of the eigenvalues of the corresponding eigenvalue problem and φ satisfies (3.1). *Moreover, $y(x) \geq 0$ if $\gamma_1 \geq 0$ and $0 < \lambda < \lambda_0$, where λ_0 is the first positive zero of $\alpha_1 \varphi(b, \lambda) + \beta_1 p(b) \varphi'(b, \lambda)$.*

Proof. From Theorem 3.1, it is easy to see that the unique solution of (3.7)-(3.8) can be written as

$$y(x) = \frac{\gamma_1 \psi(x)}{\alpha_1 \psi(b) + \beta_1 p(b)\psi'(b)}, \tag{3.10}$$

provided that $\alpha_1\psi(b,\lambda) + \beta_1 p(b)\psi'(b,\lambda) \neq 0$; that is, λ is none of the eigenvalue of the corresponding eigenvalue problem (Section 2). Since $\psi(0) = 0$, $\lim_{x\to 0^+} p(x)\psi'(x) = 1$, and $\psi(x,\lambda)$ does not change sign for $0 < \lambda < \lambda_0$, we get that $y(x) \geq 0$ for $0 < \lambda < \lambda_0$, provided that $\gamma_1 \geq 0$. □

Lemma 3.4. *For the linear differential operator associated with*

$$Ly(x) = q(x)f(x), \quad 0 < x \leq b, \tag{3.11}$$

$$y(0) = 0, \alpha_1 y(b) + \beta_1 p(b)y'(b) = \gamma_1, \tag{3.12}$$

with $f \in L_q^2(0,b)$, the generalized Green's function for the corresponding homogeneous boundary value problem is given by

$$G(x,t,\lambda) = \sum_{n=0}^{\infty} \frac{\psi(x,\lambda_n)\psi(t,\lambda_n)}{\lambda_n - \lambda}, \tag{3.13}$$

where $\psi(x,\lambda_i)$ are the normalized eigenfunctions corresponding to the eigenvalue λ_i. $G(x,t,\lambda)$ satisfies the homogeneous boundary condition provided that $\lambda \neq \lambda_0, \lambda_1, \ldots$. Solution of the non-homogeneous LSBVP (3.11)-(3.12) *is*

$$y(x) = \frac{\gamma_1 \psi(x,\lambda)}{\alpha_1 \psi(b) + \beta_1 p(b)\psi'(b)} + \int_0^b q(t)f(t)G(x,t,\lambda)dt. \tag{3.14}$$

The series on the right is absolutely convergent.

Proof. The solution $y(x)$ of (3.11)-(3.12) can be written as sum of the solution of (3.11) with boundary condition $y(0) = 0, \alpha_1 y(b) + \beta_1 p(b)y'(b) = 0$ and solution of (3.7) with boundary condition $y(0) = 0$, $\alpha_1 y(b) + \beta_1 p(b)y'(b) = \gamma_1$,

$$y(x) = \frac{\gamma_1 \psi(x,\lambda)}{\alpha_1 \psi(b,\lambda) + \beta_1 p(b)\psi'(b,\lambda)} + \int_0^b q(t)f(t)G(x,t,\lambda)dt, \tag{3.15}$$

where $G(x,t,\lambda)$ is Green's function defined by (3.3). Now using the analysis of ([11, page 38], it is easy to show that the generalized Green's function is given by

$$G(x,t,\lambda) = \sum_{n=0}^{\infty} \frac{\psi(x,\lambda_n)\psi(t,\lambda_n)}{\lambda_n - \lambda}, \tag{3.16}$$

and absolute convergence of the series on the right-hand side follows from the analysis of ([11, page 38]. This completes the proof. □

Lemma 3.5. *If* $f \in L_q^2(0,b)$, $\gamma_1 \geq 0$ *and* $f \geq 0$, *then solution of* (3.11)-(3.12) *is nonnegative provided that* $0 < \lambda < \lambda_0$.

Proof. We first show that $G(x,t) \geq 0$ for all $0 \leq x, t \leq b$ if $0 < \lambda < \lambda_0$. Fixing t, $G(x,t)$ satisfies $LG(x,t) = 0, 0 < x \leq t^-$, where $' \equiv \partial/\partial x$. Since $G(0,t) = 0$, $\alpha_1 G(t^-,t) + \beta_1 p(t^-)G'(t^-,t) \geq 0$ for $0 < \lambda < \lambda_0$, from Lemma 3.3 $G(x,t) \geq 0$ for $0 \leq x \leq t^-$, provided that $0 < \lambda < \lambda_0$. By the symmetry, continuity, and $G(t,t) \geq 0$ for $0 < \lambda < \lambda_0$, it follows that $G(x,t) \geq 0$ for $0 \leq x, t \leq b$, provided that $0 < \lambda < \lambda_0$. The result follows. □

Corollary 3.6. *If* $y(x)$ *satisfies* $Ly(x) = q(x)f(x) \geq 0$ *for* $0 < x \leq b$ *and* $y(0) = 0$, $\alpha_1 y(b) + \beta_1 p(b) y'(b) = \gamma_1 \geq 0$, *then* $y(x) \geq 0$, *provided that* $\lambda < \lambda_0$.

Proof. The proof follows from Lemmas 3.2 and 3.5. □

Corollary 3.7. *The solution of the boundary value problem in Lemma 3.4 is unique.*

Proof. The proof follows from Corollary 3.6. □

4. Nonlinear Sturm-Liouville's Problem

In this section, we establish the existence uniqueness of solution of the nonlinear problem (1.1)-(1.2). For this, first we prove that the sequences generated by (1.4) are monotonic sequences (Lemmas 4.2 and 4.3). Then using the bound for py' (Lemmas 4.9 and 4.10), the uniform convergence of these sequences to a solution of the nonlinear problem is established (Theorem 4.11). Finally the uniqueness of the solution is established in Theorem 4.14.

The nonlinear boundary value problem

$$\begin{aligned} -\left(p(x)y'(x)\right)' &= q(x)f\left(x, y, py'\right), \quad 0 < x \leq b, \\ y(0) = a, &\quad \alpha_1 y(b) + \beta_1 p(b) y'(b) = \gamma_2 \end{aligned} \tag{4.1}$$

can be transformed to

$$\begin{aligned} -\left(p(x)u'(x)\right)' &= q(x)f\left(x, u+a, pu'\right), \quad 0 < x \leq b, \\ u(0) = 0, &\quad \alpha_1 u(b) + \beta_1 p(b) u'(b) = \gamma_2 - a\alpha_1, \end{aligned} \tag{4.2}$$

with $u = y - a$. Further, the functions $f(x, u+a, pu')$ and $f(x, y, py')$ satisfy the same Lipschitz condition, so we may work with the boundary value problem

$$\begin{aligned} -\left(p(x)y'(x)\right)' &= q(x)f\left(x, y, py'\right), \quad 0 < x \leq b, \\ y(0) = 0, &\quad \alpha_1 y(b) + \beta_1 p(b) y'(b) = \gamma_1. \end{aligned} \tag{4.3}$$

Next, we define upper solution $u_0(x)$ and lower solution $v_0(x)$ such that $u_0 \geq v_0$, which work as initial iterates for our constructive approach.

Definition 4.1. A function $u_0(x) \in C[0,b] \cap C^2(0,b]$ is an upper solution if

$$-\left(p(x)u_0'(x)\right)' \geq q(x)f\left(x, u_0, pu_0'\right), \quad 0 < x \leq b,$$
$$u_0(0) = 0, \quad \alpha_1 u_0(b) + \beta_1 p(b)u_0'(b) \geq \gamma_1, \tag{4.4}$$

and a function $v_0(x) \in C[0,b] \cap C^2(0,b]$ is a lower solution if

$$-\left(p(x)v_0'(x)\right)' \leq q(x)f\left(x, v_0, pv_0'\right), \quad 0 < x \leq b,$$
$$v_0(0) = 0, \quad \alpha_1 v_0(b) + \beta_1 p(b)v_0'(b) \leq \gamma_1. \tag{4.5}$$

Lemma 4.2. *If $\lambda < 0$, $\lambda k(x) \leq K_1$, $|\mu(x)| \leq L_1$, $Ly \geq 0$ for $0 < x \leq b$, $y(0) = 0$, and $\alpha_1 y(b) + \beta_1 p(b) y'(b) \geq 0$, then*

$$(K_1 - \lambda k(x))y - \left(\mu(x) + L_1\left(\operatorname{sign} y'\right)\right)py' \geq 0, \quad 0 < x \leq b, \tag{4.6}$$

provided that

$$1 + \lambda \int_0^b \frac{dt}{p(t)g(t)} \int_0^b s(t)g(t)dt - \sup\left(\frac{pg}{b}\right) \int_0^b \frac{dt}{p(t)g(t)} > 0, \quad 0 < x \leq b,$$
$$(K_1 - \lambda k(x)) - \left(L_1 - \left|\mu(x)\right|\right)\Phi(p, q, s, g) \geq 0, \quad 0 < x \leq b, \tag{4.7}$$

hold. Here,

$$\Phi(p, q, s, g) = \sup\left(\frac{pg}{b}\right) I(p, q, s, g) - \lambda \int_0^b s(t)g(t)dt, \tag{4.8}$$

$$I(p, q, s, g) = \left(1 + \lambda \int_0^b \frac{dt}{p(t)g(t)} \int_0^b s(t)g(t)dt - \sup\left(\frac{pg}{b}\right) \int_0^b \frac{dt}{p(t)g(t)}\right)^{-1}. \tag{4.9}$$

Proof. The solution of the equation $Ly = qf \geq 0$, $y(0) = 0$, and $\alpha_1 y(b) + \beta_1 p(b)y'(b) = \gamma_1 \geq 0$ is given by (3.15) where $G(x, t, \lambda)$ is defined by (3.3). Substituting $y(x)$ from (3.15) into (4.6), it is easy to see that we require the following inequalities in order to complete the proof

$$(K_1 - \lambda k(x))\psi - \left(\mu(x) + L_1\left(\operatorname{sign} y'\right)\right)p\psi' \geq 0, \quad 0 < x \leq b, \tag{4.10}$$

$$(K_1 - \lambda k(x))\phi - \left(\mu(x) + L_1\left(\operatorname{sign} y'\right)\right)p\phi' \geq 0, \quad 0 < x \leq b. \tag{4.11}$$

Here ψ satisfies the IVP at $x = 0$; that is, $L\psi = 0$, $\psi(0) = 0$ and $\lim_{x \to 0^+} p(x)\psi(x) = 1$, and ϕ satisfies the IVP at $x = b$; that is, $L\phi = 0$, $\phi(b) = \beta_1$ and $p(b)\phi'(b) = -\alpha_1$. The solutions ψ and ϕ cannot have either point of maxima (at the point of maxima the $L\psi = 0$ or $L\phi = 0$ will be contradicted) or point of minima (since to have minima, maxima is bound to occur). So, finally we have $\psi'(x) \geq 0$ and $\phi'(x) \leq 0$ on $[0,b]$. As $|\mu(x)| \leq L_1$, it is enough to prove the following inequalities:

$$(K_1 - \lambda k(x))\psi - (L_1 - |\mu(x)|)p\psi' \geq 0, \quad 0 < x \leq b, \tag{4.12}$$

$$(K_1 - \lambda k(x))\phi + (L_1 - |\mu(x)|)p\phi' \geq 0, \quad 0 < x \leq b. \tag{4.13}$$

Next, we prove the inequality (4.12), and the other one can be proved in a similar manner.

By the mean value theorem, there exist $\tau \in (0,b)$ such that $\psi(b) = b\psi'(\tau)$. Writing $L\psi = 0$ in the following form:

$$-(p(x)g(x)\psi(x)')' - \lambda s(x)g(x)\psi(x) = 0, \tag{4.14}$$

and integrating it first from τ to x and then x to b, we get that

$$p(x)\psi'(x) \leq \psi(x)\Phi(p,q,s,g) \quad \text{on } 0 < x \leq b. \tag{4.15}$$

Here $\Phi(p,q,s,g)$ is given by (4.8). Now, the result follows from (4.7), (4.12), and (4.15). □

Lemma 4.3. *If* $0 < \lambda < \lambda_0$, $\lambda k(x) \leq K_1$, $|\mu(x)| \leq L_1$, $Ly \geq 0$ *for* $0 < x \leq b$, $y(0) = 0$, *and* $\alpha_1 y(b) + \beta_1 p(b) y'(b) \geq 0$, *then*

$$(K_1 - \lambda k(x))y - (\mu(x) + L_1(\operatorname{sign} y'))py' \geq 0, \quad 0 < x \leq b, \tag{4.16}$$

provided that

$$\begin{gathered} 1 - \sup\left(\frac{pg}{b}\right)\int_0^b \frac{dt}{pg} > 0, \quad 0 < x \leq b, \\ (K_1 - \lambda k(x))\left(1 - \sup\left(\frac{pg}{b}\right)\int_0^b \frac{dt}{pg}\right) - (L_1 - |\mu(x)|)\sup\left(\frac{pg}{b}\right) \geq 0, \quad 0 < x \leq b \end{gathered} \tag{4.17}$$

or

$$\begin{gathered} 1 - \lambda\int_0^b s(t)g(t)dt \int_0^b \frac{dt}{p(t)g(t)} > 0, \\ (K_1 - \lambda k(x))\left(1 - \lambda\int_0^b sgdt\int_0^b \frac{dt}{pg}\right) - (L_1 - |\mu(x)|)\lambda\int_0^b sgdt \geq 0, \quad 0 < x \leq b, \end{gathered} \tag{4.18}$$

hold.

Proof. Similar to the proof of Lemma 4.2, we need to establish two inequalities (4.10)-(4.11) for $0 < \lambda < \lambda_0$. Here ψ and ϕ cannot have the point of minima in $(0,b)$, because at the point of minima, the differential equation $L\psi = 0$ or $L\phi = 0$ will be contradicted. So either $\psi'(x) \geq 0$

and $\phi'(x) \le 0$ or $\psi(x)$ and $\phi(x)$ both are concave downwards on $[0,b]$. Thus we can divide the proof in two cases:

Case i. ψ and ϕ both are concave downwards.

We prove for ψ as similar analysis provides result for ϕ. Let the point of maxima be $x_0 \in (0,b)$. Then $\psi'(x) > 0$ for $x < x_0$ and $\psi'(x) < 0$ for $x > x_0$. On both sides of x_0, the inequality (4.10) will be reduced into the following two inequalities:

$$\begin{aligned} &(K_1 - \lambda k(x))\psi - (L_1 + \mu(x))p\psi' \ge 0, \quad \psi'(x) \ge 0, \\ &(K_1 - \lambda k(x))\psi + (L_1 - \mu(x))p\psi' \ge 0, \quad \psi'(x) \le 0. \end{aligned} \tag{4.19}$$

For a point x on the left side of x_0, we integrate (4.14) from x to x_0 twice and get

$$p(x)\psi'(x) \le \frac{\lambda\psi(x)\int_0^b s(t)g(t)dt}{1 - \lambda\int_0^b s(t)g(t)dt\int_0^b (1/p(t)g(t))dt}. \tag{4.20}$$

Similarly for any point x on the right side of x_0, we get

$$-p(x)\psi'(x) \le \frac{\lambda\psi(x)\int_0^b s(t)g(t)dt}{1 - \lambda\int_0^b s(t)g(t)dt\int_0^b (1/p(t)g(t))dt}. \tag{4.21}$$

Now, the result follows from the fact that $|\mu(x)| \le L_1$ and from (4.18) to (4.21).

Case ii. When $\psi'(x) \ge 0$ and $\phi'(x) \le 0$.

To establish the inequality (4.16), we require to establish the inequalities (4.12)-(4.13). We prove the inequality (4.12), and the proof for (4.13) is quite similar. By the mean value theorem, there exists $\tau \in (0,b)$ such that $\psi(b) = b\psi'(\tau)$. Integrating (4.14) first from τ to x and then from x to b, we get

$$p(x)\psi'(x) \le \frac{\psi(x)\sup(pg/b)}{1 - \sup(pg/b)\int_0^b (dt/pg)}, \tag{4.22}$$

and the result follows from (4.12), (4.17), and (4.22). This completes the proof. □

Lemma 4.4. *If u_n is an upper solution of* (4.3) *and u_{n+1} is defined by* (1.4)–(1.5), *then $u_n \ge u_{n+1}$ for $\lambda < \lambda_0$.*

Proof. Let $w = u_n - u_{n+1}$. w satisfies $Lw = -(pu_n')' - qf(x, u_n, py_n') \ge 0, 0 < x \le b, w(0) = 0, \alpha_1 w(b) + \beta_1 p(b)w'(b) \ge 0$, and the result follows from Corollary 3.6. □

Proposition 4.5. *Let u_0 be an upper solution of* (4.3), *and let $f(x, y, py')$ satisfy the following*

(F1) *$f(x, y, py')$ is continuous on*

$$D_0 = \{(x, y, py') : [0,b] \times [v_0, u_0] \times \mathbb{R}\}, \tag{4.23}$$

(F2) $\exists\ K_1 \equiv K_1(D_0)$ *such that for all* $(x, y, v), (x, w, v) \in D_0$,

$$K_1(y - w) \leq f(x, y, v) - f(x, w, v) \quad \text{for } y \geq w, \text{ and} \tag{4.24}$$

(F3) $\exists\ 0 \leq L_1 \equiv L_1(D_0)$ *such that for all* $(x, y, v_1), (x, y, v_2) \in D_0$,

$$\left| f(x, y, v_1) - f(x, y, v_2) \right| \leq L_1 |v_1 - v_2|, \tag{4.25}$$

and (4.7), (4.17), *or* (4.18) *hold. Then the functions* u_n *defined by* (1.4)–(1.5) *are such that, for all* $n \in \mathbb{N}$, *(i)* u_n *is upper solution of* (4.3) *and (ii)* $u_n \geq u_{n+1}$.

Proof. Since u_0 is an upper solution from Lemma 4.4, we have $u_0 \geq u_1$. Assume that the claim is true for $n - 1$; that is, u_{n-1} is an upper solution and $u_{n-1} \geq u_n$.

Let $w = u_{n-1} - u_n$. We have

$$-(pu_n')' - qf(x, u_n, pu_n') \geq q\left\{ (K_1 - \lambda k(x))w - \left(\mu(x) + L_1(\operatorname{sign} w')\right)pw' \right\}, \tag{4.26}$$

and from Lemmas 4.2 and 4.3 we get $-(pu_n')' - qf(x, u_n, pu_n') \geq 0, 0 < x \leq b$.

Thus, u_n is an upper solution for all $n \in \mathbb{N}$. From Lemma 4.4 we have $u_n \geq u_{n+1}$. Hence, the result follows.

Similar results (Lemma 4.6, Proposition 4.7) follow for lower solutions. □

Lemma 4.6. *If* v_n *is a lower solution of* (4.3) *and* v_{n+1} *is defined by* (1.4)–(1.5) *then* $v_n \leq v_{n+1}$ *for* $\lambda < \lambda_0$.

Proposition 4.7. *Let* v_0 *be a lower solution of* (4.3), *let* $f(x, y, py')$ *satisfies (F1)–(F3) and* (4.7), (4.17), *or* (4.18) *hold. Then the functions* v_n *defined by* (1.4)–(1.5) *are such that, for all* $n \in \mathbb{N}$, *(i)* v_n *is lower solution of* (4.3) *and (ii)* $v_n \leq v_{n+1}$.

Proposition 4.8. *If* $f(x, y, py')$ *satisfies*

(F4) $f(x, u_0, pu_0') - f(x, v_0, pv_0') - \mu(x)(pu_0' - pv_0') - \lambda k(x)(u_0 - v_0) \geq 0$ *for* $0 < x \leq b$ *such that* $\lambda k(x) \leq K_1$ *and* $|\mu(x)| \leq L_1$,

and in addition let (F1)–(F3) and (4.7), (4.17), *or* (4.18) *hold, then for all* $n \in \mathbb{N}$ *the functions* u_n *and* v_n *defined by* (1.4)–(1.5) *satisfy* $v_n \leq u_n$.

Proof. Let $w_i = u_i - v_i$, then w_i satisfies $Lw_i = q(x)h_{i-1}$ for all $i \in N$ such that

$$h_i(x) = f(x, u_i, pu_i') - f(x, v_i, pv_i') - \mu(x)(pu_i' - pv_i') - \lambda k(x)(u_i - v_i), \quad 0 < x \leq b. \tag{4.27}$$

Since $v_0 \leq u_0$, we prove that $v_1 \leq u_1$. Since w_1 is solution of $Lw_1 = qh_0 \geq 0$, $w_1(0) = 0$ and $\alpha_1 w_1(0) + \beta_1 p(b) w_1'(b) = 0$, from Corollary 3.6 we have $w_1 \geq 0$. Let $n \geq 2$, let $h_{n-2} \geq 0$, and

$u_{n-1} \geq v_{n-1}$, then we prove that $h_{n-1} \geq 0$ and $u_n \geq v_n$. Consider

$$\begin{aligned} h_{n-1} &= f(x, u_{n-1}, pu'_{n-1}) - f(x, v_{n-1}, pv'_{n-1}) - \mu(x)pw'_{n-1} - \lambda k(x)w_{n-1} \\ &\geq (K_1 - \lambda k(x))w_{n-1} - (\mu + L(\operatorname{sign} w'_{n-1}))pw'_{n-1}. \end{aligned} \tag{4.28}$$

Since w_{n-1} is a solution of $Lw_{n-1} = qh_{n-2} \geq 0$, $w_{n-1}(0) = 0$, and $\alpha_1 w_{n-1}(0) + \beta_1 p(b)w'_{n-1}(b) = 0$; hence, from Lemmas 4.2 and 4.3, we have $h_{n-1} \geq 0$. Thus, from Corollary 3.6 on $Lw_n = qh_{n-1} \geq 0$, $w_n(0) = 0$ and $\alpha_1 w_n(0) + \beta_1 p(b)w'_n(b) = 0$, we have $w_n \geq 0$, that is, $u_n \geq v_n$. This completes the proof. □

Lemma 4.9. *If $f(x, y, py')$ satisfies*

(F5) *for all $(x, y, v) \in D_0$, $|f(x, y, v)| \leq \varphi(|v|)$ where $\varphi : [0, \infty) \to (0, \infty)$ is continuous and satisfies*

$$\int_0^b q(s)ds < \int_{l_0}^{\infty} \frac{ds}{\varphi(s)}, \tag{4.29}$$

where $l_0 = \sup_{[0,b]}|p(x)u_0(x)/b|$, then there exists $R_0 > 0$ such that any solution of

$$-(py')' \geq qf(x, y, py'), \quad 0 < x \leq b, \tag{4.30}$$

$$y(0) = 0, \quad \alpha_1 y(b) + \beta_1 p(b)y'(b) \geq \gamma_1, \tag{4.31}$$

with $y \in [v_0, u_0]$ for all $x \in [0, b]$, satisfies $\|py'\|_\infty < R_0$.

Proof. We divide the proof in three parts.

Case i. If solution is not monotone throughout the interval, then we consider the interval $(x_0, x] \subset (0, b)$ such that $y'(x_0) = 0$ and $y'(x) > 0$ for $x > x_0$. Integrating (4.30) from x_0 to x we get

$$\int_0^{py'} \frac{ds}{\varphi(s)} \leq \int_0^b q(s)ds. \tag{4.32}$$

From (F5) we can choose $R_0 > 0$ such that

$$\int_0^{py'} \frac{ds}{\varphi(s)} \leq \int_0^b q(s)ds \leq \int_{l_0}^{R_0} \frac{ds}{\varphi(s)} \leq \int_0^{R_0} \frac{ds}{\varphi(s)}, \tag{4.33}$$

which gives

$$p(x)y'(x) \leq R_0. \tag{4.34}$$

Now we consider the case in which $y'(x) < 0$ for $x < x_0$, $y'(x_0) = 0$, and proceeding in the similar way we get

$$-p(x)y'(x) \leq R_0, \tag{4.35}$$

and the result follows.

Case ii. If y is monotonically increasing in $(0,b)$, that is, $y' > 0$ in $(0,b)$, then by the mean value theorem there exists a point $\tau \in (0,b)$ such that

$$y'(\tau) = \frac{y(b) - y(0)}{b} \leq \left|\frac{u_0}{b}\right|. \tag{4.36}$$

Now, integrating (4.30) from τ to x, we get

$$\int_0^{py'} \frac{ds}{\varphi(s)} \leq \int_0^b q(t)dt + \int_0^{l_0} \frac{ds}{\varphi(s)}. \tag{4.37}$$

Further, from (F5) we can choose R_0 such that

$$\int_0^{py'} \frac{ds}{\varphi(s)} \leq \int_0^b q(s)ds + \int_0^{l_0} \frac{ds}{\varphi(s)} \leq \int_0^{R_0} \frac{ds}{\varphi(s)}. \tag{4.38}$$

which gives $p(x)y'(x) \leq R_0$.

Case iii. If y is monotonically decreasing in $(0,b)$; that is, $y' < 0$ in $(0,b)$, then argument similar to Case ii yields

$$\int_0^{-py'} \frac{ds}{\varphi(s)} \leq \int_0^b q(s)ds + \int_0^{l_0} \frac{ds}{\varphi(s)} \leq \int_0^{R_0} \frac{ds}{\varphi(s)}, \tag{4.39}$$

and we get

$$-p(x)y'(x) \leq R_0, \tag{4.40}$$

and the result follows. □

Lemma 4.10. *If $f(x,y,py')$ satisfies (F5), then there exists $R_0 > 0$ such that any solution of*

$$\begin{gathered} -\left(py'\right)' \leq qf\left(x,y,py'\right), \quad 0 < x \leq b, \\ y(0) = 0, \quad \alpha_1 y(b) + \beta_1 p(b)y'(b) \leq \gamma_1, \end{gathered} \tag{4.41}$$

with $y \in [v_0, u_0]$ for all $x \in [0,b]$, satisfies $\|py'\|_\infty < R_0$.

Proof. Proof follows from the analysis of Lemma 4.9. □

Theorem 4.11. *Let u_0 and v_0 be upper and lower solutions. Let $f(x,y,py')$ satisfy (F1) to (F5) and* (4.7), (4.17), *or* (4.18) *hold. Then, boundary value problem* (4.3) *has at least one solution in the region D_0. If $\lambda < \lambda_0$ is chosen such that $\lambda k(x) \le K_1$ and $|\mu(x)| \le L_1$, where λ_0 is the first positive eigenvalue of the corresponding eigenvalue problem, then the sequences $\{u_n\}$ and $\{v_n\}$ generated by* (1.4)–(1.5) *with initial iterate u_0 and v_0 converge monotonically and uniformly towards solutions $\tilde{u}(x)$ and $\tilde{v}(x)$ of* (4.3). *Any solution $z(x)$ in D_0 must satisfy $\tilde{v}(x) \le z(x) \le \tilde{u}(x)$.*

Proof. From Lemmas 4.2–4.10, Propositions 4.5–4.8, and we get two monotonic sequences $\{u_n\}$ and $\{v_n\}$ which are bounded by u_0 and v_0; respectively, and by Dini's Theorem their uniform convergence is assured. Let $\{u_n\}$ and $\{v_n\}$ converge uniformly to $\tilde{u}$ and $\tilde{v}$.

By Lemmas 4.9 and 4.10, it is easy to see that the sequences $\{pu'_n\}$ and $\{pv'_n\}$ are uniformly bounded. Now, from

$$\left|py'_n(x_1) - py'_n(x_2)\right| = \left|\int_{x_1}^{x_2} (py'_n)' dt\right|, \tag{4.42}$$

uniform convergence of $\{y_n\}$, properties (A1)–(A4), and (F1), it is easy to prove that $\{py'_n\}$ is equicontinuous. Hence, by Arzela-Ascoli's Theorem there exist a uniform convergent subsequence $\{py'_{n_k}\}$ of $\{py'_n\}$. Since limit is unique so original sequence will also converge uniformly to the same limit say py'. It is easy to see that, if $y_n \to \tilde{y}$, then $py'_n \to p\tilde{y}'$. Therefore sequences $\{pu'_n\}$ and $\{pv'_n\}$ converge uniformly to $p\tilde{u}'$ and $p\tilde{v}'$, respectively.

Let $G(x,t)$ be Green's function for the linear boundary value problem $Ly_n = 0$, $y_n(0) = 0$, and $\alpha_1 y_n(b) + \beta_1 p(b) y'_n(b) = 0$. Then solution of (1.4)–(1.5) can be written as

$$y_n = Cx^2 + \int_0^b G(x,t)\{F(t, y_{n-1}, py'_{n-1}) + H(t)\}dt, \tag{4.43}$$

where $H(t) = 2C(tp'(t) + p(t)) + 2Ct\mu(t)q(t)p(t) + \lambda Ct^2 s(t)$ and $C = \gamma_1/(\alpha_1 b^2 + 2\beta_1 bp(b))$.

Now, uniform convergence of $\{y_n\}$, $\{py'_n\}$ and continuity of $f(x,y,py')$ imply that $\{(1/q)F(x,y_n,py'_n)\}$ converges uniformly in $[0,b]$. Hence, $\{(1/q)F(x,y_n,py'_n)\}$ converges in the sense of mean in $L_q^2(0,b)$. Taking limit as $n \to \infty$ and using Lemma 2.4 ([11, page 27]), we get

$$y = Cx^2 + \int_0^b G(x,t)\{F(t,y,py') + H(t)\}dt, \tag{4.44}$$

which is the solution of the boundary value problem (4.3).

Any solution $z(x)$ in D_0 plays the role of $u_0(x)$. Hence $z(x) \ge \tilde{v}(x)$. Similarly, $z(x) \le \tilde{u}(x)$. This completes the proof. □

Remark 4.12. The case when $\lambda = 0$ corresponds to the case when $f(x,y,py') \equiv f(x,py')$. In such cases the boundary value problem (4.3) can be reduced to two initial value problems $-z' = qf(x,z)$, $z(0) = -\alpha_1$ and $py' = z$, $y(0) = \beta_1$. From the assumptions on $p(x)$, $q(x)$, and $f(x,y,py')$, one can easily conclude existence uniqueness of solutions of the nonlinear boundary value problem.

Remark 4.13. Suppose, in addition to the hypothesis of Theorem 4.11, $|f(x, y, py')| \leq N_0$ in D_0. Then lower solution v_0 and upper solution u_0 may be obtained as solution of the following linear boundary value problems:

$$\begin{aligned}
&-(pv_0')' + N_0 q(x) = 0, \quad 0 < x \leq b, \\
&v_0(0) = 0, \quad \alpha_1 v_0(b) + \beta_1 p(b) v_0'(b) = \gamma_1, \\
&-(pu_0')' - N_0 q(x) = 0, \quad 0 < x \leq b, \\
&u_0(0) = 0, \quad \alpha_1 u_0(b) + \beta_1 p(b) u_0'(b) = \gamma_1.
\end{aligned} \tag{4.45}$$

Theorem 4.14. *Suppose that $f(x, y, py')$ satisfies (F1), (F3), and $\exists$ constants $K_1(D_0) < \lambda_0$ such that*

$$K_1(u - v) \leq f(x, u, py') - f(x, v, py'). \tag{4.46}$$

Then the boundary value problem (4.3) *has unique solution.*

Proof. Let u and v be two solutions of (4.3), then we get

$$\begin{aligned}
&-(p(u - v)')' = q(x)\{f(x, u, pu') - f(x, v, pv')\}, \quad 0 < x \leq b, \\
&\text{or} -(p(u - v)')' + L_1 q(x)(pu' - pv') - K_1 q(x)(u - v) \geq 0, \quad 0 < x \leq b, \\
&(u - v)(0) = 0, \quad \alpha_1 (u - v)(b) + \beta_1 p(b)(u - v)'(b) = 0.
\end{aligned} \tag{4.47}$$

Since $K_1 < \lambda_0$, from Corollary 3.6 we get $u - v \geq 0$ or $u \geq v$. Similarly $v \geq u$. Therefore, the solution of (4.3) is unique. □

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