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Research Article

An Optimal Double Inequality between Seiffert and Geometric Means

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For $\alpha,\beta\in(0,1/2)$ we prove that the double inequality $G(\alpha a+(1-\alpha)b,\alpha b+(1-\alpha)a)<P(a,b)<G(\beta a+(1-\beta)b,\beta b+(1-\beta)a)$ holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leq(1-\sqrt{1-4/\pi^2})/2$ and $\beta\geq(3-\sqrt{3})/6$. Here, $G(a,b)$ and $P(a,b)$ denote the geometric and Seiffert means of two positive numbers a and b, respectively.

1. Introduction

For $a,\ b>0$ with $a\neq b$ the Seiffert mean $P(a,b)$ was introduced by Seiffert [1] as follows:

$$P(a,b)=\frac{a-b}{4\arctan\sqrt{a/b}-\pi}. \tag{1.1}$$

Recently, the bivariate mean values have been the subject of intensive research. In particular, many remarkable inequalities for the Seiffert mean can be found in the literature [1–9].

Let $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(a-b)/(\log a-\log b)$, $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b)=(a+b)/2$, $C(a,b)=(a^2+b^2)/(a+b)$, and $M_p(a,b)=[(a^p+b^p)/2]^{1/p}(p\neq 0)$ and $M_0(a,b)=\sqrt{ab}$ be the harmonic, geometric, logarithmic, identric, arithmetic, contraharmonic, and pth power means of two different positive numbers a and b, respectively. Then it is well known that

$$\min\{a,b\} < H(a,b) = M_{-1}(a,b) < G(a,b) = M_0(a,b) < L(a,b) \\ < I(a,b) < A(a,b) = M_1(a,b) < C(a,b) < \max\{a,b\} \tag{1.2}$$

for all $a,b > 0$ with $a \neq b$.

For all $a,b > 0$ with $a \neq b$, Seiffert [1] established that $L(a,b) < P(a,b) < I(a,b)$; Jagers [4] proved that $M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b)$ and $M_{2/3}(a,b)$ is the best possible upper power mean bound for the Seiffert mean $P(a,b)$; Seiffert [7] established that $P(a,b) > A(a,b)G(a,b)/L(a,b)$ and $P(a,b) > 2A(a,b)/\pi$; Sándor [6] presented that $(A(a,b) + G(a,b))/2 < P(a,b) < \sqrt{A(a,b)(A(a,b)+G(a,b))/2}$ and $\sqrt[3]{A^2(a,b)G(a,b)} < P(a,b) < (G(a,b) + 2A(a,b))/3$; Hästö [3] proved that $P(a,b) > M_{\log 2/\log \pi}(a,b)$ and $M_{\log 2/\log \pi}(a,b)$ is the best possible lower power mean bound for the Seiffert mean $P(a,b)$.

Very recently, Wang and Chu [8] found the greatest value α and the least value β such that the double inequality $A^{\alpha}(a,b)H^{1-\alpha}(a,b) < P(a,b) < A^{\beta}(a,b)H^{1-\beta}(a,b)$ holds for $a,b > 0$ with $a \neq b$; For any $\alpha \in (0,1)$, Chu et al. [10] presented the best possible bounds for $P^{\alpha}(a,b)G^{1-\alpha}(a,b)$ in terms of the power mean; In [2] the authors proved that the double inequality $\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b)$ holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/\pi$ and $\beta \geq 5/6$; Liu and Meng [5] proved that the inequalities

$$\alpha_1 C(a,b) + (1-\alpha_1)G(a,b) < P(a,b) < \beta_1 C(a,b) + (1-\beta_1)G(a,b), \\ \alpha_2 C(a,b) + (1-\alpha_2)H(a,b) < P(a,b) < \beta_2 C(a,b) + (1-\beta_2)H(a,b) \tag{1.3}$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/9$, $\beta_1 \geq 1/\pi$, $\alpha_2 \leq 1/\pi$ and $\beta_2 \geq 5/12$.

For fixed $a,b > 0$ with $a \neq b$ and $x \in [0,1/2]$, let

$$g(x) = G(xa + (1-x)b, xb + (1-x)a). \tag{1.4}$$

Then it is not difficult to verify that $g(x)$ is continuous and strictly increasing in $[0,1/2]$. Note that $g(0) = G(a,b) < P(a,b)$ and $g(1/2) = A(a,b) > P(a,b)$. Therefore, it is natural to ask what are the greatest value α and least value β in $(0,1/2)$ such that the double inequality $G(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < P(a,b) < G(\beta a + (1-\beta)b, \beta b + (1-\beta)a)$ holds for all $a,b > 0$ with $a \neq b$. The main purpose of this paper is to answer these questions. Our main result is the following Theorem 1.1.

Theorem 1.1. *If $\alpha,\beta \in (0,1/2)$, then the double inequality*

$$G(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < P(a,b) < G(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \tag{1.5}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq (1 - \sqrt{1 - 4/\pi^2})/2$ and $\beta \geq (3 - \sqrt{3})/6$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $\lambda = (1 - \sqrt{1-4/\pi^2})/2$ and $\mu = (3-\sqrt{3})/6$. We first prove that inequalities

$$P(a,b) > G(\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a), \tag{2.1}$$

$$P(a,b) < G(\mu a + (1-\mu)b, \mu b + (1-\mu)a) \tag{2.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Without loss of generality, we assume that $a > b$. Let $t = \sqrt{a/b} > 1$ and $p \in (0, 1/2)$, then from (1.1) one has

$$\begin{aligned} &\log G(pa + (1-p)b, pb + (1-p)a) - \log P(a,b) \\ &\quad = \frac{1}{2}\log\left[\left(pt^2 + (1-p)\right)\left((1-p)t^2 + p\right)\right] - \log\frac{t^2-1}{4\arctan t - \pi}. \end{aligned} \tag{2.3}$$

Let

$$f(t) = \frac{1}{2}\log\left[\left(pt^2 + (1-p)\right)\left((1-p)t^2 + p\right)\right] - \log\frac{t^2-1}{4\arctan t - \pi}, \tag{2.4}$$

then simple computations lead to

$$f(1) = 0, \tag{2.5}$$

$$\lim_{t\to+\infty} f(t) = \frac{1}{2}\log\left[p(1-p)\right] + \log\pi, \tag{2.6}$$

$$f'(t) = \frac{t(t^2+1)}{(t^2-1)(4\arctan t - \pi)(pt^2 + (1-p))((1-p)t^2+p)} f_1(t), \tag{2.7}$$

where

$$f_1(t) = \frac{4(t^2-1)(pt^2+1-p)\left[(1-p)t^2+p\right]}{t(t^2+1)^2} - 4\arctan t + \pi. \tag{2.8}$$

$$f_1(1) = 0, \tag{2.9}$$

$$\lim_{t\to+\infty} f_1(t) = +\infty, \tag{2.10}$$

$$f_1'(t) = \frac{4f_2(t^2)}{t^2(t^2+1)^4}, \tag{2.11}$$

where $f_2(t) = p(1-p)t^5 - (3p-2)(3p-1)t^4 + 2(5p^2-5p+1)t^3 + 2(5p^2-5p+1)t^2 - (3p-2)(3p-1)t + p(1-p)$.

Note that

$$f_2(1) = 0, \tag{2.12}$$

$$\lim_{t\to+\infty} f_2(t) = +\infty, \tag{2.13}$$

$$\begin{aligned} f_2'(t) &= 5p(1-p)t^4 - 4(3p-2)(3p-1)t^3 + 6\left(5p^2-5p+1\right)t^2 \\ &\quad + 4\left(5p^2-5p+1\right)t - (3p-2)(3p-1), \end{aligned} \tag{2.14}$$

$$f_2'(1) = 0, \tag{2.15}$$

$$\lim_{t\to+\infty} f_2'(t) = +\infty, \tag{2.16}$$

$$f_2''(t) = 20p(1-p)t^3 - 12(3p-2)(3p-1)t^2 + 12\left(5p^2-5p+1\right)t + 4\left(5p^2-5p+1\right), \tag{2.17}$$

$$f_2''(t) = -8\left(6p^2-6p+1\right), \tag{2.18}$$

$$\lim_{t\to+\infty} f_2''(t) = +\infty, \tag{2.19}$$

$$f_3'''(t) = 60p(1-p)t^2 - 24(3p-2)(3p-1)t + 12\left(5p^2-5p+1\right), \tag{2.20}$$

$$f_2'''(1) = -36\left(6p^2-6p+1\right), \tag{2.21}$$

$$\lim_{t\to+\infty} f_2'''(t) = +\infty, \tag{2.22}$$

$$f_2^{(4)}(t) = 120p(1-p)t - 24(3p-2)(3p-1), \tag{2.23}$$

$$f_2^{(4)}(1) = -48\left(7p^2-7p+1\right), \tag{2.24}$$

$$\lim_{t\to+\infty} f_2^{(4)}(t) = +\infty. \tag{2.25}$$

We divide the proof into two cases.

Case 1 ($p = \lambda = (1-\sqrt{1-4/\pi^2})/2$). Then (2.6), (2.18), (2.21), and (2.24) become

$$\lim_{t\to+\infty} f(t) = 0, \tag{2.26}$$

$$f_2''(1) = -\frac{8(\pi^2-6)}{\pi^2} < 0, \tag{2.27}$$

$$f_2'''(1) = -\frac{36(\pi^2-6)}{\pi^2} < 0, \tag{2.28}$$

$$f_2^{(4)}(1) = -\frac{48(\pi^2-7)}{\pi^2} < 0. \tag{2.29}$$

From (2.23) we clearly see that $f_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$, then (2.25) and inequality (2.29) lead to the conclusion that there exists $\lambda_1 > 1$ such that $f_2^{(4)}(t) < 0$ for $t \in [1, \lambda_1)$ and $f_2^{(4)}(t) > 0$ for $t \in (\lambda_1, +\infty)$. Thus, $f_2'''(t)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in $[\lambda_1, +\infty)$.

It follows from (2.22) and inequality (2.28) together with the piecewise monotonicity of $f_2'''(t)$ that there exists $\lambda_2 > \lambda_1 > 1$ such that $f_2''(t)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$. Then (2.19) and inequality (2.27) lead to the conclusion that there exists $\lambda_3 > \lambda_2 > 1$ such that $f_2'(t)$ is strictly decreasing in $[1, \lambda_3]$ and strictly increasing in $[\lambda_3, +\infty)$.

From (2.15) and (2.16) together with the piecewise monotonicity of $f_2'(t)$ we know that there exists $\lambda_4 > \lambda_3 > 1$ such that $f_2(t)$ is strictly decreasing in $[1, \lambda_4]$ and strictly increasing in $[\lambda_4, +\infty)$. Then (2.11)–(2.13) lead to the conclusion that there exists $\lambda_5 > \lambda_4 > 1$ such that $f_1(t)$ is strictly decreasing in $[1, \sqrt{\lambda_5}]$ and strictly increasing in $[\sqrt{\lambda_5}, +\infty)$.

It follows from (2.7)–(2.10) and the piecewise monotonicity of $f_1(t)$ that there exists $\lambda_6 > \sqrt{\lambda_5} > 1$ such that $f(t)$ is strictly decreasing in $[1, \lambda_6]$ and strictly increasing in $[\lambda_6, +\infty)$.

Therefore, inequality (2.1) follows from (2.3)–(2.5) and the piecewise monotonicity of $f(t)$.

Case 2 ($p = \mu = (3 - \sqrt{3})/6$). Then (2.18), (2.21) and (2.24) become

$$f_2''(1) = 0, \tag{2.30}$$

$$f_2'''(1) = 0, \tag{2.31}$$

$$f_2^{(4)}(1) = 8 > 0. \tag{2.32}$$

From (2.23) we clearly see that $f_2^{(4)}(t)$ is strictly increasing in $[1, +\infty)$, then inequality (2.32) leads to the conclusion that $f_2'''(t)$ is strictly increasing in $[1, +\infty)$.

Therefore, inequality (2.2) follows from (2.3)–(2.5), (2.7)–(2.9), (2.11), (2.12), (2.15), and inequalities (2.30) and (2.31) together with the monotonicity of $f_2'''(t)$.

Next, we prove that $\lambda = (1 - \sqrt{1 - 4/\pi^2})/2$ is the best possible parameter such that inequality (2.1) holds for all $a, b > 0$ with $a \neq b$. In fact, if $(1 - \sqrt{1 - 4/\pi^2})/2 = \lambda < p < 1/2$, then (2.6) leads to

$$\lim_{t \to +\infty} f(t) = \frac{1}{2} \log \left[p(1 - p)\right] + \log \pi > 0. \tag{2.33}$$

Inequality (2.33) implies that there exists $T = T(p) > 1$ such that

$$f(t) > 0 \tag{2.34}$$

for $t \in (T, +\infty)$.

It follows from (2.3) and (2.4) together with inequality (2.34) that $P(a, b) < G(pa + (1 - p)b, pb + (1 - p)a)$ for $a/b \in (T^2, +\infty)$.

Finally, we prove that $\mu = (3-\sqrt{3})/6$ is the best possible parameter such that inequality (2.2) holds for all $a, b > 0$ with $a \neq b$. In fact, if $0 < p < \mu = (3-\sqrt{3})/6$, then from (2.18) we get $f_2''(1) < 0$, which implies that there exists $\delta > 0$ such that

$$f_2''(t) < 0 \tag{2.35}$$

for $t \in [1, 1+\delta)$.

Therefore, $P(a,b) > G(pa + (1-p)b, pb + (1-p)a)$ for $a/b \in (1, (1+\delta)^2)$ follows from (2.3)–(2.5), (2.7)–(2.9), (2.11), (2.12), and (2.15) together with inequality (2.35). □

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