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*Research Article*

# Existence and Global Attractivity of Positive Periodic Solutions for a Two-Species Competitive System with Stage Structure and Impulse

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A class of nonautonomous two-species competitive system with stage structure and impulse is considered. By using the continuation theorem of coincidence degree theory, we derive a set of easily verifiable sufficient conditions that guarantee the existence of at least a positive periodic solution, and, by constructing a suitable Lyapunov functional, the uniqueness and global attractivity of the positive periodic solution are presented. Finally, an illustrative example is given to demonstrate the correctness of the obtained results.

## 1. Introduction

In recent years, with the increasing applications of theory of differential equations in mathematical ecology, various mathematical models have been proposed in the study of population [1–25]. But most of the previous results focused on the dynamical behaviors (including the stability, attractiveness, persistence, and periodicity of solution) of the systems which have fixed parameters and there is no impulse. Considering that harvest of many populations are not continuous and the periodic environmental factor, it is reasonable to investigate the systems with periodic coefficients and impulse. Impulsive differential systems display a combination of characteristics of both the continuous-time and discrete-time systems [26–30]. In 2006, Chen [1] studied the following non-autonomous almost periodic competitive two-species model with stage structure in one species:

$$\begin{aligned}
\dot{x}_1(t) &= -a_1(t)x_1(t) + b_1(t)x_2(t), \\
\dot{x}_2(t) &= a_2(t)x_1(t) - b_2(t)x_2(t) - c(t)x_2^2(t) - \beta_1(t)x_2(t)x_2(t)x_3(t), \\
\dot{x}_3(t) &= x_3(t)\left[d(t) - e(t)x_3(t) - \beta_2(t)x_2(t)\right],
\end{aligned} \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ are immature and mature population densities of one species, respectively; $x_3(t)$ represents the population density of another species; $a_i(t),\ b_i(t),\ \beta_i(t)\ (i = 1,2)$, $c(t),\ d(t),\ e(t)$ are all continuous, almost periodic functions. The competition is between $x_2(t)$ and $x_3(t)$. Chen [1] obtained sufficient conditions for the existence of a unique, globally attractive, strictly positive almost periodic solution for system (1.1).

Considering that the harvest is an annual harvest pulse, to describe a system more accurately, we should consider the impulsive differential equation. Motivated by this point of view, we revised system (1.1) into the following form:

$$\begin{aligned}
\dot{x}_1(t) &= -a_1(t)x_1(t) + b_1(t)x_2(t), \quad t \neq t_k, \\
\dot{x}_2(t) &= a_2(t)x_2(t) - b_2(t)x_2(t) - c(t)x_2^2(t) - \beta_1(t)x_2(t)x_3(t), \quad t \neq t_k, \\
\dot{x}_3(t) &= x_3(t)\left[d(t) - e(t)x_3(t) - \beta_2(t)x_2(t)\right], \quad t \neq t_k, \\
\Delta x_i(t_k) &= x_i\left(t_k^+\right) - x_i\left(t_k^-\right) = -\gamma_{ik}x_i(tk), \quad i = 1,2,3,\ k = 1,2,\dots,q,
\end{aligned} \tag{1.2}$$

where $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ are the impulses at moments $t_k$ and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$; $x_1(t)$ and $x_2(t)$ are immature and mature population densities of one species, respectively, and $x_3(t)$ represents the population density of another species. The competition is between $x_2(t)$ and $x_3(t)$.

Throughout the paper, we always assume the following.

($H_1$) $a_i(t)$, $b_i(t)$, $\beta_i(t)$ $(i = 1,2)$, $c(t)$, $d(t)$, $e(t)$ are all continuous $\omega$ periodic; that is, $a_i(t+\omega) = a_i(t)$, $b_i(t+\omega) = b_i(t)$, $\beta_i(t+\omega) = \beta_i(t)$ $(i = 1,2)$, $c(t+\omega) = c(t)$, $d(t+\omega) = d(t)$, $e(t+\omega) = e(t)$ for any $t \in R$.

($H_2$) $a_i(t)$, $b_i(t)$, $\beta_i(t)$ $(i = 1,2)$, $c(t)$, $d(t)$, $e(t)$ are all positive.

($H_3$) $0 < \gamma_{ik} < 1$, $i = 1,2,3$ for all $k \in N$, and there exists a positive integer $q$ such that $t_{k+q} = t_k + \omega$, $\gamma_{i(k+q)} = \gamma_{ik}$, $i = 1,2,3$.

The principle object of this paper is by using Mawhin's continuation theorem of coincidence degree theory and by constructing the Lyapunov functions to investigate the stability and existence of periodic solutions of (1.2). To the best of my knowledge, it is the first time to deal with the existence and stability of periodic solutions of (1.2).

The organization of the paper is as follows. In Section 2, we introduce some notations and definitions and state some preliminary results needed in later sections. We then establish, in Section 3, some simple criteria for the existence of positive periodic solutions of system (1.2) by using the continuation theorem of coincidence degree theory proposed by Gaines and Mawhin [31]. The uniqueness and global attractivity of the positive periodic solution are presented in Section 4. In Section 5, an illustrative example is given to demonstrate the correctness of the obtained results.

## 2. Preliminaries

We will introduce some notations and definitions and state some preliminary results. Consider the impulsive system

$$\begin{gathered}\dot{x}(t) = f(t,x), \quad t \neq t_k, \; k = 1,2,\ldots, \\ \Delta x(t)|_{t=t_k} = I_k\left(x\left(t_k^-\right)\right),\end{gathered} \tag{2.1}$$

where $x \in R^n$, $f : R \times R^n \to R^n$ is continuous and $f(t+\omega, x) = f(t,x)$; $I_k : R^n \to R^n$ are continuous, and there exists a positive integer $q$ such that $t_{k+q} = t_k + \omega, I_{k+q}(x) = I_k(x)$ with $t_k \in R, t_{k+1} > t_k, \lim_{k\to\infty} = \infty, \Delta x(t)|_{t=t_k} = x(t_k^+) - x(t_k^-)$. For $t_k \neq 0$ $(k = 1,2,\ldots)$, $[0,\omega] \cap \{t_k\} = \{t_1, t_2, \ldots, t_q\}$. As we know, $\{t_k\}$ are called points of jump.

Let us recall some definitions. For the Canchy problem,

$$\begin{gathered}\dot{x}(t) = f(t,x), \quad t \in [0,\omega], \; t \neq t_k, \\ \Delta x(t)|_{t=t_k} = I_k\left(x\left(t_k^-\right)\right), \quad x(0) = x_0.\end{gathered} \tag{2.2}$$

*Definition 2.1.* A map $x : [0, \omega] \to R^n$ is said to be a solution of (2.2), if it satisfied the following conditions:

(i) $x(t)$ is a piecewise continuous map with first-class discontinuity points in $t_k \cap [0, \omega]$, and at each discontinuity point it is continuous on the left;

(ii) $x(t)$ satisfies (2.2).

*Definition 2.2.* A map $x : [0, \omega] \to R^n$ is said to be an $\omega$ periodic solution of (2.1), if

(i) $x(t)$ satisfies (i) and (ii) of Definition 2.1 in the interval $[0,\omega]$ and

(ii) $x(t)$ satisfies $x(t+\omega-0) = x(t-0)$, $t \in R$.

Obviously, if $x(t)$ is a solution of (2.2) defined on $[0,\omega]$, such that $x(0) = x(\omega)$, then, by the periodicity of (2.2) in $t$, the function $x^*(t)$ defined by

$$x^*(t) = \begin{cases} x(t - j\omega), & t \in \left[j\omega, (j+1)\omega\right] \setminus \{t_k\}, \\ x^*(t) \text{ is left continuous at } t = t_k \end{cases} \tag{2.3}$$

is a $\omega$ periodic solution of (2.1).

For system (1.2), seeking the periodic solutions is equivalent to seeking solutions of the following boundary value problem:

$$\begin{gathered}\dot{x}_1(t) = -a_1(t)x_1(t) + b_1(t)x_2(t), \quad t \neq t_k, \; t \in [0,\omega], \; k = 1,2,\ldots,q, \\ \dot{x}_2(t) = a_2(t)x_2(t) - b_2(t)x_2(t) - c(t)x_2^2(t) - \beta_1(t)x_2(t)x_3(t), \quad t \neq t_k, \; t \in [0,\omega], \; k = 1,2,\ldots,q, \\ \dot{x}_3(t) = x_3(t)\left[d(t) - e(t)x_3(t) - \beta_2(t)x_2(t)\right], \quad t \neq t_k, \; t \in [0,\omega], \; k = 1,2,\ldots,q, \\ \Delta x_i(t_k) = x_i\left(t_k^+\right) - x_i\left(t_k^-\right) = -\gamma_{ik}x_i(t_k), \quad i = 1,2,3, \; x_i(0) = x_i(\omega), \; k = 1,2,\ldots,q.\end{gathered} \tag{2.4}$$

## 3. Existence of Positive Periodic Solutions

In this section, based on the Mawhin's continuation theorem, we shall study the existence of at least one periodic solution of (1.1). To do so, we shall make some preparations.

Let $X, Y$ be normed vector spaces; $L : \operatorname{Dom} L \subset X \to Y$ is a linear mapping; $N : X \to Y$ is a continuous mapping. The mapping $L$ will be called a Fredholm mapping of index zero if $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in $Y$. If $L$ is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, it follows that $L \mid \operatorname{Dom} L \cap \operatorname{Ker} P : (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by $K_P$. If $\Omega$ is an open bounded subset of $X$, the mapping $N$ will be called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Now we introduce Mawhin's continuation theorem [31] as follows.

**Lemma 3.1** (Continuation Theorem [31]). *Let $L$ be a Fredholm mapping of index zero, and let $N$ be $L$-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$.*

(b) *$QNx \neq 0$ for each $x \in \operatorname{Ker} L \bigcap \partial\Omega$, and $\deg\{JQN, \Omega \bigcap \partial \operatorname{Ker} L, 0\} \neq 0$.*

*Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \bigcap \overline{\Omega}$.*

For convenience and simplicity in the following discussion, we always use the notations below throughout the paper:

$$\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)dt, \qquad f^L = \min_{t\in[0,\omega]} f(t), \qquad f^M = \max_{t\in[0,\omega]} f(t), \qquad \overline{|f|} = \frac{1}{\omega}\int_0^\omega |f(t)|dt, \tag{3.1}$$

where $f(t)$ is a $\omega$ continuous periodic function. For any nonnegative integer $p$, let $C^{(p)}[0,\omega; t_1, t_2, \ldots, t_q] = \{x : [0,\omega] \to R^m \mid x^{(p)}(t)$ exist for $t \neq t_1, \ldots, t_q$; $x^{(p)}(t+0)$, and let $x^{(p)}(t-0)$ exist at $t_1, t_2, \ldots, t_q$, and $x^{(j)}(t_k) = x^{(j)}(t_k - 0)$, $k = 1, \ldots, m$, $j = 0, 1, 2, \ldots, p\}$ with the norm $\|x\|_p = \max\{\sup_{t\in[0,\omega]}\|x^{(j)}(t)\|\}_{j=1}^{p}$, where $\|\cdot\|$ is any norm of $R^m$. It is easy to see that $C^{(p)}[0,\omega; t_1, t_2, \ldots, t_q]$ is a Banach space.

Now we are now in a position to state and prove the existence of periodic solutions of (2.4).

**Theorem 3.2.** *In addition to* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, $(\mathrm{H}_3)$, *assume further that the following hold:*

$$\begin{aligned}
&(\mathrm{H}_4)\ \min\{P_1, P_2, P_3\} > 0,\\
&(\mathrm{H}_5)\ \overline{a}_1\omega > \sum_{k=1}^{q}\ln(1-\gamma_{1k}),\\
&(\mathrm{H}_6)\ \overline{d}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{3k}) > \overline{\beta}_2\omega e^{B_4}, \quad \overline{\beta}_1\overline{\beta}_2 \neq \overline{ce},
\end{aligned} \tag{3.2}$$

*where*

$$P_1 = \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) - \overline{c}\omega e^{B_6},$$
$$P_2 = \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) - \overline{\beta}_1\omega e^{B_{29}}, \tag{3.3}$$
$$P_3 = \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) - \overline{c}\omega,$$

*and* $B_4$, $B_6$, $B_{29}$ *are defined by* (3.27), (3.32), *and* (3.61), *respectively. Then the system* (1.2) *has at least a* $\omega$ *periodic solution.*

*Proof.* According to the discussion above in Section 2, we need only to prove that the boundary value problem (2.4) has a solution. Since solutions of (2.4) remained positive for all $t \geq 0$, we let

$$u_1(t) = \ln[x_1(t)], \qquad u_2(t) = \ln[x_2(t)], \qquad u_3(t) = \ln[x_3(t)], \tag{3.4}$$

then system (2.4) can be translated to

$$\dot{u}_1(t) = -a_1(t) + b_1(t)e^{(u_2(t)-u_1(t))}, \quad t \neq t_k, \ t \in [0,\omega], \ k = 1,2,\ldots,q,$$
$$\dot{u}_2(t) = a_2(t) - b_2(t) - c(t)e^{u_2(t)} - \beta_1(t)e^{u_3(t)}, \quad t \neq t_k, \ t \in [0,\omega], \ k = 1,2,\ldots,q, \tag{3.5}$$
$$\dot{u}_3(t) = d(t) - e(t)e^{u_3(t)} - \beta_2(t)e^{u_2(t)}, \quad t \neq t_k, \ t \in [0,\omega], \ k = 1,2,\ldots,q,$$
$$\Delta u_i(t_k) = \ln(1-\gamma_{ik}), \quad i = 1,2,3, \quad u_i(0) = u_i(\omega).$$

It is easy to see that if system (3.5) has one $\omega$ periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t))^T$, then $(x_1^*(t), x_2^*(t), x_3^*(t))^T = (e^{u_1^*(t)}, e^{u_2^*(t)}, e^{u_3^*(t)})^T$ is a positive solution of system (1.2). Therefore, to complete the proof, it suffices to show that system (3.5) has at least one $\omega$ periodic solution.

In order to use the continuation theorem of coincidence degree theory, we take

$$X = \{u \in C[0,\omega; t_1, t_2, \ldots, t_q]\}, \quad Y = X \times R^{3\times(q+1)}. \tag{3.6}$$

Then $X$ is a Banach space with norm $\|\cdot\|_0$, and $Y$ is also a Banch space with norm $\|z\| = \|x\|_0 + \|y\|$, $x \in X, Y \in R^{3q}$.

Let the following hold:

$$\begin{aligned}
&\operatorname{dom} L=\left\{x=(u_1,u_2,u_3)^T\in C[0,\omega];t_1,t_2,\dots,t_q\right\},\\
&L:\operatorname{Dom} L\subset X\longrightarrow Y,\quad x\longrightarrow\left(x',\Delta x(t_k)_{k=1}^{q}\right),\\
&N:X\longrightarrow Y,\\
&Nu=\left(\left(\begin{array}{c}-a_1(t)+b_1(t)\exp(u_2(t)-u_1(t))\\ a_2(t)-b_2(t)-c(t)e^{u_2(t)}-\beta_1(t)e^{u_3(t)}\\ d(t)-e(t)e^{u_3(t)}-\beta_2(t)e^{u_2(t)}\end{array}\right),\left(\begin{array}{c}\ln(1-\gamma_{11})\\ \ln(1-\gamma_{21})\\ \ln(1-\gamma_{31})\end{array}\right),\right.\\
&\qquad\left.\left(\begin{array}{c}\ln(1-\gamma_{12})\\ \ln(1-\gamma_{22})\\ \ln(1-\gamma_{32})\end{array}\right),\dots,\left(\begin{array}{c}\ln(1-\gamma_{1q})\\ \ln(1-\gamma_{2q})\\ \ln(1-\gamma_{3q})\end{array}\right),0\right)
\end{aligned}\tag{3.7}$$

Obviously,

$$\begin{aligned}
&\operatorname{Ker} L=\left\{u:u(t)=h\in R^3,\ t\in[0,\omega]\right\},\\
&\operatorname{Im} L=\left\{z=\left(f,a_1,a_2,\dots,a_q,d\right)\in Y:\int_0^\omega f(s)ds+\sum_{k=1}^{q}a_k+d=0\right\}\\
&\qquad=X\times R^{3\times q}\times\{0\},\\
&\dim\operatorname{Ker} L=3=\operatorname{codim}\operatorname{Im} L.
\end{aligned}\tag{3.8}$$

So, $\operatorname{Im} L$ is closed in $Y$; $L$ is a Fredholm mapping of index zero. Define two projectors

$$\begin{aligned}
&Px=\frac{1}{\omega}\int_0^\omega x(t)dt,\\
&Qz=Q\left(f,a_1,a_2,\dots,a_q,d\right)=\left(\frac{1}{\omega}\left[\int_0^\omega f(s)ds+\sum_{k=1}^{q}a_k+d,\right],0,0,\dots,0\right).
\end{aligned}\tag{3.9}$$

It is easy to show that $P$ and $Q$ are continuous and satisfy $\operatorname{Im} P=\operatorname{Ker} L$, $\operatorname{Im} L=\operatorname{Ker} Q=\operatorname{Im}(I-Q)$.

Further, by direct computation, we can find that the inverse $K_P$ of $L$, $K_P:\operatorname{Im} L\to\operatorname{Ker} P\cap\operatorname{Dom} L$ has the following form:

$$K_P(z)=\int_0^t f(s)ds+\sum_{t_k<t}a_k-\frac{1}{\omega}\int_0^\omega\int_0^t f(s)ds\,dt-\sum_{k=1}^{q}a_k+\frac{1}{\omega}\sum_{k=1}^{q}a_k t_k.\tag{3.10}$$

Moreover, it is easy to check that

$$QNu = \left( \begin{pmatrix} \frac{1}{\omega}\int_0^t F_1(s)ds + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1-\gamma_{1k}) \\ \frac{1}{\omega}\int_0^t F_2(s)ds + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1-\gamma_{2k}) \\ \frac{1}{\omega}\int_0^t F_3(s)ds + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1-\gamma_{3k}) \end{pmatrix}, 0, 0, \dots, 0 \right),$$

$$\begin{aligned} K_P(I-Q)Nu = &\begin{pmatrix} \int_0^t F_1(s)ds + \sum_{t>t_k}\ln(1-\gamma_{1k}) \\ \int_0^t F_2(s)ds + \sum_{t>t_k}\ln(1-\gamma_{2k}) \\ \int_0^t F_3(s)ds + \sum_{t>t_k}\ln(1-\gamma_{3k}) \end{pmatrix} \\ &- \begin{pmatrix} \frac{1}{\omega}\int_0^\omega\int_0^t F_1(s)ds\,dt - \sum_{k=1}^{q}\ln(1-\gamma_{1k}) + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1-\gamma_{1k}) \\ \frac{1}{\omega}\int_0^\omega\int_0^t F_2(s)ds\,dt - \sum_{k=1}^{q}\ln(1-\gamma_{2k}) + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1-\gamma_{2k}) \\ \frac{1}{\omega}\int_0^\omega\int_0^t F_3(s)ds\,dt - \sum_{k=1}^{q}\ln(1-\gamma_{3k}) + \frac{1}{\omega}\sum_{k=1}^{q}\ln(1-\gamma_{3k}) \end{pmatrix} \\ &- \left(\frac{t}{\omega} - \frac{1}{2}\right) \begin{pmatrix} \int_0^\omega F_1(s)ds + \sum_{k=1}^{q}\ln(1-\gamma_{1k}) \\ \int_0^\omega F_2(s)ds + \sum_{k=1}^{q}\ln(1-\gamma_{2k}) \\ \int_0^\omega F_3(s)ds + \sum_{k=1}^{q}\ln(1-\gamma_{3k}) \end{pmatrix}, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} F_1(s) &= -a_1(s) + b_1(s)e^{(u_2(s)-u_1(s))}, \\ F_2(s) &= a_2(s) - b_2(s) - c(s)e^{u_2(s)} - \beta_1(s)e^{u_3(s)}, \\ F_2(s) &= d(s) - e(s)e^{u_3(s)} - \beta_2(s)e^{u_2(s)}. \end{aligned} \tag{3.12}$$

Obviously, $QN$ and $K_P(I-Q)N$ are continuous. Using the Ascoli-Arzela theorem, it is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, $N$ is $L$-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we are at the point to search for an appropriate open, bounded subset $\Omega$ for the application of the continuation theorem. Corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$\begin{gathered}
\dot{u}_1(t) = \lambda\left[-a_1(t) + b_1(t)e^{(u_2(t)-u_1(t))}\right], \quad t \neq t_k,\ t \in [0,\omega],\ k = 1,2,\ldots,q, \\
\dot{u}_2(t) = \lambda\left[a_2(t) - b_2(t) - c(t)e^{u_2(t)} - \beta_1(t)e^{u_3(t)}\right], \quad t \neq t_k,\ t \in [0,\omega],\ k = 1,2,\ldots,q, \\
\dot{u}_3(t) = \lambda\left[\ d_1(t) - e(t)e^{u_3(t)} - \beta_2(t)e^{u_2(t)})\right], \quad t \neq t_k,\ t \in [0,\omega],\ k = 1,2,\ldots,q, \\
\Delta u_i(t_k) = \lambda \ln(1-\gamma_{ik}), \quad i = 1,2,3, \quad u_i(0) = u_i(\omega).
\end{gathered} \tag{3.13}$$

Suppose that $u(t) = (u_1(t), u_2(t), u_3(t))^T \in X$ is an arbitrary solution of system (3.13) for a certain $\lambda \in (0,1)$, integrating both sides of (3.13) over the interval $[0,\omega]$ with respect to $t$, we obtain

$$\begin{gathered}
\int_0^\omega \left[b_1(t)e^{(u_2(t)-u_1(t))}\right]dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) = \int_0^\omega a_1(t)dt, \\
\int_0^\omega \left[c(t)e^{u_2(t)} + \beta_1(t)e^{u_3(t)}\right]dt = \int_0^\omega (a_2(t) - b_2(t))dt + \sum_{k=1}^{q} \ln(1-\gamma_{2k}), \\
\int_0^\omega \left[e(t)e^{u_3(t)} + \beta_2(t)e^{u_2(t)}\right]dt = \int_0^\omega d(t)dt + \sum_{k=1}^{q} \ln(1-\gamma_{3k}).
\end{gathered} \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\begin{aligned}
\int_0^\omega |\dot{u}_1(t)|dt &< \int_0^\omega a_1(t)dt + \int_0^\omega \left[b_1(t)e^{(u_2(t)-u_1(t))}\right]dt \\
&= 2\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k}),
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\int_0^\omega |\dot{u}_2(t)|dt &< \int_0^\omega [a_2(t) + b_2(t)]dt + \int_0^\omega \left[c(t)e^{u_2(t)} + \beta_1(t)e^{u_3(t))}\right]dt \\
&= 2\overline{a}_2\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}),
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
\int_0^\omega |\dot{u}_3(t)|dt &< \int_0^\omega d(t)dt + \int_0^\omega \left[\ e(t)e^{u_3(t)} + \beta_2(t)e^{u_2(t)})\right]dt \\
&= 2\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k}).
\end{aligned} \tag{3.17}$$

Let the following hold:

$$u_i(\xi_i) = \min_{t\in[0,\omega]} u_i(t), \qquad u_i(\eta_i) = \max_{t\in[0,\omega]} u_i(t), \quad i = 1,2,3. \tag{3.18}$$

From the second and the third equations of (3.14), we can obtain

$$\begin{aligned}
&\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) > \int_0^{\omega} c(t)e^{u_2(t)}dt \geq \int_0^{\omega} c(t)e^{u_2(\xi_2)}dt, \\
&\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k}) > \int_0^{\omega} e(t)e^{u_3(t)}dt \geq \int_0^{\omega} e(t)e^{u_3(\xi_3)}dt = \overline{e}\omega e^{u_3(\xi_3)},
\end{aligned} \tag{3.19}$$

then

$$u_2(\xi_2) < \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k})}{\overline{c}\omega}\right], \tag{3.20}$$

$$u_3(\xi_3) < \ln\left[\frac{\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k})}{\overline{e}\omega}\right]. \tag{3.21}$$

Thus

$$\begin{aligned}
u_2(t) &= u_2(\xi_2) + \int_{\xi_2}^{t} \dot{u}_2(t)dt \leq u_2(\xi_2) + \int_0^{\omega} |\dot{u}_2(t)|dt \\
&< \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k})}{\overline{c}\omega}\right] + 2\overline{a}_2\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) =: B_1.
\end{aligned} \tag{3.22}$$

In the following, we will consider four cases.

*Case 1* (if $u_1(t) > 0,\ u_2(t) > 0$). From the first equation of (3.14), we have

$$\begin{aligned}
&\overline{a}_1\omega < \int_0^{\omega} b_1(t)e^{u_2(t)}dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) \leq \overline{b}_1\omega e^{u_2(\eta_2)} + \sum_{k=1}^{q} \ln(1-\gamma_{3k}), \\
&\overline{a}_1\omega > \int_0^{\omega} b_1(t)e^{-u_1(t)}dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) \geq \int_0^{\omega} b_1(t)e^{-u_1(\eta_1)}dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}),
\end{aligned} \tag{3.23}$$

that is,

$$u_2(\eta_2) > \ln\left[\frac{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}{\overline{b}_1\omega}\right],$$
$$u_1(\eta_1) > \ln\left[\frac{\overline{b}_1\omega}{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}\right]. \tag{3.24}$$

Then

$$\begin{aligned} u_2(t) &= u_2(\eta_2) - \int_t^{\eta_2}\dot{u}_2(t)dt \geq u_2(\eta_2) - \int_0^{\omega}|\dot{u}_2(t)|dt \\ &\geq \ln\left[\frac{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}{\overline{b}_1\omega}\right] - 2\overline{a}_2\omega - \sum_{k=1}^{q}\ln(1-\gamma_{2k}) =: B_2, \end{aligned} \tag{3.25}$$

$$\begin{aligned} u_1(t) &= u_1(\eta_1) - \int_t^{\eta_1}\dot{u}_1(t)dt \geq u_1(\eta_1) - \int_0^{\omega}|\dot{u}_1(t)|dt \\ &\geq \ln\left[\frac{\overline{b}_1\omega}{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}\right] - 2\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k}) =: B_3. \end{aligned} \tag{3.26}$$

Thus, from (3.22) and (3.25), we obtain

$$|u_2(t)| \leq \max\{|B_1|, |B_2|\} =: B_4. \tag{3.27}$$

By the first and the third equations of (3.14), we get

$$\int_0^{\omega}\left[b_1(t)e^{B_4-u_1(\xi_1)}\right]dt + \sum_{k=1}^{q}\ln(1-\gamma_{1k}) > \overline{a}_1\omega,$$
$$\int_0^{\omega}\left[e(t)e^{u_3(\eta_3)} + \beta_2(t)e^{B_4}\right]dt > \overline{d}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{3k}), \tag{3.28}$$

then

$$u_1(\xi_1) < \ln\left[\frac{\overline{b}_1\omega e^{B_4}}{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}\right], \tag{3.29}$$

$$u_3(\eta_3) > \ln\left[\frac{\overline{d}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{3k}) - \beta_2\omega e^{B_4}}{\overline{e}\omega}\right]. \tag{3.30}$$

From (3.15), (3.17), (3.21), and (3.30), we have

$$\begin{aligned} u_1(t) &= u_1(\xi_1) + \int_{\xi_1}^{t} \dot{u}_1(t)dt \le u_1(\xi_1) + \int_0^{\omega} |\dot{u}_1(t)|dt \\ &< \ln\left[\frac{\overline{b}_1 \omega e^{B_4}}{\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k})}\right] + 2\overline{a}_1\omega + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) =: B_5, \end{aligned} \tag{3.31}$$

$$\begin{aligned} u_3(t) &= u_3(\xi_3) + \int_{\xi_3}^{t} \dot{u}_3(t)dt \le u_3(\xi_3) + \int_0^{\omega} |\dot{u}_3(t)|dt \\ &< \ln\left[\frac{\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k})}{\overline{e}\omega}\right] + 2\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k}) =: B_6, \end{aligned} \tag{3.32}$$

$$\begin{aligned} u_3(t) &= u_3(\eta_3) - \int_{t}^{\eta_3} \dot{u}_3(t)dt \ge u_3(\eta_3) - \int_0^{\omega} |\dot{u}_3(t)|dt \\ &> \ln\left[\frac{\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k}) - \overline{\beta}_2\omega e^{B_4}}{\overline{e}\omega}\right] - 2\overline{d}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{3k}) =: B_7. \end{aligned} \tag{3.33}$$

Thus,

$$|u_1(t)| \le B_5, \qquad |u_3(t)| \max\{|B_6|, |B_7|\} =: B_8. \tag{3.34}$$

*Case 2* (if $u_1(t) > 0,\ u_2(t) < 0$). By the first equation of (3.14), we have

$$\overline{a}_1\omega < \int_0^{\omega} b_1(t)e^{u_2(t)}dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) \le \overline{b}_1\omega e^{u_2(\eta_2)} + \sum_{k=1}^{q} \ln(1-\gamma_{3k}), \tag{3.35}$$

namely,

$$u_2(\eta_2) > \ln\left[\frac{\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k})}{\overline{b}_1\omega}\right]. \tag{3.36}$$

Then

$$\begin{aligned} u_2(t) &= u_2(\eta_2) - \int_{t}^{\eta_2} \dot{u}_2(t)dt \ge u_2(\eta_2) - \int_0^{\omega} |\dot{u}_2(t)|dt \\ &\ge \ln\left[\frac{\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k})}{\overline{b}_1}\right] - 2\overline{a}_2\omega - \sum_{k=1}^{q} \ln(1-\gamma_{2k}) =: B_2. \end{aligned} \tag{3.37}$$

From (3.22) and (3.37), we obtain

$$|u_2(t)| \le \max\{|B_1|, |B_2|\} =: B_3. \tag{3.38}$$

By the first equation of (3.14), we also have

$$\begin{aligned} &\int_0^{\omega} \frac{b_1(t)e^{B_3}}{e^{u_1(t)}} dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) \ > \overline{a}_1\omega, \\ &\int_0^{\omega} \frac{b_1(t)e^{-B_3}}{e^{u_1(t)}} dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) < \overline{a}_1\omega, \end{aligned} \tag{3.39}$$

Then

$$\begin{aligned} &\frac{\overline{b}_1\omega e^{B_3}}{e^{u_1(\xi_1)}} + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) > \overline{a}_1\omega, \\ &\frac{\overline{b}_1\omega e^{-B_3}}{e^{u_1(\eta_1)}} + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) < \overline{a}_1\omega. \end{aligned} \tag{3.40}$$

that is,

$$\begin{aligned} u_1(\xi_1) &< \ln\left[\frac{\overline{b}_1\omega e^{B_3}}{\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k})}\right] =: B_8, \\ u_1(\eta_1) &< \ln\left[\frac{\overline{b}_1\omega e^{B_3}}{\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k})}\right] =: B_9. \end{aligned} \tag{3.41}$$

Thus,

$$\begin{aligned} u_1(t) &= u_1(\xi_1) + \int_{\xi_1}^{t} \dot{u}_1(t)dt \le u_1(\xi_1) + \int_0^{\omega} |\dot{u}_1(t)|dt \\ &< B_8 + 2\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k}) =: B_{10}, \\ u_1(t) &= u_1(\eta_1) + \int_{t}^{\eta_1} \dot{u}_1(t)dt \ge u_1(\eta_1) + \int_0^{\omega} |\dot{u}_1(t)|dt \\ &> B_9 - 2\overline{a}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{1k}) =: B_{11}. \end{aligned} \tag{3.42}$$

From (3.42), we have

$$|u_1(t) \le \max\{|B_{10}|, |B_{11}|\} =: B_{12}. \tag{3.43}$$

By the second equation of (3.14), we have

$$\begin{aligned}
&\int_0^\omega c(t)e^{B_3}dt + \int_0^\omega \beta_1(t)e^{u_3(t)}dt > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}), \\
&\int_0^\omega c(t)e^{-B_3}dt + \int_0^\omega \beta_1(t)e^{u_3(t)}dt < \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}).
\end{aligned} \tag{3.44}$$

Then

$$\begin{aligned}
&\overline{c}\omega e^{B_3} + \overline{\beta}_1\omega e^{u_3(\eta_3)} > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}), \\
&\overline{c}\omega e^{-B_3} + \overline{\beta}_1\omega e^{u_3(\xi_3)} < \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) =: B_{13},
\end{aligned} \tag{3.45}$$

that is,

$$\begin{aligned}
&u_3(\eta_3) > \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) - \overline{c}\omega e^{B_6}}{\overline{\beta}_1\omega}\right], \\
&u_3(\xi_3) < \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) - \overline{c}\omega e^{-B_3}}{\overline{\beta}_1\omega}\right] =: B_{14}.
\end{aligned} \tag{3.46}$$

Therefore, we get

$$\begin{aligned}
u_3(t) &= u_3(\xi_3) + \int_{\xi_3}^{t} \dot{u}_3(t)dt \le u_3(\xi_3) + \int_0^\omega |\dot{u}_3(t)|dt \\
&< B_{14} + 2\overline{d}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{3k}) =: B_{15}, \\
u_3(t) &= u_3(\eta_3) - \int_t^{\eta_3} \dot{u}_3(t)dt \ge u_3(\eta_3) - \int_0^\omega |\dot{u}_3(t)|dt \\
&> B_{14} - 2\overline{d}_1\omega - \sum_{k=1}^{q} \ln(1-\gamma_{3k}) =: B_{16}.
\end{aligned} \tag{3.47}$$

Hence, we have

$$|u_3(t)| \le \max\{|B_{15}|, |B_{16}|\} =: B_{17}. \tag{3.48}$$

*Case 3* (if $u_1(t) < 0,\ u_2(t) > 0$). By the first equation of (3.14), we have

$$\int_0^{\omega} b_1(t)e^{-u_1(t)}dt + \sum_{k=1}^{q}\ln(1-\gamma_{1k}) < \int_0^{\omega} a_1(t)dt,$$
$$\int_0^{\omega} \left[b_1(t)e^{(u_2(\eta_2)-u_1(t))}\right]dt + \sum_{k=1}^{q}\ln(1-\gamma_{1k}) > \int_0^{\omega} a_1(t)dt. \tag{3.49}$$

Then

$$\overline{b}_1\omega e^{-u_1(\eta_1)} + \sum_{k=1}^{q}\ln(1-\gamma_{1k}) < \overline{a}_1\omega,$$
$$\overline{b}_1\omega e^{-B_{19}}e^{u_2(\eta_2)} + \sum_{k=1}^{q}\ln(1-\gamma_{1k}) > \overline{a}_1\omega. \tag{3.50}$$

namely,

$$u_1(\eta_1) > \ln\left[\frac{\overline{b}_1\omega}{\overline{a}_1 - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}\right] =: B_{18},$$
$$u_2(\eta_2) > \ln\left[\frac{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}{\overline{b}_1\omega e^{-B_{19}}}\right] =: B_{20}. \tag{3.51}$$

Therefore,

$$\begin{aligned}
u_1(t) &= u_1(\eta_1) - \int_t^{\eta_1}\dot{u}_1(t)dt \geq u_1(\eta_1) - \int_0^{\omega}|\dot{u}_1(t)|dt \\
&> B_{18} - 2\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k}) =: B_{19}, \\
u_2(t) &= u_2(\xi_2) + \int_{\xi_2}^{t}\dot{u}_2(t)dt \leq u_2(\xi_2) + \int_0^{\omega}|\dot{u}_2(t)|dt \\
&< \ln\left[\frac{\overline{|a_2-b_2|}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{2k})}{\overline{c}\omega}\right] + 2\overline{a}_2\omega + \sum_{k=1}^{q}\ln(1-\gamma_{2k}) =: B_{21}, \\
u_2(t) &= u_2(\eta_2) - \int_t^{\eta_2}\dot{u}_2(t)dt \geq u_2(\eta_1) - \int_0^{\omega}|\dot{u}_2(t)|dt \\
&> B_{20} - 2\overline{a}_2\omega - \sum_{k=1}^{q}\ln(1-\gamma_{2k}) =: B_{22}.
\end{aligned} \tag{3.52}$$

So

$$B_{19} < u_1(t) < 0, |u_2(t)| \le \max\{|B_{21}|, |B_{22}|\} =: B_{23}. \tag{3.53}$$

By the third equation of (3.14), we obtain

$$\int_0^{\omega} e(t)e^{u_3(\eta_3)}dt + \int_0^{\omega} \beta_2(t)e^{B_{23}}dt > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}), \tag{3.54}$$

that is,

$$u_3(\eta_3) > \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{2k}) - \overline{\beta}_2\omega e^{B_{23}}}{\overline{e}\omega}\right] =: B_{24}. \tag{3.55}$$

Thus,

$$|u_3(t)| \le \max\{|B_{23}|, |B_{24}|\} =: B_{25}. \tag{3.56}$$

*Case 4* (if $u_1(t) < 0,\ u_2(t) < 0$). By the second equation of (3.14), we have

$$\int_0^{\omega} c(t)dt + \int_0^{\omega} \beta_1(t)e^{u_3(t)}dt > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}). \tag{3.57}$$

Then

$$\overline{c}\omega + \overline{\beta}_1\omega e^{u_3(\eta_3)} > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}), \tag{3.58}$$

that is,

$$u_3(\eta_3) \ge \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{2k}) - \overline{c}\omega}{\overline{\beta}_1\omega}\right] =: B_{26}. \tag{3.59}$$

Therefore,

$$\begin{aligned} u_3(t) &= u_3(\xi_3) + \int_{\xi_3}^{t} \dot{u}_3(t)dt \le u_3(\xi_3) + \int_0^{\omega} |\dot{u}_3(t)|dt \\ &< \ln\left[\frac{\overline{d}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{3k})}{\overline{e}\omega}\right] + 2\overline{d}\omega + \sum_{k=1}^{q}\ln(1-\gamma_{3k}) =: B_{27}, \end{aligned}$$

$$u_3(t) = u_3(\eta_3) - \int_t^{\eta_3} \dot{u}_3(t)dt \geq u_3(\eta_3) - \int_0^{\omega} |\dot{u}_3(t)|dt$$
$$> B_{26} - 2\overline{d}\omega - \sum_{k=1}^{q} \ln(1-\gamma_{3k}) =: B_{28}. \tag{3.60}$$

Thus,

$$|u_3(t)| \leq \max\{|B_{27}|, |B_{28}|\} =: B_{29}. \tag{3.61}$$

By the second equation of (3.14), we obtain

$$\int_0^{\omega} c(t)e^{u_2(\eta_2)}dt + \int_0^{\omega} \beta_1(t)e^{B_{29}}dt > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}), \tag{3.62}$$

that is,

$$\overline{c}e^{u_2(\eta_2)} + \overline{\beta}_1\omega e^{B_{29}} > \overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}). \tag{3.63}$$

Thus,

$$u_2(\eta_2) > \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) - \overline{\beta}_1\omega e^{B_{29}}}{\overline{c}\omega}\right]. \tag{3.64}$$

Then, from (3.16) and (3.20), we get

$$u_2(t) = u_2(\xi_2) + \int_{\xi_2}^{t} \dot{u}_2(t)dt \leq u_2(\xi_2) + \int_0^{\omega} |\dot{u}_2(t)|dt$$
$$< \ln\left[\frac{\overline{|a_2 - b_2|}\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k})}{\overline{c}\omega}\right] + 2\overline{a}_2\omega + \sum_{k=1}^{q} \ln(1-\gamma_{2k}) =: B_{31},$$
$$u_2(t) = u_2(\eta_2) - \int_t^{\eta_2} \dot{u}_2(t)dt \geq u_2(\eta_2) - \int_0^{\omega} |\dot{u}_2(t)|dt \tag{3.65}$$
$$> B_{30} - 2\overline{a}_2\omega\omega - \sum_{k=1}^{q} \ln(1-\gamma_{2k}) =: B_{32}.$$

Thus,

$$|u_2(t)| \leq \max\{|B_{31}|, |B_{32}|\} =: B_{33}. \tag{3.66}$$

By the first equation of (3.14), we have

$$\int_0^{\omega} b_1(t)e^{u_2(t)-u_1(\eta_1)}dt + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) < \int_0^{\omega} a_1(t)dt. \tag{3.67}$$

Then

$$\overline{b}_1\omega e^{-B_{33}} e^{-u_1(\eta_1)} + \sum_{k=1}^{q} \ln(1-\gamma_{1k}) < \overline{a}_1\omega. \tag{3.68}$$

Thus,

$$u_1(\eta_1) > \ln\left[\frac{\overline{b}_1\omega e^{-B_{32}}}{\overline{a}_1\omega - \sum_{k=1}^{q}\ln(1-\gamma_{1k})}\right] =: B_{34}. \tag{3.69}$$

Hence, we have

$$B_{34} < u_1(t) < 0. \tag{3.70}$$

Based on the discussion above, we can easily obtain

$$\begin{aligned} u_1(t) &\le \max\{B_5, B_{12}, |B_{19}|, |B_{34}|\}, \\ u_2(t) &\le \max\{B_3, B_4, B_{23}, B_{33}\}, \\ u_3(t) &\le \max\{B_8, B_{17}, B_{25}, B_{29}\}. \end{aligned} \tag{3.71}$$

Obviously, $B_i$ $(i = 1, 2, \ldots, 34)$ are independent of $\lambda \in (0,1)$. Similar to the proof of Theorem 2.1 of [17], we can easily find a sufficiently large $M > 0$ so that we denote the set

$$\Omega = \left\{u(t) = (u_1(t), u_2(t), u_3(t))^T \in x : \|u\| < M,\ u(t_k^+) \in \Omega,\ k = 1, 2, \ldots, q\right\}. \tag{3.72}$$

It is clear that $\Omega$ satisfies the requirement (a) in Lemma 3.1.

When $(u_1(t), u_2(t), u_3(t))^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$ and $u = \{(u_1, u_2, u_3)^T\}$ is a constant vector in $R^3$ with $\|u\| = \|(u_1(t), u_2(t), u_3(t))^T\| = M$, then we have

$$QNu = \left(\begin{pmatrix} \overline{a}_1 + \overline{b}_1 e^{u_2-u_1} + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q}\ln(1-\gamma_{1k}) \\ \overline{a}_2 - b_2 - \overline{c}e^{u_2}\overline{\beta}_1 e^{u_3} + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q}\ln(1-\gamma_{2k}) \\ \overline{d} - \overline{e}e^{u_3} - \overline{\beta}_2 e^{u_2} + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q}\ln(1-\gamma_{3k}) \end{pmatrix}, 0, \ldots, 0\right) \ne 0. \tag{3.73}$$

Letting $J : \operatorname{Im} Q \to \operatorname{Ker} L, (r, 0, \dots, 0, 0) \to r$ and, by direct calculation, we get

$$\begin{aligned}
&\deg\left\{JQN(u_1,u_2,u_3)^T; \partial\Omega \bigcap \ker L; 0\right\} \\
&= \operatorname{signdet}\begin{pmatrix} -\overline{b}_1 e^{u_2-u_1} & \overline{b}_1 e^{u_2-u_1} & 0 \\ 0 & -\overline{c} e^{u_2} & -\overline{\beta}_1 e^{u_3} \\ 0 & -\overline{\beta}_2 e^{u_2} & -\overline{e} e^{u_3} \end{pmatrix} \\
&= \operatorname{sign}\left\{\left(\overline{b}_1\overline{\beta}_1\overline{\beta}_2 - \overline{b}_1\overline{c}\overline{e}\right)e^{2u_2-u_1+u_3}\right\} \neq 0.
\end{aligned} \tag{3.74}$$

This proves that condition (b) in Lemma 3.1 is satisfied. By now, we have proved that $\Omega$ verifies all requirements of Lemma 3.1, then it follows that $Lu = Nu$ has at least one solution $(u_1(t), u_2(t), u_3(t))^T$ in $\operatorname{Dom} L \cap \overline{\Omega}$; that is, to say, (3.5) has at least one $\omega$ periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$. Then we know that $((x_1(t), x_2(t), x_3(t))^T = (e^{u_1(t)}, ex^{u_2(t)}, e^{u_3(t)})^T$ is an $\omega$ periodic solution of system (2.4) with strictly positive components. This completes the proof. □

## 4. Uniqueness and Global Attractivity of Periodic Solutions

Under the hypotheses $(H_1)$, $(H_2)$, $(H_3)$, we consider the following ordinary differential equation without impulsive:

$$\begin{aligned}
\dot{z}_1(t) &= z_1(t)\left[-a_1(t) + b_1(t)\frac{\prod_{0<t_k<t}(1-\gamma_{2k})z_2(t)}{\prod_{0<t_k<t}(1-\gamma_{1k})z_1(t)}\right], \\
\dot{z}_2(t) &= z_2(t)\left[a_2(t) - b_2(t) - c(t)\prod_{0<t_k<t}(1-\gamma_{2k})z_2(t) - \beta_1(t)\prod_{0<t_k<t}(1-\gamma_{3k})z_3(t)\right], \\
\dot{z}_3(t) &= z_3(t)\left[d(t) - e(t)\prod_{0<t_k<t}(1-\gamma_{3k})z_3(t) - \beta_2(t)\prod_{0<t_k<t}(1-\gamma_{2k})z_2(t)\right],
\end{aligned} \tag{4.1}$$

with the initial conditions $z_i(0) > 0$, $i = 1, 2, 3$.

The following lemmas will be helpful in the proofs of our results. The proof of the following Lemma 4.1 is similar to that of Theorem 1 in [18], and it will be omitted.

**Lemma 4.1.** *Assume that* $(H_1)$, $(H_2)$, $(H_3)$ *hold, then one has the following.*

(i) *If* $z(t) = (z_1(t), z_2(t), z_3(t))^T$ *is a solution of* (4.1) *on* $[0, +\infty)$, *then* $x_i(t) = \prod_{0<t_k<t}(1 - \gamma_{ik})z_i(t)$ $(i = 1, 2, 3)$ *is a solution of* (2.4) *on* $[0, +\infty)$.

(ii) *If* $x(t) = (x_1(t), x_2(t), x_3(t))^T$ *is a solution of* (2.4) *on* $[0, +\infty)$, *then* $z_i(t) = \prod_{0<t_k<t}(1 - \gamma_{ik})^{-1}x_i(t)$ $(i = 1, 2, 3)$ *is a solution of* (4.1) *on* $[0, +\infty)$.

**Lemma 4.2.** *Let* $z(t) = (z_1(t), z_2(t), z_3(t))^T$ *denote any positive solution of system* (4.1) *with initial conditions* $z_i(0) > 0$, $i = 1, 2, 3$. *Assume that the following condition holds,*

$$(H_7)\ a_2^M > b_2^L, \qquad d^M > e^L. \tag{4.2}$$

*Then there exists a* $T_3 > 0$ *such that*

$$0 < z_i(t) \le M_i, \quad (i = 1, 2, 3), \text{ for } t \ge T_3, \tag{4.3}$$

*where*

$$\begin{aligned} M_1 > M_1^* &= \frac{\prod_{0<t_k<t}(1-\gamma_{2k})M_2}{a_1^L \prod_{0<t_k<t}(1-\gamma_{1k})}, \\ M_2 > M_2^* &= \frac{a_2^M - b_2^L}{c^L \prod_{0<t_k<t}(1-\gamma_{2k})}, \\ M_3 > M_3^* &= \frac{d^M - e^L}{e^L \prod_{0<t_k<t}(1-\gamma_{3k})}. \end{aligned} \tag{4.4}$$

*Proof.* From the second equation of (4.1), we can obtain

$$\begin{aligned} \dot{z}_2(t) &\le z_2(t)\left[a_2(t) - b_2(t) - c(t)\prod_{0<t_k<t}(1-\gamma_{2k})z_2(t)\right] \\ &\le z_2(t)\left[a_2^M - b_2^L - c^L \prod_{0<t_k<t}(1-\gamma_{2k})z_2(t)\right]. \end{aligned} \tag{4.5}$$

By (4.5), we can derive the following.

($A_1$) If $z_2(0) \le M_2$, then $z_2(t) \le M_2$, $t \ge 0$.

($A_2$) If $z_2(0) > M_2$, let $-\alpha_1 = M_2[a_2^M - b_2^L - c^L \prod_{0<t_k<t}(1-\gamma_{2k})M_2]$, $(\alpha_1 > 0)$. Then there exists $\varepsilon_1 > 0$ such that $t \in [0, \varepsilon_1)$, then $z_2(t) > M_2$, and also we have

$$\dot{z}_2(t) < -\alpha_1 < 0. \tag{4.6}$$

From what has been discussed above, we can easily conclude that, if $z_2(0) > M_2$, then $z_2(t)$ is strictly monotone decreasing with speed at least $\alpha_1$. Therefore, there exists a $T_1 > 0$ such that $t > T_1$, then $z_2(t) \le M_2$.

From the third equation of (4.1), we can obtain

$$\begin{aligned} \dot{z}_3(t) &\le z_3(t)\left[d(t) - e(t)\prod_{0<t_k<t}(1-\gamma_{3k})z_3(t)\right] \\ &\le z_3(t)\left[d^M - e^L \prod_{0<t_k<t}(1-\gamma_{3k})z_3(t)\right]. \end{aligned} \tag{4.7}$$

By (4.7), we can derive the following

($B_1$) If $z_3(0) \le M_3$, then $z_3(t) \le M_3$, $t \ge 0$.

($B_2$) If $z_3(0) > M_3$, let $-\alpha_2 = M_3[d^M - e^L \prod_{0<t_k<t}(1-\gamma_{3k})M_3]$, $(\alpha_2 > 0)$. Then there exists $\varepsilon_2 > 0$ such that $t \in [0, \varepsilon_2)$, then $z_3(t) > M_3$, and also we have

$$\dot{z}_3(t) < -\alpha_2 < 0. \tag{4.8}$$

From what has been discussed above, we can easily conclude that, if $z_3(0) > M_3$, then $z_3(t)$ is strictly monotone decreasing with speed at least $\alpha_2$. Therefore, there exists a $T_2 > 0$ such that $t > T_2$, then $z_3(t) \le M_3$.

From the first equation of (4.1), we can obtain

$$\begin{aligned} \dot{z}_1(t) &\le z_1(t)\left[-a_1(t) + b_1(t)\frac{\prod_{0<t_k<t}(1-\gamma_{2k})M_2}{\prod_{0<t_k<t}(1-\gamma_{1k})z_1(t)}\right] \\ &= -a_1(t)z_1(t) + \frac{\prod_{0<t_k<t}(1-\gamma_{2k})M_2}{\prod_{0<t_k<t}(1-\gamma_{1k})} \\ &\le -a_1^L z_1(t) + \frac{\prod_{0<t_k<t}(1-\gamma_{2k})M_2}{\prod_{0<t_k<t}(1-\gamma_{1k})}. \end{aligned} \tag{4.9}$$

Then we have

$$z_1(t) \le M_1, \quad \text{for } t \ge T_1. \tag{4.10}$$

Set $T_3 = \max\{T_1, T_2\}$, then we have

$$0 < z_i(t) \le M_i, \quad (i = 1,2,3), \quad \text{for } t \ge T_3. \tag{4.11}$$

The proof is complete. □

**Lemma 4.3.** *Let* ($H_1$), ($H_2$), ($H_3$) *hold. Assume that the following condition holds.*

$$(H_8)\ a_2^L > b_2^M + \beta_1^M \prod_{0<t_k<t}(1-\gamma_{3k}), \qquad d^L > e^M - \beta_2^M \prod_{0<t_k<t}(1-\gamma_{2k}). \tag{4.12}$$

*Then there exists positive constants $T > 0$ and $m_i$ $(i = 1,2,3)$ such that, for all $t > T$,*

$$m_i < z_i(t), \quad (i = 1,2,3), \quad \text{for } t \ge T, \tag{4.13}$$

*in which*

$$m_1 < m_1^* = \frac{b_1^L \prod_{0<t_k<t}(1-\gamma_{2k})m_2}{a_1^M \prod_{0<t_k<t}(1-\gamma_{1k})},$$

$$m_2 < m_2^* = \frac{a_2^L - b_2^M - \beta_1^M \prod_{0<t_k<t}(1-\gamma_{3k})}{c^M \prod_{0<t_k<t}(1-\gamma_{2k})},$$

$$m_3 < m_3^* = \frac{d^L - e^M - \beta_2^M \prod_{0<t_k<t}(1-\gamma_{2k}) M_2}{e^M \prod_{0<t_k<t}(1-\gamma_{3k})}. \tag{4.14}$$

*Proof.* By the second equation of (4.1), It is easy to obtain that, for $t \geq T_3$,

$$\dot{z}_2(t) \geq z_2(t)\left[a_2^L - b_2^M - c^M \prod_{0<t_k<t}(1-\gamma_{2k}) z_2(t) - \beta_1^M \prod_{0<t_k<t}(1-\gamma_{3k}) M_3\right], \tag{4.15}$$

where $T_3$ is defined in Lemma 4.1.

($C_1$) If $z_2(T_3) \geq m_2$, then $z_2(t) \geq m_2$, $t \geq T_3$.

($C_2$) If $z_2(T_3) < m_2$ and let

$$\mu_1 = z_2(T_3)\left[a_2^L - b_2^M - c^M \prod_{0<t_k<t}(1-\gamma_{2k}) m_2\right], \tag{4.16}$$

then there exists $\varepsilon_3 > 0$ such that $t \in [T_3, T_3 + \varepsilon_3)$, then $z_2(t) > m_2$, and also we have

$$\dot{z}_2(t) > \mu_1 > 0. \tag{4.17}$$

Then we know that if $z_2(T_3) < m_2$, $z_2(t)$ will strictly monotonically increase with speed $\mu_2$. Thus, there exists $T_4 > T_3$ such that if $t \geq T_4$, then $z_2(t) \geq m_2$.

By the third equation of (4.1), It is easy to obtain that for $t \geq T_3$,

$$\dot{z}_3(t) \geq z_3(t)\left[d^L - e^M \prod_{0<t_k<t}(1-\gamma_{3k}) z_3(t) - \beta_2^M \prod_{0<t_k<t}(1-\gamma_{2k}) M_2\right], \tag{4.18}$$

where $T_3$ is defined in Lemma 4.2.

($D_1$) If $z_2(T_3) \geq m_3$, then $z_3(t) \geq m_3$, $t \geq T_3$.

($D_2$) If $z_2(T_3) < m_3$, and let

$$\mu_2 = z_3(T_3)\left[d^L - e^M \prod_{0<t_k<t}(1-\gamma_{3k}) m_3 - \beta_2^M \prod_{0<t_k<t}(1-\gamma_{2k}) M_2\right], \tag{4.19}$$

then there exists $\varepsilon_4 > 0$ such that $t \in [T_3, T_3 + \varepsilon_4)$, then $z_3(t) > m_3$, and also we have

$$\dot{z}_3(t) > \mu_2 > 0. \tag{4.20}$$

Then we know that if $z_3(T_3) < m_3$, $z_3(t)$ will strictly monotonically increase with speed $\mu_2$. Thus, there exists $T_5 > T_3$ such that, if $t \geq T_5$, then $z_3(t) \geq m_3$.

Finally, by the third equation of (4.1), we obtain

$$\begin{aligned}\dot{z}_1(t) &\geq z_1(t)\left[-a_1^M + b_1^L \frac{\prod_{0<t_k<t}(1-\gamma_{2k})m_2}{\prod_{0<t_k<t}(1-\gamma_{1k})z_1(t)}\right] \\ &= -a_1^M z_1(t) + b_1^L \frac{\prod_{0<t_k<t}(1-\gamma_{2k})m_2}{\prod_{0<t_k<t}(1-\gamma_{1k})z_1(t)}.\end{aligned} \tag{4.21}$$

Thus, we have

$$z_1(t) \geq m_1, \tag{4.22}$$

for $t \geq T_4$. Set $T = \max\{T_4, T_5\}$, then we have

$$z_i(t) > m_i, \quad (i = 1, 2, 3), \quad \text{for } t \geq T. \tag{4.23}$$

In the sequel, we formulate the uniqueness and global attractivity of the $\omega$ periodic solution $x^*(t)$ in Theorem 4.4. It is immediate that if $x^*(t)$ is global attractivity, then $x^*(t)$ is in fact unique. □

**Theorem 4.4.** *In addition to* $(\mathrm{H}_1) - (\mathrm{H}_8)$, *assume further* $(\mathrm{H}_9)$ $\lim_{t\to\infty} \inf B_i(t) > 0$, *where*

$$\begin{aligned}B_1(t) &= \left[c(t) - \beta_2(t) - \frac{b^M}{m_1\prod_{0<t_k<t}(1-\gamma_{1k})}\right]\prod_{0<t_k<t}(1-\gamma_{2k}), \\ B_2(t) &= \left[e(t) - \beta_1(t)\right]\prod_{0<t_k<t}(1-\gamma_{3k}).\end{aligned} \tag{4.24}$$

*Then system* (2.4) *has a unique positive* $\omega$ *periodic solution* $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^T$ *which is global attractivity.*

*Proof.* According to the conclusion of Theorem 3.2, we only need to show that the positive periodic solution of (2.4) is global asymptotical stable. Let $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^T$ be a positive $\omega$ periodic solution of system (2.4) let $x(t) = (x_1(t), x_2(t), x_3(t))^T$ be any positive solution of system (2.4). Then $z^*(t) = (z_1^*(t), z_2^*(t), z_3^*(t))^T$, $(z_1^*(t) = \prod_{0<t_k<t}(1-\gamma_{1k})x_1^*(t)$, $z_2^*(t) = \prod_{0<t_k<t}(1-\gamma_{2k})x_2^*(t)$, $z_3^*(t) = \prod_{0<t_k<t}(1-\gamma_{3k})x_3^*(t))$ is the positive $\omega$ periodic solution of (4.1), and $z(t)$ is the positive solution of (4.1). It follows from Lemma 4.2 and 4.3 that there exists positive constants $T > 0$, $M_i$ and $m_i$ (defined by Lemmas 4.2 and 4.3, resp.) such that, for all $t > T$,

$$m_i < z_i^*(t) \leq M_i, \quad m_i < z_i(t) \leq M_i, \; i = 1, 2, 3. \tag{4.25}$$

Define

$$V(t) = \left|\ln z_1^*(t) - \ln z_1(t)\right| + \left|\ln z_2^*(t) - \ln z_2(t)\right| + \left|\ln z_3^*(t) - \ln z_3(t)\right|. \tag{4.26}$$

Calculating the upper-right derivative of $V(t)$ along the solution of (4.1), it follows for $t \geq T$ that

$$\begin{aligned}
D^+V(t) &= \sum_{i=1}^{3}\left(\frac{z_i^{*\prime}(t)}{z_i^*(t)} - \frac{z_i{}'(t)}{z_i(t)}\right)\operatorname{sgn}\left(z_i^*(t) - z_i(t)\right) \\
&= \operatorname{sgn}\left(z_1^*(t) - z_1(t)\right)\left[b_1(t)\frac{\prod_{0<t_k<t}(1-\gamma_{2k})}{\prod_{0<t_k<t}(1-\gamma_{1k})}\left(\frac{z_2^*(t)}{z_1^*(t)} - \frac{z_2(t)}{z_1(t)}\right)\right] \\
&\quad + \operatorname{sgn}\left(z_2^*(t) - z_2(t)\right)\Bigg[-c(t)\prod_{0<t_k<t}(1-\gamma_{2k})\left(z_2^*(t) - z_2(t)\right) \\
&\qquad\qquad -\beta_1(t)\prod_{0<t_k<t}(1-\gamma_{3k})\left(z_3^*(t) - z_3(t)\right)\Bigg] \\
&\quad + \operatorname{sgn}\left(z_3^*(t) - z_3(t)\right)\Bigg[-e(t)\prod_{0<t_k<t}(1-\gamma_{3k})\left(z_3^*(t) - z_3(t)\right) \\
&\qquad\qquad -\beta_2(t)\prod_{0<t_k<t}(1-\gamma_{2k})\left(z_2^*(t) - z_2(t)\right)\Bigg] \\
&\leq -c(t)\prod_{0<t_k<t}(1-\gamma_{2k})\left|z_2^*(t) - z_2(t)\right| + \beta_1(t)\prod_{0<t_k<t}(1-\gamma_{3k})\left|z_3^*(t) - z_3(t)\right| \\
&\quad - e(t)\prod_{0<t_k<t}(1-\gamma_{3k})\left|z_3^*(t) - z_3(t)\right| + \beta_2(t)\prod_{0<t_k<t}(1-\gamma_{2k})\left|z_2^*(t) - z_2(t)\right| + D_1(t),
\end{aligned} \tag{4.27}$$

where

$$D_1(t) = \begin{cases} b_1(t)\dfrac{\prod_{0<t_k<t}(1-\gamma_{2k})}{\prod_{0<t_k<t}(1-\gamma_{1k})}\left(\dfrac{z_2^*(t)}{z_1^*(t)} - \dfrac{z_2(t)}{z_1(t)}\right), & z_1^*(t) > z_1(t), \\[2ex] b_1(t)\dfrac{\prod_{0<t_k<t}(1-\gamma_{2k})}{\prod_{0<t_k<t}(1-\gamma_{1k})}\left(\dfrac{z_2^*(t)}{z_1^*(t)} - \dfrac{z_2(t)}{z_1(t)}\right), & z_1^*(t) < z_1(t). \end{cases} \tag{4.28}$$

In the sequel, we will estimate $D_1(t)$ under the following two cases.

(i) If $z_1^*(t) \geq z_1(t)$, then

$$\begin{aligned}
D_1(t) &\leq \frac{b_1(t)\prod_{0<t_k<t}(1-\gamma_{2k})}{z_1^*(t)\prod_{0<t_k<t}(1-\gamma_{1k})}\left(z_2^*(t) - z_2(t)\right) \\
&\leq \frac{b^M\prod_{0<t_k<t}(1-\gamma_{2k})}{m_1\prod_{0<t_k<t}(1-\gamma_{1k})}\left|z_2^*(t) - z_2(t)\right|.
\end{aligned} \tag{4.29}$$

(ii) If $z_1^*(t) < z_1(t)$, then

$$D_1(t) \le \frac{b_1(t)\prod_{0<t_k<t}(1-\gamma_{2k})}{z_1(t)\prod_{0<t_k<t}(1-\gamma_{1k})}\left(z_2(t)-z_2^*(t)\right) \le \frac{b^M\prod_{0<t_k<t}(1-\gamma_{2k})}{m_1\prod_{0<t_k<t}(1-\gamma_{1k})}\left|z_2^*(t)-z_2(t)\right|. \tag{4.30}$$

Combining the conclusions of (4.29) and (4.30), we obtain

$$D_1(t) \le \frac{b^M\prod_{0<t_k<t}(1-\gamma_{2k})}{m_1\prod_{0<t_k<t}(1-\gamma_{1k})}\left|z_2^*(t)-z_2(t)\right|. \tag{4.31}$$

It follows from (4.31) that

$$\begin{aligned}
D^+V(t) \le {} & -c(t)\prod_{0<t_k<t}(1-\gamma_{2k})\left|z_2^*(t)-z_2(t)\right| + \beta_1(t)\prod_{0<t_k<t}(1-\gamma_{3k})\left|z_3^*(t)-z_3(t)\right| \\
& - e(t)\prod_{0<t_k<t}(1-\gamma_{3k})\left|z_3^*(t)-z_3(t)\right| + \beta_2(t)\prod_{0<t_k<t}(1-\gamma_{2k})\left|z_2^*(t)-z_2(t)\right| \\
& + \frac{b^M\prod_{0<t_k<t}(1-\gamma_{2k})}{m_1\prod_{0<t_k<t}(1-\gamma_{1k})}\left|z_2^*(t)-z_2(t)\right| \\
= {} & \left[\frac{b^M}{m_1\prod_{0<t_k<t}(1-\gamma_{1k})} - c(t) + \beta_2(t)\right]\prod_{0<t_k<t}(1-\gamma_{2k})\left|z_2^*(t)-z_2(t)\right| \\
& + \left[\beta_1(t)-e(t)\right]\prod_{0<t_k<t}(1-\gamma_{3k})\left|z_3^*(t)-z_3(t)\right| \\
\le {} & -\left(B_1(t)\left|z_2^*(t)-z_2(t)\right| + B_2(t)\left|z_3^*(t)-z_3(t)\right|\right),
\end{aligned} \tag{4.32}$$

where $B_1(t)$ and $B_2(t)$ are defined in Theorem 4.4.

By hypothesis ($H_8$), there exist constants $\alpha_i$, $(i=2,3)$ and $T^* > T$ such that

$$B_i(t) \ge \alpha_i > 0, \quad (i=2,3), \quad \text{for } t \ge T^*. \tag{4.33}$$

Integrating both sides of (4.32) on interval $[T^*, t]$ yields

$$V(t) + \sum_{i=2}^{3}\int_{T^*}^{t} B_i(t)\left|z_i^*(t)-z_i(t)\right|ds \le V(T^*). \tag{4.34}$$

It follows from (4.33) and (4.34) that

$$\sum_{i=2}^{3}\int_{T^*}^{t} B_i(t)\left|z_i^*(t)-z_i(t)\right|ds \le V(T^*) < \infty, \quad \text{for } t \ge T^*. \tag{4.35}$$

Since $z_i^*(t)$ and $z_i(t)$ $(i=2,3)$ are bounded for $t \ge T^*$, so $|z_i^*(t)-z_i(t)|$ $(i=2,3)$ are uniformly continuous on $[T^*,\infty)$. By Barbalat's Lemma [32], we have

$$\lim_{t\to\infty}\left|z_i^*(t)-z_i(t)\right| = \lim_{t\to\infty}\left[\prod_{0<t_k<t}(1-\gamma_{ik})^{-1}\left|x_i^*(t)-x_i(t)\right|\right] = 0, \quad (i=2,3). \tag{4.36}$$

Thus,

$$\lim_{t\to\infty}\left|x_i^*(t)-x_i(t)\right| = 0, \quad (i=2,3). \tag{4.37}$$

By (4.37) and the first equation of (2.4), one can easily obtain that

$$\lim_{t\to\infty}\left|x_1^*(t)-x_1(t)\right| = 0. \tag{4.38}$$

By Theorems 7.4 and 8.2 in [33], we know that the positive periodic solution $x^*(t) = (x_1^*(t),x_2^*(t),x_3^*(t))^T$ of (2.4) is uniformly asymptotically stable. The proof of Theorem 4.4 is complete. □

## 5. An Example

As an application of our main results, we consider the following system:

$$\begin{aligned}
\dot{x}_1(t) &= -2x_1(t)+x_2(t), \quad t\neq t_k,\\
\dot{x}_2(t) &= (4+\cos t)x_2(t)-(2+\cos t)x_2(t)-(1-\sin t)x_2^2(t)\\
&\quad -\frac{1}{2e^{200\pi+1}}+\sin t x_2(t)x_3(t), \quad t\neq t_k,\\
\dot{x}_3(t) &= x_3(t)\left[50+\sin t-\left(50e^{200\pi}+1+\sin t\right)x_3(t)-\left(\frac{49}{e^{50}3\pi}-\cos t\right)x_2(t)\right], \quad t\neq t_k,\\
\Delta x_1(t_k) &= x_1\left(t_k^+\right)-x_1\left(t_k^-\right) = -\frac{1}{2}x_1(t_k), \quad k=1,2,\ldots,\\
\Delta x_2(t_k) &= x_2\left(t_k^+\right)-x_2\left(t_k^-\right) = -\frac{1}{3}x_2(t_k), \quad k=1,2,\ldots,\\
\Delta x_3(t_k) &= x_3\left(t_k^+\right)-x_3\left(t_k^-\right) = -\frac{1}{4}x_3(t_k), \quad k=1,2,\ldots,
\end{aligned} \tag{5.1}$$

in which $t_{k+2} = t_k + 2\pi$, $[0, 2\pi] \cap \{t_k\} = \{t_1, t_2\}$, $a_1(t) = 2$, $b_1(t) = 1$, $a_2(t) = 4 + \cos t$, $b_2(t) = 2 + \cos t$, $\beta_1(t) = (1/2e^{200\pi+1}) + \sin t$, $\beta_2(t) = (49/e^{50}3\pi) - \cos t$, $c(t) = 1 - \sin t$, $d(t) = 50 + \sin t$, and $e(t) = 50e^{200\pi} + 1 + \sin t$. By direct computation, we can obtain

$$\begin{gathered}
\overline{a}_1 = 2, \qquad \overline{a}_2 = 4, \qquad \overline{c} = 1, \qquad \overline{|a_2 - b_2|} = 2, \qquad \overline{d} = 50, \qquad \overline{\beta}_1 = \frac{1}{2e^{200\pi+1}}, \\
\overline{\beta}_2 = \frac{49}{e^{50}3\pi}, \qquad \overline{e} = 50e^{200\pi} + 1, \qquad \overline{b}_1 = 1, \\
B_1 = \ln \frac{4\pi + 2\ln(2/3)}{2\pi} + 16\pi + 2\ln\frac{2}{3} \doteq 50.07, \\
B_2 = \ln \frac{4\pi - 2\ln(1/2)}{2\pi} - 16\pi - 2\ln\frac{2}{3} \doteq -47.07, \\
B_6 = B_{27} = \ln \frac{100\pi - 2\ln(3/4)}{2\pi(50e^{200\pi})} + 200\pi + 2\ln\frac{3}{4} \doteq -0.5754, \\
B_{26} = \ln \frac{2\pi + 2\ln(2/3)}{\pi / e^{200\pi+1}} \doteq 629.55, \\
B_{28} = B_{26} - 200\pi - 2\ln\frac{3}{4} \doteq 2.9362.
\end{gathered} \tag{5.2}$$

Then $B_4 \doteq 50.07$, $B_{29} \doteq 2.9362$. It is easy to check that (5.1) satisfies all the conditions of Theorems 3.2 and 4.4; hence, (5.1) has a positive $2\pi$ periodic solution which is global attractivity.

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