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Research Article

Quasimultipliers on F-Algebras

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We investigate the extent to which the study of quasimultipliers can be made beyond Banach algebras. We will focus mainly on the class of F-algebras, in particular on complete k-normed algebras, $0 < k \leq 1$, not necessarily locally convex. We include a few counterexamples to demonstrate that some of our results do not carry over to general F-algebras. The bilinearity and joint continuity of quasimultipliers on an F-algebra A are obtained under the assumption of strong factorability. Further, we establish several properties of the strict and quasistrict topologies on the algebra $QM(A)$ of quasimultipliers of a complete k-normed algebra A having a minimal ultra-approximate identity.

1. Introduction

A quasimultiplier is a generalization of the notion of a left (right, double) multiplier and was first introduced by Akemann and Pedersen in [1, Section 4]. The first systematic account of the general theory of quasimultipliers on a Banach algebra with a bounded approximate identity was given in a paper by McKennon [2] in 1977. Further developments have been made, among others, by Vasudevan and Goel [3], Kassem and Rowlands [4], Lin [5, 6], Dearden [7], Argün and Rowlands [8], Grosser [9], Yılmaz and Rowlands [10], and Kaneda [11, 12].

In this paper, we consider the notion of quasimultipliers on certain topological algebras and give an account, how far one can get beyond Banach algebras, using combination of standard methods. In particular, we are able to establish some results of the above authors in the framework of F-algebras or complete k-normed algebras.

2. Preliminaries

Definition 2.1. Let E be a vector space over the field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$).

(1) A function $q : E \to \mathbb{R}$ is called an *F-seminorm* on E if it satisfies

(F_1) $q(x) \geq 0$ for all $x \in E$,
(F_2) $q(x) = 0$ if $x = 0$,
(F_3) $q(\alpha x) \leq q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$,
(F_4) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$,
(F_5) if $\alpha_n \to 0$ in $\mathbb{K}$, then $q(\alpha_n x) \to 0$ for all $x \in E$.

(2) An F-seminorm q on E is called an *F-norm* if, for any $x \in E$, $q(x) = 0$ implies that $x = 0$.

(3) An F-seminorm (or F-norm) q on E is called *k-homogeneous* ([13, page 160]; [14, pages 90, 95]), where $0 < k \leq 1$, if it also satisfies

(F'_3) $q(\alpha x) = |\alpha|^k q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$.

(4) A k-homogeneous F-seminorm (resp., F-norm) on E is called, in short, a *k-seminorm* (resp., *k-norm*).

Definition 2.2. (1) A vector space with an F-norm q is called an *F-normed space* and is denoted by (E, q); if it is also complete, it is called an F-space. Clearly, any F-normed space (E, q) is a metrizable TVS with metric given by $d(x, y) = q(x - y)$, $x, y \in E$.

(2) An F-seminorm (or F-norm) q on an algebra A is called *submultiplicative* if

$$q(xy) \leq q(x)q(y), \quad \forall x, y \in A. \tag{2.1}$$

An algebra with a submultiplicative F-norm q is called an *F-normed algebra;* if it is also complete, it is called an F-algebra. An algebra with a submultiplicative k-norm q is called an *k-normed algebra.* A complete k-normed algebra is also called a *k-Banach algebra* in the literature.

Theorem 2.3. (a) *If (E, τ) is TVS, then its topology τ can be defined by a family of F-seminorms (see [15, pages 48–51]; [16, pages 2-3]).*

(b) *If (E, τ) is a metrizable TVS, then τ may be defined by a single F-norm (see [13, 15, 17]).*

(c) *If (E, τ) is a Hausdorff locally bounded TVS, then τ may be a single k-norm for some k,* $0 < k \leq 1$ *(see [13, 14]).*

Note that if (E, q) is an F-normed space, then, for any $\varepsilon > 0$, the set $\{x \in E : q(x) \leq \varepsilon\}$ is a neighbourhood of 0 in E, but it need not be a bounded set. In case, if $\{x \in E : q(x) \leq \varepsilon\}$ is bounded for some $\varepsilon > 0$, then (E, q) becomes a Hausdorff locally bounded TVS and hence, by Theorem 2.3(c), a k-normed space for some k, $0 < k \leq 1$.

Definition 2.4. Let A be an algebra over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and τ a topology on A such that (A, τ) is a TVS. Then the pair (A, τ) is called a *topological algebra* if it has a separately continuous multiplication. A topological algebra A is said to be *locally bounded* if it has a bounded neighbourhood of 0 (see [18, page 39]). If (A, τ) is a complete Hausdorff locally bounded topological algebra, then its topology can be defined by a submultiplicative k-norm q, $0 < k \leq 1$ [18, page 41].

By a famous result of Arens (see [18, page 24]), every Baire metrizable topological algebra has jointly continuous multiplication; in particular, every F-algebra has jointly continuous multiplication.

For the general theory and undefined terms, the reader is referred to [13, 15–17, 19] for topological vector spaces, to [13, 14, 20] for F-normed and k-normed spaces, and to [18, 21, 22, pages 32–35] for various classes of topological algebras.

If E and F are topological vector spaces over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), then the set of all continuous linear mappings $T : E \to F$ is denoted by $\mathrm{CL}(E, F)$. Clearly, $\mathrm{CL}(E, F)$ is a vector space over $\mathbb{K}$ with the usual pointwise operations. Further, if $F = \mathrm{E}$, $\mathrm{CL}(E) = \mathrm{CL}(E, E)$ is an algebra under composition (i.e., $(ST)(x) = S(T(x))$, $x \in E$) and has the identity $I : E \to E$ given by $I(x) = x$ $(x \in E)$.

We now state the following three versions of the uniform boundedness principle for reference purpose.

Theorem 2.5 (see [23, page 142, principle 33.1]). *Let X be a complete metric space and $\mathscr{H} = \{f_\alpha : \alpha \in J\}$ a family of continuous real-valued functions on X. If $\mathscr{H}$ is pointwise bounded from above, then on a certain closed ball $B \subseteq X$ it is uniformly bounded above, that is, there exists a constant $C > 0$ such that*

$$f_\alpha(x) \leq C, \quad \forall \alpha \in J, \ x \in B. \tag{2.2}$$

Theorem 2.6 (see [14, page 39]; [19, page 465]). *Let E be an F-space and F any topological vector space. Let $\mathscr{H} \subseteq \mathrm{CL}(E, F)$ be a collection such that $\mathscr{H}$ is pointwise bounded on E. Then $\mathscr{H}$ is equicontinuous; hence, for any bounded set D in E, $\cup\{T(D) : T \in \mathscr{H}\}$ is a bounded set in F.*

The following version is for bilinear mappings.

Theorem 2.7. *Let E and F be F-spaces and G any TVS.*

(a) *A collection $\mathscr{H}$ of bilinear mappings from $E \times F$ into G is equicontinuous if and only if each $f \in \mathscr{H}$ is separately continuous and $\mathscr{H}$ is pointwise bounded on $E \times F$. In particular, every separately continuous bilinear map $f : E \times F \longrightarrow G$ is jointly continuous (see [13, page 172]; [19, page 489]).*

(b) *Let $f_n : E \times F \longrightarrow G$ be a sequence of separately continuous bilinear mappings such that $\lim_{n\to\infty} f_n(x, y) = f(x, y)$ exists for each $(x, y) \in E \times F$. Then $\{f_n\}$ is equicontinuous and f is bilinear and jointly continuous (see [19, page 490]; [24, page 328]).*

Definition 2.8. (1) A net $\{e_\lambda : \lambda \in I\}$ in a topological algebra A is called an approximate *identity* if

$$\lim_\lambda e_\lambda a = a = \lim_\lambda a e_\lambda, \quad \forall a \in A. \tag{2.3}$$

(2) An approximate identity $\{e_\lambda : \lambda \in I\}$ in an F-normed algebra (A, q) is said to be *minimal* if $q(e_\lambda) \leq 1$ for all $\lambda \in I$.

(3) An algebra A is said to be *left* (resp., *right*) *faithful* if, for any $a \in A$, $aA = \{0\}$ (resp., $Aa = \{0\}$) implies that $a = 0$; A is called *faithful* if it is both left and right faithful. One mentions that A is faithful in each of the following cases:

(i) A is a topological algebra with an approximate identity (e.g., A is a locally C^*-algebra);

(ii) A is a topological algebra with an orthogonal basis [25].

Definition 2.9. A topological algebra A is called

(1) *factorable* if, for each $a \in A$, there exist $b, c \in A$ such that $a = bc$,

(2) *strongly factorable* if, for any sequence $\{a_n\}$ in A with $a_n \to 0$, there exist $a \in A$ and a sequence $\{b_n\}$ (resp., $\{c_n\}$) in A with $b_n \to 0$ (resp., $c_n \to 0$) such that $a_n = ab_n$ (resp., $a_n = c_n a$) for all $n \geq 1$.

Clearly, every strongly factorable algebra is factorable. Factorization in Banach and topological algebras plays an important role in the study of multipliers and quasimultipliers. There are several versions of the famous Hewitt-Cohen's factorization theorem in the literature (see, e.g., the book [26] and its references). Using the terminology of [27], we state the following version in the nonlocally convex case.

Theorem 2.10 (see [27]). *Let A be a fundamental F-algebra with a uniformly bounded left approximate identity. Then A is strongly factorable.*

Definition 2.11. Let (A, q) be an F-normed space (in particular, an F-normed algebra). For any $T \in \mathrm{CL}(A)$, let

$$\|T\|_q = \sup\left\{ \frac{q(T(x))}{q(x)} : x \in A, x \neq 0 \right\}. \tag{$*$}$$

It is easy to see that if q is a k-norm, $0 < k \leq 1$, (resp., a seminorm) on A, then $\|\cdot\|_q$ is a k-norm (resp., a seminorm) on $\mathrm{CL}(A)$; further, in these cases, we have alternate formulas for $\|T\|_q$ as

$$\begin{aligned} \|T\|_q &= \sup\{q(T(x)) : x \in A, q(x) = 1\} \\ &= \sup\{q(T(x)) : x \in \mathrm{A}, q(x) \leq 1\}, \end{aligned} \tag{2.4}$$

for each $T \in \mathrm{CL}(A)$ (see [14, pages 101-102]; [28, pages 3–5]; [19, page 87]).

Remark 2.12. In an earlier version of this paper, the authors had erroneously made the blank assumption that, for q an F-norm on A, $\|T\|_q$ given by $(*)$ always exists for each $T \in \mathrm{CL}(A)$. We are grateful to referee for pointing out that this assumption cannot be justified in view of the following counterexamples.

(1) First, $\|T\|_q$ need not be finite for a general F-norm. For example, let $A = \mathbb{R}^2$, $q(x_1, x_2) = |x_1| + |x_2|^{1/2}$, $T(x_1, x_2) = (x_2, x_1)$. Then q is an F-norm on A, but $\|T\|_q = \infty$: for any $n \in \mathbb{N}$,

$$\|T\|_q \geq \frac{q[T(n, n^2)]}{q(n, n^2)} = \frac{q(n^2, n)}{q(n, n^2)} = \frac{n^2 + n^{1/2}}{2n} \longrightarrow \infty. \tag{2.5}$$

(2) Even when considering the subspace of those T for which $\|T\|_q < \infty$, then $\|\cdot\|_q$ need not always be an F-norm, since (F_5) need not hold. For example, for a fixed sequence (p_n) with $0 < p_n \leq 1$, $p_n \to 0$, consider the F-algebra A of sequences $(x_n) \subseteq \mathbb{R}$ with $|x_n|^{p_n} \to 0$ and $q((x_n)) = \sup_{n\geq 1}|x_n|^{p_n}$. Then $\|T\|_q < \infty$ for all multipliers of A, but $\|\cdot\|_q$ is not an F-norm, that is, it makes the space $\mathrm{CL}(A)$ into an additive topological group but not into a topological vector space (as it would lack the continuity of scalar multiplication in the absence of (F_5) (cf. [14, Example 1.2.3, page 8]).

In view of the above remark, we will need to assume that (A, q) is a k-normed space (or a k-normed algebra) whenever $\|T\|_q$ is considered for $T \in \mathrm{CL}(A)$ or $\mathrm{CL}(A \times A, A)$.

Some useful properties of $(\mathrm{CL}(A), \|\cdot\|_q)$ are summarized as follows.

Theorem 2.13 (see [14, pages 101-102]). *Let (A, q) be a k-normed space (in particular, a k-normed algebra) with $0 < k \leq 1$. Then:*

(a) *a linear mapping $T : A \to A$ is continuous $\Leftrightarrow \|T\|_q < \infty$;*

(b) *$\|\cdot\|_q$ is a k-norm on $\mathrm{CL}(A)$;*

(c) *$q(T(x)) \leq \|T\|_q \cdot q(x)$ for all $x \in A$;*

(d) *for any $S, T \in \mathrm{CL}(A)$, $\|ST\|_q \leq \|S\|_q \|T\|_q$; hence $(\mathrm{CL}(A), \|\cdot\|_q)$ is a k-normed algebra;*

(e) *if A is complete, then $(\mathrm{CL}(A), \|\cdot\|_q)$ is a complete k-normed algebra.*

Remark 2.14. The referee has enquired if the present theory can be considered for the class of locally convex F-algebras. It is well known (e.g., [22, page 33]; [21, page 9]) that, for this class of topological algebras, the topology can be generated by an increasing countable family of seminorms $\{q_n\}$, which need not be submultiplicative but satisfy the weaker condition $q_n(xy) \leq q_{n+1}(x)q_{n+1}(y)$; however, for locally m-convex F-algebras, the seminorms can be chosen to be submultiplicative. In view of this, we believe that a study of quasimultipliers can possibly be made on locally m-convex algebra F-algebras parallel to the one given by Phillips (see [29, pages 177–180]) for multipliers.

3. Multipliers on F-Normed Algebras

In this section, we recall definitions and results on various notions of multipliers on an algebra A (as given in [30–33]) which we shall require later in the study of quasimultipliers (see also [25, 29, 34–40]). In fact, we shall see that the proofs of most of the results on quasimultipliers are based on the properties of left, right, and double multipliers.

Definition 3.1 (see [31]). Let A be an algebra over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$).

(1) A mapping $T : A \to A$ is called a

(i) *multiplier* on A if $aT(b) = T(a)b$ for all $a, b \in A$,

(ii) *left multpilier* on A if $T(ab) = T(a)b$ for all $a, b \in A$,

(iii) *right multiplier* on A if $T(ab) = aT(b)$ for all $a, b \in A$.

(2) A pair (S, T) of mappings $S, T : A \to A$ is called a *double multiplier* on A if $aS(b) = T(a)b$ for all a, $b \in A$.

Some authors use the term *centralizer* instead of *multiplier* (see, e.g., [30, 31]).

Let $M(A)$ (resp., $M_\ell(A), M_r(A)$) denote the set of all multipliers (resp., left multipliers, right multipliers) on A and $M_d(A)$ the set of all double multipliers on an algebra A. For any $a \in A$, let $L_a, R_a : A \to A$ be given by

$$L_a(x) = ax, \quad R_a(x) = xa, \quad x \in A. \tag{3.1}$$

Clearly, $L_a \in M_\ell(A)$, $R_a \in M_r(A)$, and $(L_a, R_a) \in M_d(A)$.

For convenience, we summarize some basic properties of multipliers in the following theorems for later references.

Theorem 3.2 (see [31]). *Let A be an algebra. Then,*

(a) $M_\ell(A) \cap M_r(A) \subseteq M(A)$;

(b) *if A is faithful, then $M(A) \subseteq M_\ell(A) \cap M_r(A)$ and hence $M(A) = M_\ell(A) \cap M_r(A)$;*

(c) *if A is commutative and faithful, then $M_\ell(A) = M_r(A) = M(A)$;*

(d) *$M_\ell(A)$ and $M_r(A)$ are algebras with composition as multiplication (i.e., $(T_1T_2)(x) = T_1(T_2(x))$) and have the identity $I : A \to A$, $I(x) = x$ $(x \in A)$;*

(e) *$M(A)$ is a vector space; if, in addition, A is faithful, then $M(A)$ is a commutative algebra (without A being commutative) with identity I.*

Theorem 3.3 (see [31]). *Let A be a faithful algebra. Then,*

(a) *if $(S,T) \in M_d(A)$, then* (i) *S and T are linear and* (ii) *$S \in M_\ell(A)$ and $T \in M_r(A)$. In particular, every $T \in M(A)$ is linear;*

(b) *$M_d(A)$ is an algebra with identity (I, I) under the operations*

$$\begin{gathered}(S_1,T_1) + (S_2,T_2) = (S_1 + S_2, T_1 + T_2), \qquad \lambda(S_1,T_1) = (\lambda S_1, \lambda T_1) \quad (\lambda \in \mathbb{K}), \\ (S_1,T_1)(S_2,T_2) = (S_1S_2, T_2T_1);\end{gathered} \tag{3.2}$$

(c) *let $(S_1,T_1), (S_2,T_2) \in M_d(A)$. If $S_1 = S_2$, then $T_1 = T_2$; if $T_1 = T_2$ then $S_1 = S_2$;*

(d) *if A is commutative, then $M_d(A)$ is commutative and $M_d(A) = M(A)$; in fact, if $(S,T) \in M_d(A)$, then $S = T$.*

Definition 3.4. One defines mappings $\mu_\ell : A \to M_\ell(A)$, $\mu_r : A \to M_r(A)$, and $\mu_d : A \to M_d(A)$ by

$$\mu_\ell(a) = L_a, \quad \mu_r(a) = R_a, \quad \mu_d(a) = (L_a, R_a), \quad a \in A. \tag{3.3}$$

Theorem 3.5 (see [31]). *Let A be an algebra, and let μ_ℓ, μ_r, and μ_d be the mappings as defined above. Then,*

(a) *μ_ℓ, μ_r, and μ_d are linear;*

(b) *μ_ℓ and μ_d are algebra homomorphisms, while μ_r is an algebra antihomomorphism;*

(c) *μ_ℓ (resp., μ_r) is 1-1 $\Leftrightarrow$ A is left (resp., right) faithful; μ_d is 1-1 $\Leftrightarrow$ A is faithful;*

(d) *μ_ℓ (resp., μ_r) is onto $\Leftrightarrow$ A has left (resp., right) identity; μ_d is onto $\Leftrightarrow$ A has an identity.*

Theorem 3.6 (see [31]). *Let A be an algebra.*

(a) *For any $a \in A$ and $T \in M_\ell(A)$, $TL_a = L_{T(a)} \in \mu_\ell(A)$; hence $\mu_\ell(A)$ is a left ideal in $M_\ell(A)$.*

(b) *For any $a \in A$ and $T \in M_r(A)$, $TR_a = R_{T(a)} \in \mu_r(A)$; hence $\mu_r(A)$ is a left ideal in $M_r(A)$.*

(c) *Suppose that A is faithful. Then, for any $a \in A$ and $(S,T) \in M_d(A)$,*

$$(L_a, R_a)(S,T) = \left(L_{T(a)}, R_{T(a)}\right) \in \mu_d(A), \qquad (S,T)(L_a, R_a) = \left(L_{S(a)}, R_{S(a)}\right) \in \mu_d(A); \tag{3.4}$$

hence $\mu_d(A)$ is a two-sided ideal in $M_d(A)$.

Regarding the continuity of multipliers, we state the following.

Theorem 3.7. (a) *Suppose that A is a strongly factorable F-normed algebra. If $T \in M_\ell(A)$ (resp., $M_r(A)$), then T is linear and continuous (see [32, 33]).*

(b) *Suppose that A is a faithful F-algebra. If $(S,T) \in M_d(A)$, then S and T are linear and continuous; in particular each $T \in M(A)$ is linear and continuous (see [31, 33]).*

Convention 1. In the remaining part of this paper, unless stated otherwise, A is a topological algebra and $M(A)$ (resp., $M_\ell(A)$, $M_r(A)$) denotes the set of all continuous linear multipliers (resp., left multipliers, right multipliers) on A and $M_d(A)$ denotes the set of all double multipliers (S,T) on A with both S and T continuous and linear.

Definition 3.8 (see [31, 33]). Let A be a topological algebra. The *uniform operator topology* u (resp., the *strong operator topology* s) on $M_d(A)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all the sets of the form

$$N(D,W) = \{(S,T) \in M_d(A) : S(D) \subseteq W,\ T(D) \subseteq W\}, \tag{3.5}$$

where D is a bounded (resp., finite) subset of A and W is a neighborhood of 0 in A. Clearly, $s \leq u$. Note that the u and s topologies can also be defined on the multiplier algebras $M(A)$, $M_\ell(A)$ and $M_r(A)$ in an analogous way. (The topology s is also called the *strict topology* in the literature and denoted by β.) There is an extensive literature on the s and u topologies (see, e.g., [30, 33–35, 37–39, 41–45]).

Theorem 3.9 (see [33]). *Let A be a faithful F-algebra, and let $M_t(A)$ denote any one of the algebras $M_\ell(A)$, $M_r(A)$, $M(A)$, and $M_d(A)$. Then,*

(a) *$(M_t(A), u)$ and $(M_t(A), s)$ are topological algebras with separately continuous multiplication;*

(b) *$(M_t(A), u)$ and $(M_t(A), s)$ are complete;*

(c) *s and u have the same bounded sets;*

(d) *if $(M_t(A), s)$ is metrizable, then $s = u$ on $M_t(A)$;*

(e) *if A has a two-sided approximate identity, then A is s-dense in $M_t(A)$.*

Remark 3.10. Let (A, q) be an F-normed algebra.

(1) If (A, q) is a k-normed algebra, the u-topology on $M_\ell(A)$, $M_r(A)$, and $M(A)$ is given by the k-norm

$$\|T\|_q = \sup\left\{ \frac{q(T(x))}{q(x)} : x \in A,\ x \neq 0 \right\}; \tag{3.6}$$

the u-topology on $M_d(A)$ is given by the k-norm

$$\|(S, T)\|_q = \max\left\{ \|S\|_q, \|T\|_q \right\}, \quad (S, T) \in M_d(A). \tag{3.7}$$

(2) The s-topology on $M_\ell(A)$, $M_r(A)$, and $M(A)$ is given by the family of $\{p_a : a \in A\}$ of F-seminorms, where

$$p_a(T) = q(T(a)), \quad T \in M_\ell(A),\ M_r(A) \text{ or } M(A); \tag{3.8}$$

the s-topology on $M_d(A)$ is given by the family $\{r_a : a \in A\}$ of F-seminorms, where

$$r_a(S, T) = \max\{q(S(a)), q(T(a))\}, \quad (S, T) \in M_d(A). \tag{3.9}$$

Theorem 3.11. *Let (A, q) be a k-normed algebra having a minimal approximate identity $\{e_\lambda : \lambda \in I\}$. Then,*

(a) *for any $a \in A$, $\|L_a\|_q = \|R_a\|_q = q(a)$; so each of the maps $\mu_\ell : A \to (M_\ell(A), u)$, $\mu_r : A \to (M_r(A), u)$, $\mu_d : A \to (M_d(A), u)$ is an isometry and hence continuous;*

(b) *for any $(S, T) \in M_d(A)$, $\|S\|_q = \|T\|_q$;*

(c) *if (A, q) is complete, then A is a u-closed two-sided ideal in $M_d(A)$, under the identification $\mu_d : a \to (L_a, R_a)$.*

Proof. (a) Let $a \in A$. Then

$$\|L_a\|_q = \sup_{b \neq 0} \frac{q(L_a(b))}{q(b)} = \sup_{b \neq 0} \frac{q(ab)}{q(b)} \leq \sup_{b \neq 0} \frac{q(a) \cdot q(b)}{q(b)} = q(a). \tag{3.10}$$

On the other hand,

$$\|L_a\|_q = \sup_{b\neq 0}\frac{q(ab)}{q(b)} \geq \frac{q(ae_\lambda)}{q(e_\lambda)} \geq q(ae_\lambda), \quad \forall\lambda \in I; \tag{3.11}$$

so

$$\|L_a\|_q \geq \lim_{\lambda} q(ae_\lambda) = q\left(\lim_{\lambda} ae_\lambda\right) = q(a). \tag{3.12}$$

Hence $\|\mu_\ell(a)\|_q = \|L_a\|_q = q(a)$. Similarly, $\|\mu_r(a)\|_q = \|R_a\|_q = q(a)$. Thus

$$\left\|\mu_d(a)\right\|_q = \max\left\{\|L_a\|_q, \|R_a\|_q\right\} = q(a). \tag{3.13}$$

(b) Let $(S,T) \in M_d(A)$. Using (a), we have

$$\begin{aligned}\|S\|_q &= \sup_{a\neq 0}\frac{q(S(a))}{q(a)} = \sup_{a\neq 0}\frac{\left\|R_{S(a)}\right\|_q}{q(a)} = \sup_{a\neq 0}\sup_{b\neq 0}\frac{1}{q(a)}\cdot\frac{q\left[R_{S(a)}(b)\right]}{q(b)} \\ &= \sup_{a\neq 0}\sup_{b\neq 0}\frac{q[bS(a)]}{q(a)\cdot q(b)} = \sup_{a\neq 0}\sup_{b\neq 0}\frac{q[T(b)a]}{q(a)\cdot q(b)} \\ &\leq \sup_{a\neq 0}\sup_{b\neq 0}\frac{q(T(b))\cdot q(a)}{q(a)\cdot q(b)} = \sup_{b\neq 0}\frac{q(T(b))}{q(b)} = \|T\|_q.\end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned}\|T\|_q &= \sup_{a\neq 0}\frac{\left\|L_{T(a)}\right\|_q}{q(a)} = \sup_{a\neq 0}\sup_{b\neq 0}\frac{q[T(a)b]}{q(a)\cdot q(b)} \\ &= \sup_{a\neq 0}\sup_{b\neq 0}\frac{q[a\cdot S(b)]}{q(a)\cdot q(b)} \leq \|S\|_q.\end{aligned} \tag{3.15}$$

Thus, $\|S\|_q = \|T\|_q$.

(c) In view of Theorem 3.6(c), we only need to show that $\mu_d(A)$ is u-closed in $M_d(A)$. Let $(S,T) \in M_d(A)$ with $(S,T) \in u\text{-cl}(\mu_d(A))$. Choose $\{a_\alpha : \alpha \in J\} \subseteq A$ such that $\{L_{a_\alpha}, R_{a_\alpha}\} \xrightarrow{u} (S,T)$. By part (a), μ_d is an isometry. Hence $\{a_\alpha : \alpha \in J\}$ is a Cauchy net in A. Then

$$(S,T) = u - \lim_{\alpha} \mu_d(a_\alpha) \in \mu_d(A). \tag{3.16}$$

Thus $\mu_d(A)$ is u-closed in $M_d(A)$. □

4. Quasimultipliers on F-Algebras and k-Normed Algebras

In this section, we consider the notion of quasimultipliers on F-algebras and complete k-normed algebra and extend several basic results of McKennon [2], Kassem and Rowlands [4], Argün and Rowlands [8], and Yılmaz and Rowlands [10] from Banach algebras to these classes of topological algebras.

Definition 4.1 (see [2, 4]). Let A be an algebra. A mapping $m : A \times A \to A$ is said to be a *quasimultiplier* on A if

$$m(ab,c) = am(b,c), \qquad m(a,bc) = m(a,b)c, \tag{4.1}$$

for all $a,b,c \in A$.

The following Lemma shows in particular that every left multiplier, right multiplier, multiplier, and double multiplier on an algebra A can be viewed as quasimultiplier on A.

Lemma 4.2. *Let A be a faithful algebra.*

(a) *For any $c \in A$, define $m = m_c : A \times A \to A$ by*

$$m_c(a,b) = acb, \quad \forall (a,b) \in A \times A. \tag{4.2}$$

(b) *For any $T \in M_\ell(A)$, define an associated map $m = m_T : A \times A \to A$ by*

$$m_T(a,b) = aT(b), \quad \forall (a,b) \in A \times A. \tag{4.3}$$

(c) *For any $T \in M_r(A)$, define an associated map $m = m_T : A \times A \to A$ by*

$$m_T(a,b) = T(a)b, \quad \forall (a,b) \in A \times A. \tag{4.4}$$

(d) *For any $T \in M(A)$, define an associated map $m = m_T : A \times A \to A$ by*

$$m_T(a,b) = aT(b), \quad \forall (a,b) \in A \times A. \tag{4.5}$$

(e) *For any $(S,T) \in M_d(A)$, define an associated map $m = m_{(S,T)} : A \times A \to A$ by*

$$m_{(S,T)}(a,b) = aS(b), \quad \forall (a,b) \in A \times A. \tag{4.6}$$

Then each of the maps $m : A \times A \to A$ defined above is a quasimultiplier on A.

Proof. We only prove (e). Let $(S,T) \in M_d(A)$, for any $a,b,c,d \in A$,

$$\begin{aligned} m(ab,c) &= m_{(S,T)}(ab,c) = (ab)S(c) = a[bS(c)] \\ &= a\left[m_{(S,T)}(b,c)\right] = am(b,c). \end{aligned} \tag{4.7}$$

In a similar way, $m(a,bc) = m(a,b)c$. □

Theorem 4.3. *Suppose that (A,τ) is a strongly factorable F-algebra. Then,*

(a) *A map $m : A \times A \to A$ is a quasimultiplier on A if and only if*

$$m(ab,cd) = am(b,c)d, \quad \forall a,b,c,d \in A; \tag{4.8}$$

(b) *every quasimultiplier m on A is bilinear;*

(c) *every quasimultiplier m on A is jointly continuous.*

Proof. (a) If m is a quasimultiplier on A, then clearly, for any $a,b,c,d \in A$,

$$m(ab,cd) = am(a,cd) = am(b,c)d. \tag{4.9}$$

Conversely, let $a,b,c \in A$. Since A is a strongly factorable and $a_n = \{b,c,0,0,\ldots\} \to 0$, there exist $y,z,w \in A$ such that $b = wy$, $c = wz$. Then, using (4.8),

$$m(ab,c) = m(awy,wz) = (aw)\left[m(y,w)\right]z = am(wy,wz) = am(b,c). \tag{4.10}$$

Similarly, we obtain $m(a,bc) = m(a,b)c$.

(b) Let $a,b,c \in A$ and $\alpha \in \mathbb{K}$. Choose, as above, $x,y,w \in A$ such that $a = wx$, $b = wy$. Then,

$$\begin{aligned} m(a+b,c) &= m(wx+wy,c) = (x+y)m(w,c) = xm(w,c) + ym(w,c) \\ &= m(wx,c) + m(wy,c) = m(a,c) + m(b,c). \end{aligned} \tag{4.11}$$

Similarly, $m(c,a+b) = m(c,a) + m(c,b)$. Next,

$$m(\alpha a,c) = m(\alpha wx,c) = (\alpha w)m(x,bc) = \alpha m(wx,c) = \alpha m(a,c). \tag{4.12}$$

First, we show that m is separately continuous. Let $a \in A$ and $\{x_n\} \subseteq A$ with limit x. Then $\{x_n - x\}$ converges to 0. By strong factorability, there exist a sequence $\{z_n\}$ and an element z of A such that $z_n \to 0$, $x_n - x = zz_n$. Thus,

$$m(a,x_n) - m(a,x) = m(a,x_n - x) = m(a,zz_n) = m(a,z)z_n \longrightarrow 0. \tag{4.13}$$

Now, the joint continuity of m follows directly from Theorem 2.7(a). □

Theorem 4.4. *Let A be a commutative algebra. Then,*

(a) *$am(b,c) = m(b,a)c$ for any quasimultiplier on A and $a,b,c \in A$;*

(b) *if A is also faithful, then a bilinear map $m : A \times A \to A$ is a quasimultiplier on A if and only if*

$$m\left(a^2, b\right) = am(a,b), \quad m\left(a, b^2\right) = m(a,b)b \quad \forall a,b \in A. \tag{4.14}$$

Proof. (a) By hypothesis,

$$am(b,c) = m(b,c)a = m(b,ca) = m(b,ac) = m(b,a)c. \tag{4.15}$$

(b) ($\Rightarrow$) This is obvious.

($\Leftarrow$) For all $a,b,c \in A$, using (4.14),

$$\begin{aligned} m\left((a+b)^2, c\right) &= (a+b)m(a+b,c) \\ &= am(a,c) + am(b,c) + bm(a,c) + bm(b,c). \end{aligned} \tag{4.16}$$

On the other hand, we have

$$\begin{aligned} m\left((a+b)^2, c\right) &= m\left(a^2 + b^2 + 2ab, c\right) \\ &= m\left(a^2, c\right) + m\left(b^2, c\right) + 2m(ab,c) \\ &= am(a,c) + bm(b,c) + 2m(ab,c). \end{aligned} \tag{4.17}$$

Comparing (4.16) and (4.17), we obtain

$$2m(ab,c) = am(b,c) + bm(a,c). \tag{4.18}$$

Now, for all $a,b,c,d \in A$, using (4.18) twice,

$$\begin{aligned} 2m(abd,c) &= abm(d,c) + dm(ab,c) \\ &= abm(d,c) + \frac{1}{2}d[2m(ab,c)] \\ &= abm(d,c) + \frac{1}{2}dam(b,c) + \frac{1}{2}dbm(a,c). \end{aligned} \tag{4.19}$$

By commutativity of A and using (4.18) twice as above,

$$\begin{aligned}2m(abd,c) = 2m(adb,c) &= adm(b,c) + bm(ad,c)\\ &= adm(b,c) + \frac{1}{2}[bam(d,c) + bdm(a,c)]\\ &= adm(b,c) + \frac{1}{2}abm(d,c) + \frac{1}{2}dbm(a,c).\end{aligned} \tag{4.20}$$

Comparing (4.19) and (4.20), $abm(d,c) = adm(b,c)$. Since this holds for all $a \in A$ and A is faithful, $bm(d,c) = dm(b,c)$. Hence, for all $d \in A$,

$$dm(ab,c) = abm(d,c) = a[bm(d,c)] = adm(b,c) = dam(b,c). \tag{4.21}$$

Since this holds for all $d \in A$ and A is faithful, $m(ab,c) = am(b,c)$.

A similar computation shows that $m(a,bc) = m(a,b)c$. Hence m is a quasimultiplier. □

Definition 4.5. Let $QM(A)$ denote the set of all bilinear jointly continuous quasimultipliers on a topological algebra A. Clearly, $QM(A)$ is a vector space under the usual pointwise operations. Further, $QM(A)$ becomes an A-bimodule as follows. For any $m \in QM(A)$ and $a \in A$, we can define the products $a \circ m$ and $m \circ a$ as mappings from $A \times A$ into A given by

$$(a \circ m)(x,y) = m(xa,y), \quad (m \circ a)(x,y) = m(x,ay), \quad x,y \in A. \tag{4.22}$$

Then $a \circ m,\ m \circ a \in QM(A)$, so that $QM(A)$ is an A-bimodule.

Definition 4.6. Let (A,q) be an F-normed algebra. Following [2, 4, 8], we can define mappings

$$\begin{aligned}\phi_A : A \longrightarrow QM(A), &\qquad \phi_\ell : M_\ell(A) \longrightarrow QM(A),\\ \phi_r : M_r(A) \longrightarrow QM(A), &\qquad \phi_d : M_d(A) \longrightarrow QM(A),\end{aligned} \tag{4.23}$$

by

$$\begin{aligned}(\phi_A(a))(x,y) &= xay, \quad a \in A,\\ (\phi_\ell(T))(x,y) &= xT(y), \quad T \in M_\ell(A),\\ (\phi_r(T))(x,y) &= T(x)y, \quad T \in M_r(A),\\ (\phi_d(S,T))(x,y) &= xS(y), \quad (S,T) \in M_d(A),\end{aligned} \tag{4.24}$$

for all $(x,y) \in A \times A$. By Lemma 4.2, these mappings are well defined.

Definition 4.7. (1) A bounded approximate identity $\{e_\lambda : \lambda \in I\}$ in a topological algebra A is said to be *ultra-approximate* if, for all $m \in QM(A)$ and $a \in A$, the nets $\{m(a, e_\lambda) : \lambda \in I\}$ and $\{m(e_\lambda, a) : \lambda \in I\}$ are Cauchy in A (see [2]).

(2) A topological algebra A is called *m-symmetric* if, for each $S \in M_\ell(A) \cup M_r(A)$, there is a $T \in M_\ell(A) \cup M_r(A)$ such that either $(S, T) \in M_d(A)$ or $(T, S) \in M_d(A)$ (see [4]).

Theorem 4.8. *Let (A, q) be an F-algebra with a bounded approximate identity $\{e_\lambda : \lambda \in I\}$. Consider the following conditions.*

(a) *$\{e_\lambda : \lambda \in I\}$ is ultra-approximate.*

(b) *For any $a \in A$, $S \in M_\ell(A)$, and $T \in M_r(A)$, the nets $\{aS(e_\lambda)\}$ and $\{T(e_\lambda)a\}$ are Cauchy in A.*

(c) *A is m-symmetric.*

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). *If A is factorable, then* (c) $\Rightarrow$ (a)*; hence* (a), (b), *and* (c) *are equivalent.*

Proof. (a) $\Rightarrow$ (b) Suppose that $\{e_\lambda : \lambda \in I\}$ in A is ultra-approximate. Let $S \in M_\ell(A)$ and $T \in M_r(A)$. Put $m_1 = \phi_\ell(S)$ and $m_2 = \phi_r(T)$. Then $m_1, m_2 \in QM(A)$ and, for any $a \in A$, $\{m_1(a, e_\lambda)\} = \{aS(e_\lambda)\}$ and $\{m_1(e_\lambda, a)\} = \{T(e_\lambda)a\}$ which are Cauchy in A by hypothesis.

(b) $\Rightarrow$ (c) Suppose that (b) holds, let $S \in M_\ell(A) \cup M_r(A)$, and suppose that $S \in M_\ell(A)$. Since A is complete, the map $T : A \to A$ given by

$$T(a) = \lim_\lambda aS(e_\lambda), \quad a \in A, \tag{4.25}$$

is well-defined. Since S is continuous, for any $a, b \in A$,

$$aS(b) = a \lim_\lambda S(e_\lambda b) = a \lim_\lambda S(e_\lambda)b = \left[\lim_\lambda aS(e_\lambda)\right] b = T(a)b. \tag{4.26}$$

Since A is a faithful F-algebra, by Theorem 3.7(b), $(S, T) \in M_d(A)$. Hence A is m-symmetric.

(c) $\Rightarrow$ (b) Suppose that (c) holds. Let $S \in M_\ell(A)$ and $T \in M_r(A)$. By (c), there exist $T_1 \in M_r(A)$, $S_1 \in M_\ell(A)$ such that (S, T_1),$(S_1, T) \in M_d(A)$. Then, for any $a \in A$,

$$\begin{aligned} aS(e_\lambda) &= T_1(a)e_\lambda \longrightarrow T_1(a), \\ T(e_\lambda)a &= e_\lambda S_1(a) \longrightarrow S_1(a). \end{aligned} \tag{4.27}$$

Thus, both $\{aS(e_\lambda)\}$ and $\{T(e_\lambda)a\}$, being convergent, are Cauchy in A.

Suppose that A is factorable. Then (c) $\Rightarrow$ (a), as follows. Let $m \in QM(A)$ and $a \in A$. By factorability, $a = bc$ for some $b, c \in A$. Define the mappings $S, T : A \to A$ by

$$S(x) = m(c, x), \quad T(x) = m(x, b), \quad x \in A. \tag{4.28}$$

Then, for any $x, y \in A$,

$$\begin{aligned} S(xy) &= m(c, xy) = m(c, x)y = S(x)y, \\ T(xy) &= m(xy, b) = xm(y, b) = xT(y), \end{aligned} \tag{4.29}$$

and so $S \in M_\ell(A)$ and $T \in M_r(A)$. By (c), there exist $T_1 \in M_r(A)$, $S_1 \in M_\ell(A)$ such that $(S, T_1), (S_1, T) \in M_d(A)$. Then

$$\begin{aligned} m(a, e_\lambda) &= m(bc, e_\lambda) = bm(c, e_\lambda) = bS(e_\lambda) = T_1(b)e_\lambda \longrightarrow T_1(b), \\ m(e_\lambda, a) &= m(e_\lambda, bc) = m(e_\lambda, b)c = T(e_\lambda)c = e_\lambda S_1(c) \longrightarrow S_1(c). \end{aligned} \tag{4.30}$$

Hence $\{m(a, e_\lambda) : \lambda \in I\}$ and $\{m(e_\lambda, a) : \lambda \in I\}$ are Cauchy in A. So $\{e_\lambda : \lambda \in I\}$ is ultra-approximate. □

Theorem 4.9. *Let (A, q) be an F-algebra having an ultra-approximate identity $\{e_\lambda : \lambda \in I\}$. Then,*

(a) *each of the maps ϕ_A, ϕ_ℓ, ϕ_r, and ϕ_d is a bijection;*

(b) $\phi_d \mid M(A) = \phi_\ell \mid M(A) = \phi_r \mid M(A)$;

(c) $\phi_d \mid \mu_d(A) = \phi_A$.

Proof. (a) We give the proof only for $\phi_d : M_d(A) \to QM(A)$. To show that ϕ_d is onto, let $m \in QM(A)$. Since $\{e_\lambda\}$ is ultra-approximate, for each $x \in A$, the nets $\{m(x, e_\lambda) : \lambda \in I\}$ and $\{m(e_\lambda, x) : \lambda \in I\}$ are convergent. For each $x \in A$, define $S, T : A \to A$ by

$$S(x) = \lim_\lambda m(e_\lambda, x), \qquad T(x) = \lim_\lambda m(x, e_\lambda). \tag{4.31}$$

Then $(S, T) \in M_d(A)$ since, for any $x, y \in A$,

$$\begin{aligned} xS(y) &= x \lim_\lambda m(e_\lambda, y) = \lim_\lambda m(xe_\lambda, y) = m(x, y) \\ &= \lim_\lambda m(x, e_\lambda y) = \lim_\lambda m(x, e_\lambda)y = T(x)y. \end{aligned} \tag{4.32}$$

Further, we have for any $(a, b) \in A \times A$,

$$[\phi_d(S, T)](a, b) = aS(b) = a \lim_\lambda m(e_\lambda, b) = \lim_\lambda m(ae_\lambda, b) = m(a, b); \tag{4.33}$$

that is, $\phi_d(S, T) = m$.

To show that, ϕ_d is one to one, let $(S_1, T_1), (S_2, T_2) \in M_d(A)$ with $\phi_d(S_1, T_1) = \phi_d(S_2, T_2)$. Then, for any $a, b \in A$,

$$\phi_d(S_1, T_1)(a, b) = \phi_d(S_2, T_2)(a, b), \quad \text{or } aS_1(b) = aS_2(b). \tag{4.34}$$

Since A is faithful, $S_1(b) = S_2(b),\ b \in A$. So $S_1 = S_2$. Consequently, by Theorem 3.3(c), also $T_1 = T_2$. Thus $(S_1, T_1) = (S_2, T_2)$.

(b) Let $T \in M(A)$ and $x, y \in A$. Then, since $T \in M_\ell(A) \cap M_r(A)$ and $(T, T) \in M_d(A)$,

$$\phi_d(T)(x, y) = \phi_d(T, T)(x, y) = xT(y) = T(xy). \tag{4.35}$$

Also

$$\phi_\ell(T)(x, y) = xT(y) = T(xy); \qquad \phi_r(T)(x, y) = T(x)y = T(xy). \tag{4.36}$$

(c) For any $a \in A$ and $(x, y) \in A \times A$,

$$\phi_d(\mu_d(a))(x, y) = \phi_d(L_a, R_a)(x, y) = xL_a(y) = xay = \phi_A(a)(x, y). \tag{4.37}$$

Thus $\phi_d \mid \mu_d(A) = \phi_A$. □

We obtain the following lemma for later use.

Lemma 4.10. (a) *If (A, q) is a factorable F-normed algebra having an approximate identity $\{e_\lambda : \lambda \in I\}$, then*

$$\lim_\lambda q(e_\lambda a e_\lambda - a) = 0, \quad \forall a \in A. \tag{4.38}$$

(b) *If (A, q) is an F-algebra having a minimal ultra-approximate identity $\{e_\lambda : \lambda \in I\}$, then, for any $m \in QM(A)$,*

$$\lim_\lambda e_\lambda m(e_\lambda, x) = \lim_\lambda m(e_\lambda, x) \ \text{(exists)}, \quad \forall x \in A. \tag{4.39}$$

Compare with [10, page 124].

Proof. (a) Let $a \in A$. Since A is factorable, there exist $x, y \in A$ such that $a = xy$. Then

$$\begin{aligned}
\overline{\lim_\lambda}\ q(e_\lambda a e_\lambda - a) &= \overline{\lim_\lambda}\ q\left[(e_\lambda xy e_\lambda - e_\lambda xy) + (e_\lambda xy - xy)\right] \\
&\leq \overline{\lim_\lambda}\ q(e_\lambda x) \cdot q(y e_\lambda - y) + \overline{\lim_\lambda}\ q(e_\lambda xy - xy) \\
&= q(x) \cdot q(0) + q(0) = 0.
\end{aligned} \tag{4.40}$$

Since A is complete and $\{e_\lambda\}$ is ultra-approximate, for any $x \in A$, $\lim_\lambda m(e_\lambda, x) = y$ (say) exists. Since $q(e_\lambda) \leq 1$,

$$\begin{aligned}
q\left[y - e_\lambda m(e_\lambda, x)\right] &\leq q(y - e_\lambda y) + q\left[e_\lambda y - e_\lambda m(e_\lambda, x)\right] \\
&\leq q(y - e_\lambda y) + q(e_\lambda) q(y - m(e_\lambda, x)) \\
&\leq q(y - e_\lambda y) + q(y - m(e_\lambda, x)) \longrightarrow 0.
\end{aligned} \tag{4.41}$$

□

Definition 4.11. Let (A, q) be a k-normed algebra. For any $m \in QM(A)$, we define

$$\|m\|_q = \sup\left\{\frac{q[m(x,y)]}{q(x)q(y)} : x, y \in A,\ x, y \neq 0\right\}. \tag{4.42}$$

Clearly,

$$q[m(x,y)] \leq \|m\|_q\, q(x)q(y), \quad \forall x, y \in A. \tag{4.43}$$

Theorem 4.12. *Let (A, q) be a k-normed algebra. Then,*

(a) *$\|\cdot\|_q$ is a k-norm on $QM(A)$;*

(b) *if (A, q) is complete, then so is $(QM(A), \|\cdot\|_q)$.*

Proof. (a) For any $m \in QM(A)$,

$$\begin{aligned}
\|m\|_q = 0 &\Longleftrightarrow \sup_{x\neq 0, y\neq 0} \frac{q[m(x,y)]}{q(x)q(y)} = 0 \\
&\Longleftrightarrow q[m(x,y)] = 0, \quad \forall x, y \in A,\ x \neq 0,\ y \neq 0 \\
&\Longleftrightarrow q[m(x,y)] = 0, \quad \forall x, y \in A \\
&\Longleftrightarrow m = 0.
\end{aligned} \tag{4.44}$$

Also, for any $\alpha \in \mathbb{K}$, $q(\alpha x) = |\alpha|^k q(x)$ and so it follows that $\|\alpha m\|_q = |\alpha|^k \|m\|_q$.

Next, let $m, u \in QM(A)$, and let $\varepsilon > 0$. Choose $x, y \in A$ such that

$$\|m + u\|_q \leq \frac{q[(m+u)(x,y)]}{q(x)q(y)} - \varepsilon. \tag{4.45}$$

Then,

$$\begin{aligned}
\|m + u\|_q &\leq \frac{q[m(x,y) + u(x,y)]}{q(x)q(y)} - \varepsilon \\
&\leq \frac{q[m(x,y)]}{q(x)q(y)} + \frac{q[u(x,y)]}{q(x)q(y)} - \varepsilon \\
&\leq \|m\|_q + \|u\|_q - \varepsilon.
\end{aligned} \tag{4.46}$$

Since $\varepsilon > 0$ is arbitrary, $\|m + u\|_q \leq \|m\|_q + \|u\|_q$. Thus $\|\cdot\|_q$ is a k-norm on $QM(A)$.

(b) Let $\{m_i : i \in \mathbb{N}\}$ be a $\|\cdot\|_q$-Cauchy sequence in $QM(A)$. Then, for any $x, y \in A$,

$$\begin{aligned}\overline{\lim_{i,j}}\, q[m_i(x,y) - m_j(x,y)] &= \overline{\lim_{i,j}}\, q[(m_i - m_j)(x,y)] \\ &\leq \overline{\lim_{i,j}}\, \|m_i - m_j\|_q \cdot q(x)q(y) = 0.\end{aligned} \tag{4.47}$$

Therefore, for any x, y in A, $\{m_i(x,y)\}$ is a Cauchy sequence in A. Since A is complete, the map $m : A \times A \to A$ given by

$$m(x,y) = \lim_i m_i(x,y), \quad x, y \in A, \tag{4.48}$$

is welldefined. Clearly, m is bilinear and, by Theorem 2.7(b), m is jointly continuous. Further, for any $a, b, x, y \in A$,

$$m(ax, yb) = \lim_i m_i(ax, yb) = \lim_i [am_i(x,y)b] = a\left[\lim_i m_i(x,y)\right]b = am(x,y)b. \tag{4.49}$$

Hence, $m \in QM(A)$. Next, $\|m_i - m\|_q \to 0$ as follows. Let $\varepsilon > 0$. Since $\{m_i\}$ is $\|\cdot\|_q$-Cauchy, there exists an integer $N \geq 1$ such that

$$\|m_i - m_j\|_q < \frac{\varepsilon}{2}, \quad \forall \text{ pairs } i, j \geq N, \tag{4.50}$$

that is,

$$\frac{q[(m_i - m_j)(x,y)]}{q(x)q(y)} < \frac{\varepsilon}{2}, \quad \forall \text{ pairs } i, j \geq N, \ x, y \in A, \ x, y \neq 0. \tag{4.51}$$

Let $x, y \in A$, $x, y \neq 0$. Fix any $i_o \geq N$ in (4.51); since $m_j(x,y) \to m(x,y)$ in A, letting $j \to \infty$ in (4.51),

$$\frac{q[m_{i_o}(x,y) - m(x,y)]}{q(x)q(y)} \leq \frac{\varepsilon}{2}. \tag{4.52}$$

Then, for any $i \geq N$, using (4.51) and (4.52),

$$\begin{aligned}\frac{q[m_i(x,y) - m(x,y)]}{q(x)q(y)} &\leq \frac{q[m_i(x,y) - m_{i_o}(x,y)]}{q(x)q(y)} + \frac{q[m_{i_o}(x,y) - m(x,y)]}{q(x)q(y)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned} \tag{4.53}$$

Thus $m_i \xrightarrow{\|\cdot\|_q} m$. □

Theorem 4.13. *Let (A, q) be a factorable k-normed algebra having a minimal approximate identity $\{e_\lambda : \lambda \in I\}$. Then*

(a) *each of the maps ϕ_A, ϕ_ℓ, ϕ_r, and ϕ_d defined above is a linear isometry;*

(b) *for any $a, b \in A$ and $m \in QM(A)$, $\|a \circ m \circ b\|_q = q(m(a, b))$.*

Proof. (a) We give the proof only for $\phi_d : M_d(A) \to QM(A)$. Clearly, ϕ_d is linear. Let $(S, T) \in M_d(A)$. Then

$$\begin{aligned}\left\|\phi_d(S,T)\right\|_q &= \sup_{x\neq 0, y\neq 0}\frac{q[\phi_d(S,T)(x,y)]}{q(x)q(y)} = \sup_{x\neq 0, y\neq 0}\frac{q[xS(y)]}{q(x)q(y)}\\ &\leq \sup_{x\neq 0, y\neq 0}\frac{q(x)q(S(y))}{q(x)q(y)} = \sup_{y\neq 0}\frac{q(S(y))}{q(y)} = \|S\|_q.\end{aligned} \tag{4.54}$$

To prove the reverse inequality, let $\varepsilon > 0$. There exists $(y \neq 0) \in A$ such that $\|S\|_q < q(S(y))/q(y) + \varepsilon$. For any $\lambda \in I$, since $0 < q(ye_\lambda) \leq q(y)q(e_\lambda) \leq q(y)$,

$$\left\|\phi_d(S,T)\right\|_q \geq \frac{q[\phi_d(S,T)(e_\lambda, ye_\lambda)]}{q(e_\lambda)q(ye_\lambda)} \geq \frac{q[e_\lambda S(ye_\lambda)]}{q(ye_\lambda)} \geq \frac{q[e_\lambda S(y)e_\lambda]}{q(y)}; \tag{4.55}$$

hence, in view of factorability, using Lemma 4.10(a),

$$\left\|\phi_d(S,T)\right\|_q \geq \lim_\lambda \frac{q[e_\lambda S(y)e_\lambda]}{q(y)} = \frac{q(S(y))}{q(y)} > \|S\|_q - \varepsilon. \tag{4.56}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\|\phi_d(S,T)\|_q = \|S\|_q = \|(S,T)\|_q$, and so ϕ_d is an isometry.

(b) By (a), ϕ_A is an isometry and so

$$\begin{aligned}\|a\circ m\circ b\|_q &= \sup_{x\neq 0, y\neq 0}\frac{q[(a\circ m\circ b)(x,y)]}{q(x)q(y)} = \sup_{x\neq 0, y\neq 0}\frac{q[m(xa,by)]}{q(x)q(y)}\\ &= \sup_{x\neq 0, y\neq 0}\frac{q[xm(a,b)y]}{q(x)q(y)} = \sup_{x\neq 0, y\neq 0}\frac{q[\phi_A(m(a,b))(x,y)]}{q(x)q(y)}\\ &= \left\|\phi_A(m(a,b))\right\|_q = q(m(a,b)).\end{aligned} \tag{4.57}$$

□

We next consider multiplication in $QM(A)$ in various equivalent ways.

Definition 4.14 (see [2, 4]). Let A be an F-algebra with an ultra-approximate identity $\{e_\lambda : \lambda \in I\}$ and $m_1, m_2 \in QM(A)$. Since ϕ_d is onto, there exist $(S_1, T_1), (S_2, T_2) \in M_d(\mathrm{A})$ such that

$$\phi_d(S_1, T_1) = m_1, \qquad \phi_d(S_2, T_2) = m_2. \tag{4.58}$$

By the definitions of ϕ_ℓ and ϕ_r,

$$\phi_\ell(S_1) = m_1 = \phi_r(T_1), \qquad \phi_\ell(S_2) = m_2 = \phi_r(T_2). \tag{4.59}$$

Therefore, the product of m_1, m_2 can be defined in any of the following ways:

(i) $m_1 \circ_{\phi_d} m_2 = \phi_d(S_1, T_1) \circ_{\phi_d} \phi_d(S_2, T_2) = \phi_d[(S_1, T_1)(S_2, T_2)] = \phi_d(S_1 S_2, T_2 T_1)$,

(ii) $m_1 \circ_{\phi_\ell} m_2 = \phi_\ell(S_1) \circ_{\phi_\ell} \phi_\ell(S_2) = \phi_\ell(S_1 S_2)$,

(iii) $m_1 \circ_{\phi_r} m_2 = \phi_r(T_1) \circ_{\phi_r} \phi_r(T_2) = \phi_r(T_2 T_1)$.

Note that, for any $(x, y) \in A \times A$,

$$[\phi_d(S_1 S_2, T_2 T_1)](x, y) = x(S_1 S_2)(y) = [\phi_\ell(S_1 S_2)](x, y), \tag{4.60}$$

also

$$x(S_1 S_2)(y) = (T_2 T_1)(x) y = [\phi_r(T_2 T_1)](x, y). \tag{4.61}$$

Hence, $m_1 \circ_{\phi_d} m_2 = m_1 \circ_{\phi_\ell} m_2 = m_1 \circ_{\phi_r} m_2$.

Remark 4.15. (1) If $m = \phi_\ell(T)$ with $T \in M_\ell(A)$ and $a \in A$, then, by Theorem 3.6(a),

$$m \circ_{\phi_\ell} \phi_A(a) = \phi_\ell(T) \circ_{\phi_\ell} \phi_\ell(L_a) = \phi_\ell(T L_a) = \phi_\ell\left(L_{T(a)}\right) = \phi_A(T(a)) \in \phi_A(A). \tag{4.62}$$

(2) If $m = \phi_r(T)$ with $T \in M_r(A)$ and $a \in A$, then, by Theorem 3.6(b),

$$\phi_A(a) \circ_{\phi_r} m = \phi_r(R_a) \circ_{\phi_r} \phi_r(T) = \phi_r(T R_a) = \phi_r\left(R_{T(a)}\right) = \phi_A(T(a)) \in \phi_A(A). \tag{4.63}$$

(3) If $m = \phi_d(S, T)$ and $a \in A$, then, by Theorem 3.6(c),

$$\begin{aligned} m \circ_{\phi_d} \phi_A(a) &= \phi_d[(S, T)(L_a, R_a)] = \phi_d(S L_a, R_a T) \\ &= \phi_d\left(L_{S(a)}, R_{S(a)}\right) = \phi_A(S(a)) \in \phi_A(A). \end{aligned} \tag{4.64}$$

In the sequel, we denote the product on $QM(A)$ arising from (i), (ii), or (iii) by $\odot$. Some properties of this product are given as follows.

Theorem 4.16. *Let A be a factorable complete k-normed algebra with a minimal ultra-approximate identity $\{e_\lambda : \lambda \in I\}$. Then,*

(a) *for any $m_1, m_2 \in QM(A)$,*

$$(m_1 \odot m_2)(x, y) = m_1\left(x, \lim_\lambda m_2(e_\lambda, y)\right) \quad ((x, y) \in A \times A), \tag{4.65}$$

defines a product $\odot$ on $QM(A)$ so that $(QM(A), \|\cdot\|_q)$ is a complete k-normed algebra with identity $m_o = \phi_d(I, I)$;

(b) *for any* $m \in QM(A)$ *and* $a \in A$,

$$\phi_A(a) \odot m = a \circ m, \qquad m \odot \phi_A(a) = m \circ a; \tag{4.66}$$

(c) $\phi_A(A)$ *is a two-sided ideal in* $QM(A)$;

(d) *If* A *is factorable, then both* $M_\ell(A)$ *and* $M_d(A)$ *are isometrically algebraically isomorphic to* $QM(A)$, *while* $M_r(A)$ *is isometrically algebraically anti-isomorphic to* $QM(A)$;

Proof. (a) Let $m_1, m_2 \in QM(A)$ and $(x, y) \in A \times A$. Choose $(S_1, T_1), (S_2, T_2) \in M_d(A)$ such that

$$\phi_d(S_1, T_1) = m_1, \qquad \phi_d(S_2, T_2) = m_2, \tag{4.67}$$

that is,

$$xS_1(y) = m_1(x, y), \qquad xS_2(y) = m_2(x, y). \tag{4.68}$$

Then,

$$\begin{aligned}
(m_1 \odot m_2)(x, y) &= [\phi_d(S_1S_2, T_2T_1)](x, y) = x(S_1S_2)(y) \\
&= x[S_1(S_2(y))] = x\left[S_1\left(\lim_\lambda e_\lambda S_2(y)\right)\right] \\
&= m_1\left(x, \lim_\lambda e_\lambda S_2(y)\right) = m_1\left(x, \lim_\lambda m_2(e_\lambda, y)\right).
\end{aligned} \tag{4.69}$$

It is easy to verify that $\|m_1 \odot m_2\|_q \le \|m_1\|_q \|m_2\|_q$, so that $(QM(A), \|\cdot\|_q)$ is an F-normed algebra. Further, by Theorem 4.12, $(QM(A), \|\cdot\|_q)$ is also complete.

(b) Let $m \in QM(A)$ and $a \in A$. Using (a), for any $(x, y) \in A \times A$,

$$\begin{aligned}
[m \odot \phi_A(a)](x, y) &= m\left(x, \lim_\lambda \phi_A(a)(e_\lambda, y)\right) = m\left(x, \lim_\lambda m e_\lambda a y\right) \\
&= m(x, ay) = (m \circ a)(x, y).
\end{aligned} \tag{4.70}$$

Similarly, $[\phi_A(a) \odot m](x, y) = (a \circ m)(x, y)$.

(c) To show that $\phi_A(A)$ is a two-sided ideal, let $m \in QM(A)$ and $a \in A$. By Theorem 4.9(a), there exists $(S, T) \in M_d(A)$ such that $\phi_d(S, T) = m$. Using (b), for any $(x, y) \in A \times A$,

$$\begin{aligned}
[m \odot \phi_A(a)](x, y) &= (m \circ a)(x, y) = m(x, ay) = \phi_d(S, T)(x, ay) \\
&= xS(ay) = xS(a)y = \phi_A(S(a))(x, y).
\end{aligned} \tag{4.71}$$

Hence, $m \odot \phi_A(a) = \phi_A(S(a)) \in \phi_A(A)$. Similarly, $\phi_A(a) \odot m = \phi_A(T(a)) \in \phi_A(A)$.

(d) Suppose that A is factorable. Then, by Theorem 4.13(a), each of the maps ϕ_d, ϕ_ℓ, ϕ_r, and ϕ_A is a linear isometry. If $(S_1, T_1), (S_2, T_2) \in M_d(A)$, then by definition

$$\phi_d(S_1, T_1) \circ_{\phi_d} \phi_d(S_2, T_2) = \phi_d[(S_1, T_1)(S_2, T_2)]. \tag{4.72}$$

If $S_1, S_2 \in M_\ell(A)$, then by definition

$$\phi_\ell(S_1) \circ_{\phi_\ell} \phi_\ell(S_2) = \phi_\ell(S_1 S_2). \tag{4.73}$$

If $T_1, T_2 \in M_r(A)$, then by definition

$$\phi_r(T_1) \circ_{\phi_r} \phi_r(T_2) = \phi_r(T_2 T_1). \tag{4.74}$$

Hence, $\phi_d : M_d(A) \to QM(A)$ and $\phi_\ell : M_\ell(A) \to QM(A)$ are algebraic isomorphisms and $\phi_r : M_r(A) \to QM(A)$ is algebraic anti-isomorphism. □

Remark 4.17. If A has an identity e, then A may be identified with $QM(A)$ as follows. Let $m \in QM(A)$. Then $m(e, e) \in A$ and, for any $x, y \in A$,

$$\phi_A(m(e, e))(x, y) = xm(e, e)y = m(xe, ey) = m(x, y). \tag{4.75}$$

5. Quasistrict and Strict Topologies on $QM(A)$

In this section, we consider the quasistrict and strict topologies on $QM(A)$ and extend several results from [2, 4, 8]. Throughout we will assume, unless stated otherwise, that (A, q) is a factorable complete k-normed algebra having a minimal ultra-approximate identity $\{e_\lambda : \lambda \in I\}$.

Definition 5.1. For any $m \in QM(A)$ and $a, b \in A$, we define mappings $a \circ m, m \circ a, a \circ m \circ b : A \times A \to A$ by

$$\begin{gathered} (a \circ m)(x, y) = m(xa, y), \qquad (m \circ a)(x, y) = m(x, ay), \\ (a \circ m \circ b)(x, y) = m(xa, by), \quad (x, y) \in A \times A. \end{gathered} \tag{5.1}$$

Lemma 5.2. *Let $m \in QM(A)$ and $a, b \in A$. Then*

(a) $\|a \circ m\|_q \le \|m\|_q q(a)$, $\|m \circ a\|_q \le \|m\|_q q(a)$,

(b) $\|a \circ m \circ b\|_q = q(m(a, b)) \le \|m\|_q q(a) q(b)$.

Proof. (a) By definition,

$$\begin{aligned}\|a\circ m\|_q &= \sup_{x\neq 0,y\neq 0}\frac{q[(a\circ m)(x,y)]}{q(x)q(y)} = \sup_{x\neq 0,y\neq 0}\frac{q[x\cdot m(a,y)]}{q(x)q(y)}\\ &\le \sup_{x\neq 0,y\neq 0}\frac{q(x)q(m(a,y))}{q(x)q(y)} = \sup_{y\neq 0}\frac{q(m(a,y))}{q(y)}\\ &= \sup_{y\neq 0}\frac{\|m\|_q q(a)q(y)}{q(y)} = \|m\|_q q(a).\end{aligned}\tag{5.2}$$

Similarly, $\|m\circ a\|_q \le \|m\|_q q(a)$.

(b) By Theorem 4.13(b), $q(m(a,b)) = \|a\circ m\circ b\|_q$. Further, using (a),

$$\|a\circ m\circ b\|_q \le \|a\circ m\|_q q(b) \le \|m\|_q q(a)q(b).\tag{5.3}$$

□

Definition 5.3. (1) The *quasistrict topology* γ on $QM(A)$ is determined by the family $\{\xi_{a,b}(m) : a,b\in A\}$ of k-seminorms, where

$$\xi_{a,b}(m) = \|a\circ m\circ b\|_q = q(m(a,b)),\quad m\in QM(A).\tag{5.4}$$

Compare with [2, page 109]; [4, page 558].

(2) The *strict topology* β on $QM(A)$ is determined by the family $\{\eta_a(m) : a\in A\}$ of k-seminorms, where

$$\eta_a(m) = \max\left\{\|a\circ m\|_q, \|m\circ a\|_q\right\},\quad m\in QM(A).\tag{5.5}$$

Compare with [8, page 227].

Let τ denote the topology on $QM(A)$ generated by the k-norm $\|\cdot\|_q$.

Lemma 5.4. *$\gamma\subseteq\beta\subseteq\tau$ on $QM(A)$.*

Proof. To show that $\gamma\subseteq\beta$, let $a,b\in A$. Then

$$\xi_{a,b}(m) = \|a\circ m\circ b\|_q \le \|a\circ m\|_q q(b) \le \eta_a(m)q(b),\quad m\in QM(A),\tag{5.6}$$

also

$$\xi_{a,b}(m) = \|a\circ m\circ b\|_q \le \|m\circ b\|_q q(a) \le \eta_b(m)q(a),\quad m\in QM(A).\tag{5.7}$$

Hence,

$$\xi_{a,b}(m) \le \max\{\eta_a(m), \eta_b(m)\} q(a) q(b), \quad m \in QM(A). \tag{5.8}$$

Let $\{m_\alpha : \alpha \in J\}$ be a net in $QM(A)$ with $m_\alpha \xrightarrow{\beta} m \in QM(A)$. Then, for any $a, b \in A$, $\eta_a(m_\alpha - m) \to 0$ and $\eta_b(m_\alpha - m) \to 0$. Hence,

$$\xi_{a,b}(m_\alpha - m) \le \max\{\eta_a(m_\alpha - m),\ \eta_b(m_\alpha - m)\} q(a) q(b) \longrightarrow 0. \tag{5.9}$$

Thus $m_\alpha \xrightarrow{\gamma} m$, and so $\gamma \subseteq \beta$.

To show that $\beta \subseteq \tau$, Let $a \in A$. Note that, for any $x, y \in A$,

$$\begin{aligned} (a \circ m)(x, y) &= m(xa, y) = x \cdot m(a, y), \quad m \in QM(A), \\ (m \circ a)(x, y) &= m(x, ay) = m(x, a) y, \quad m \in QM(A). \end{aligned} \tag{5.10}$$

By Lemma 5.2(a), $\|a \circ m\|_q \le \|m\|_q q(a)$ and $\|m \circ a\|_q \le \|m\|_q q(a)$; hence,

$$\eta_a(m) = \max\left\{ \|a \circ m\|_q, \|m \circ a\|_q \right\} \le \|m\|_q q(a), \quad m \in QM(A). \tag{5.11}$$

Consequently, if $\{m_\alpha\}$ is a net in $QM(A)$ with $m_\alpha \xrightarrow{\tau} m \in QM(A)$, then $\|m_\alpha - m\|_q \to 0$, and so, for any $a, b \in A$,

$$\eta_a(m_\alpha - m) \le \|m_\alpha - m\|_q q(a) \longrightarrow 0. \tag{5.12}$$

Thus $m_\alpha \xrightarrow{\beta} m$; that is, $\beta \subseteq \tau$. □

Theorem 5.5. *$\phi_A(A)$ is β-dense in $QM(A)$ and hence γ-dense in $QM(A)$.*

Proof. Let $m \in QM(A)$. Clearly $\{m(e_\lambda, e_\lambda)\}_{\lambda \in I} \subseteq A$. We claim that $\phi_A(m(e_\lambda, e_\lambda)) \xrightarrow{\beta} m$. Let $a \in A$. We need to show that

$$\eta_a[\phi_A(m(e_\lambda, e_\lambda)) - m] \longrightarrow 0. \tag{5.13}$$

For any $x, y \in A$, by joint continuity of m,

$$\begin{aligned} &q[a \circ \{\phi_A(m(e_\lambda, e_\lambda)) - m\}(x, y)] \\ &\quad = q[\{\phi_A(m(e_\lambda, e_\lambda)) - m\}(xa, y)] = q[xam(e_\lambda, e_\lambda)y - m(xa, y)] \\ &\quad = q[m(xae_\lambda, e_\lambda y) - m(xa, y)] \longrightarrow q[m(xa, y) - m(xa, y)] = 0. \end{aligned} \tag{5.14}$$

Hence $\|a \circ \{\phi_A(m(e_\lambda, e_\lambda)) - m\}\|_q \to 0$. Similarly, $\|\{\phi_A(m(e_\lambda, e_\lambda)) - m\} \circ a\|_q \to 0$. Thus,

$$\eta_a\left[\phi_A(m(e_\lambda, e_\lambda)) - m\right] \longrightarrow 0; \tag{5.15}$$

that is, $\phi_A(A)$ is β-dense in $QM(A)$. Since $\gamma \subseteq \beta$, it follows that $\phi_A(A)$ is γ-dense in $QM(A)$. □

Theorem 5.6. (a) *$(QM(A), \gamma)$ and $(QM(A), \beta)$ are sequentially complete.*

(b) *If, in addition, (A, q) is strongly factorable, then $(QM(A), \gamma)$ and $(QM(A), \beta)$ are complete.*

Proof. (a) Let $\{m_i : i \in \mathbb{N}\}$ be a γ-Cauchy sequence in $QM(A)$. For any $x, y \in A$, using Theorem 4.13(b),

$$\begin{aligned} q\left[m_i(x,y) - m_j(x,y)\right] &= q\left[(m_i - m_j)(x,y)\right] \\ &= \left\|x \circ (m_i - m_j) \circ y\right\|_q = \xi_{x,y}(m_i - m_j), \end{aligned} \tag{5.16}$$

which implies that $\{m_i(x,y)\}$ is a Cauchy sequence in A. Define $m : A \times A \to A$ by $m(x,y) = \lim_i m_i(x,y)$. Clearly, m is bilinear and, by Theorem 2.7(b), m is jointly continuous. Further, for any $a, b, x, y \in A$,

$$m(ax, yb) = \lim_i m_i(ax, yb) = a\left[\lim_i m_i(x,y)\right]b = am(x,y)b, \tag{5.17}$$

and so $m \in QM(A)$. Further, for any $a, b \in A$,

$$\xi_{a,b}(m_i - m) = \|a \circ m_i \circ b - a \circ m \circ b\|_q = q[(m_i - m)(a,b)] \longrightarrow 0. \tag{5.18}$$

Hence $m_i \xrightarrow{\gamma} m$. So $(QM(A), \gamma)$ is sequentially complete.

Next we show that $(QM(A), \beta)$ is sequentially complete. We first note that, if $m \in QM(A)$, then, for each $c \in A$, the mappings $S_c, T_c : A \to A$ given by

$$S_c(x) = m(c, x), \quad T_c(x) = m(x, c), \quad x \in A, \tag{5.19}$$

define elements in $M_\ell(A)$ and $M_r(A)$, respectively, and it is easy to see that

$$\phi_\ell(S_c) = c \circ m, \qquad \phi_r(T_c) = m \circ c. \tag{5.20}$$

Let $\{m_i : i \in \mathbb{N}\}$ be a β-Cauchy sequence in $QM(A)$, and let $c \in A$. It follows from the definition of the β-topology that the sequences $\{\phi_\ell(S_c)_i\}$ and $\{\phi_r(T_c)_i\}$, where $(S_c)_i(x) = m_i(c, x)$ and $(T_c)_i(x) = m_i(x, c)$, are τ-Cauchy in $QM(A)$. Since ϕ_ℓ and ϕ_r are topological embeddings, the sequences $\{(S_c)_i\}$ and $\{(T_c)_i\}$ are $\|\cdot\|_q$-Cauchy in $M_\ell(A)$ and

$M_r(A)$, respectively. Both $M_\ell(A)$ and $M_r(A)$ are complete (Theorem 3.9) and so there exist $S^{(c)}$ in $M_\ell(A)$ and $T^{(c)}$ in $M_r(A)$ such that

$$\left\|(S_c)_i - S^{(c)}\right\|_q \longrightarrow 0, \qquad \left\|(T_c)_i - T^{(c)}\right\|_q \longrightarrow 0. \tag{5.21}$$

Since $\gamma \subseteq \beta$, the sequence $\{m_{\rm i}\}$ is γ-Cauchy. As proved above, the space $QM(A)$ is γ-complete and so there exists an element m_o in $QM(A)$ such that

$$\lim_i m_i(x,y) = m_o(x,y), \quad \forall x,y \in A. \tag{5.22}$$

For any $a,b \in A$,

$$\left[\phi_\ell\left(S^{(c)}\right)\right](a,b) = \lim_i\left[\phi_\ell\left(S^{(c)}\right)_i\right](a,b) = \lim_i am_i(c,b) = (c \circ m_o)(a,b), \tag{5.23}$$

which implies that $\phi_\ell(S^{(c)}) = c \circ m_o$. Similarly, we can prove that $\phi_r(T^{(c)}) = m_o \circ c$. Thus, by (5.21),

$$\begin{aligned}
\|c \circ m_i - c \circ m_o\|_q &= \left\|\phi_\ell\left(S^{(c)}\right)_i - \phi_\ell\left(S^{(c)}\right)\right\|_q = \left\|\left(S^{(c)}\right)_i - S^{(c)}\right\|_q \longrightarrow 0,\\
\|m_i \circ c - m_o \circ c\|_q &= \left\|\phi_r\left(T^{(c)}\right)_i - \phi_r\left(T^{(c)}\right)\right\|_q = \left\|\left(T^{(c)}\right)_i - T^{(c)}\right\|_q \longrightarrow 0,
\end{aligned} \tag{5.24}$$

which implies that m_o is the β-limit of the sequence $\{m_i\}$ that is, $QM(A)$ is β-complete.

(b) Suppose that A is strongly factorable. Let $\{m_\alpha : \alpha \in J\}$ be a γ-Cauchy net in $QM(A)$. Replacing the sequence $\{m_i : i \in \mathbb{N}\}$ by the net $\{m_\alpha : \alpha \in J\}$ in part (a), we obtain a map $m : A \times A \to A$ given by $m(x,y) = \lim_\alpha m_\alpha(x,y)$. Then m is bilinear; further, for any $a,b,x,y \in A$,

$$m(ax,yb) = \lim_\alpha m_\alpha(ax,yb) = a\left[\lim_\alpha m_\alpha(x,y)\right]b = am(x,y)b. \tag{5.25}$$

Hence, using strong factorability as in Theorem 4.3(c), it follows that m is jointly continuous and so $m \in QM(A)$. Again, as in part (a), it follows that $m_\alpha \xrightarrow{\gamma} m$ and consequently $(QM(A),\gamma)$ is complete. That $QM(A)$ is β-complete also follows by the argument similar to the above one. $\square$

Remark 5.7. The authors do not know whether part (b) of the above theorem can be proved without the assumption of the strong factorability of A.

Theorem 5.8. *$(QM(A), \gamma)$, $(QM(A), \beta)$, and $(QM(A), \tau)$ have the same bounded sets.*

Proof. (a) Since $\gamma \subseteq \tau$, every τ-bounded set is γ-bounded. Let H be any γ-bounded set in $QM(A)$. Then, for each $a, b \in A$, there exists a constant $r = r(a, b) > 0$ such that

$$\begin{aligned} &\|a \circ m \circ b\|_q \leq r, \quad \forall m \in H, \\ \text{or} \quad &q[m(a,b)] \leq r, \quad \forall m \in H \quad (\text{using Theorem 4.13(b)}). \end{aligned} \tag{5.26}$$

For each $a \in A$ and $m \in H$, define $M_a : A \to A$ by

$$M_a(x) = m(a, x), \quad x \in A. \tag{5.27}$$

Then, $\mathcal{F} = \{M_a : m \in H\} \subseteq \mathrm{CL}(A)$. By (5.26), for any $x \in A$

$$q[M_a(x)] = q[m(a,x)] \leq r(a,x), \quad \forall m \in H; \tag{5.28}$$

hence $\mathcal{F}$ is pointwise bounded. Then, by the uniform boundedness principle (Theorem 2.6), there exists $c = c(a) > 0$ such that

$$\|M_a\|_q \leq c, \quad \forall m \in H. \tag{5.29}$$

Consider now the family $P = \{p_m : m \in H\}$ of k-seminorms on A defined by

$$p_m(a) = \|M_a\|_q = \sup_{b \neq 0} \frac{q[M_a(b)]}{q(b)} = \sup_{b \neq 0} \frac{q[m(a,b)]}{q(b)}, \quad a \in A. \tag{5.30}$$

For each $m \in H$, p_m is continuous on A since, if $\{a_n\} \subseteq A$ with $a_n \to a_o$ in A, then

$$\begin{aligned} |p_m(a_n) - p_m(a_o)| &\leq p_m(a_n - a_o) = \sup_{b \neq 0} \frac{q[M_{a_n - a_o}(b)]}{q(b)} \\ &= \sup_{b \neq 0} \frac{q[m(a_n - a_o, b)]}{q(b)} \leq \sup_{b \neq 0} \frac{\|m\|_q q(a_n - a_o) q(b)}{q(b)} \\ &= \|m\|_q q(a_n - a_o) \longrightarrow 0. \end{aligned} \tag{5.31}$$

Then, by (5.29), the family P is pointwise bounded. Applying Theorem 2.5, there exists a ball $B = B(x_o, r) = \{x \in A : q(x - x_o) \leq r\}$ and a constant $C > 0$ such that

$$p_m(x) \leq C, \quad \forall m \in H, \ x \in B(x_o, r). \tag{5.32}$$

For any fixed $a \in A$, we claim that

$$p_m(a) \leq \frac{2C \cdot q(a)}{r}. \tag{5.33}$$

If $a = 0$, this is obvious. Suppose that $a \neq 0$. For simplification, put $t = (r/q(a))^{1/k}$. Then, q is k-homogeneous, and we have $ta + x_o,\ x_o \in B(x_o, r)$, as follows:

$$\begin{aligned} q(ta + x_o - x_o) &= q(ta) = t^k \cdot q(a) \leq \frac{r}{q(a)} q(a) = r, \\ q(x_o - x_o) &= q(0) = 0 < r. \end{aligned} \tag{5.34}$$

So, by (5.32),

$$p_m(ta + x_o) \leq C, \qquad p_m(x_o) \leq C. \tag{5.35}$$

Now, using (5.35) and the properties of k-norm again,

$$\begin{aligned} p_m(a) = p_m\left(\frac{1}{t} ta\right) &= \left(\frac{1}{t}\right)^k p_m(ta) \leq \frac{q(a)}{r} p_m[ta + x_o - x_o] \\ &\leq \frac{q(a)}{r}\left[p_m(ta + x_o) + p_m(x_o)\right] \\ &\leq \frac{q(a)}{r}[C + C] = \frac{2C \cdot q(a)}{r}. \end{aligned} \tag{5.36}$$

This proves our claim. Hence, using (5.33), for any $m \in H$,

$$\begin{aligned} \|m\|_q &= \sup_{a,b \neq 0} \frac{q[m(a,b)]}{q(a)q(b)} = \sup_{a \neq 0} \frac{1}{q(a)} \sup_{b \neq 0} \frac{q[m(a,b)]}{q(b)} \\ &= \sup_{a \neq 0} \frac{1}{q(a)} p_m(a) \leq \sup_{a \neq 0} \frac{1}{q(a)} \cdot \frac{2C \cdot q(a)}{r} \leq \frac{2C}{r}. \end{aligned} \tag{5.37}$$

Consequently, H is τ-bounded.

(b) This follows from (a) since $\gamma \subseteq \beta \subseteq \tau$. □

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