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Research Article

On Delay-Independent Criteria for Oscillation of Higher-Order Functional Differential Equations

Yuangong Sun

School of Mathematics, University of Jinan, Jinan, Shandong 250022, China

Correspondence should be addressed to Yuangong Sun, sunyuangong@yahoo.cn

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We investigate the oscillation of the following higher-order functional differential equation: $x^{(n)}(t) + q(t)|x(t-\tau)|^{\lambda-1}x(t-\tau) = e(t)$, where $q(t)$ and $e(t)$ are continuous functions on $[t_0, \infty)$, $1 > \lambda > 0$ and $\tau \neq 0$ are constants. Unlike most of delay-dependent oscillation results in the literature, two delay-independent oscillation criteria for the equation are established in both the case $\tau > 0$ and the case $\tau < 0$ under the assumption that the potentials $q(t)$ and $e(t)$ change signs on $[t_0, \infty)$.

1. Introduction

Consider the following nth-order forced functional differential equation of the form:

$$x^{(n)}(t) + q(t)|x(t-\tau)|^{\lambda-1}x(t-\tau) = e(t), \tag{1.1}$$

where $n \geq 1$ is an integer, $q(t), e(t) \in C[t_0, \infty)$, $\lambda > 0$, and $\tau \neq 0$ are constants.

We are here only concerned with the nonconstant solutions of (1.1) that are defined for all large t. The oscillatory behavior is considered in the usual sense, that is, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory.

The oscillatory behavior of (1.1) with $\tau = 0$ has been studied by many authors. In early papers [1, 2], by assuming that $e(t) = h^{(n)}(t)$, where $h(t)$ is an oscillatory function satisfying $\lim_{t\to\infty} h(t) = 0$, the author proved that the forced equation would remain oscillatory if the unforced equation is oscillatory. However, the potential $q(t)$ is usually assumed to be non-negative in [1, 2].

When $q(t) < 0$, $\tau = 0$, and $\lambda > 1$, Agarwal and Grace [3] studied the oscillation of (1.1) by using a method of general means without imposing the Kartsatos condition. Following this method, the oscillation of (1.1) with $\tau = 0$ was studied in [4] for both the case $q(t) \geq 0$

and $q(t) < 0$ on $[t_0, \infty)$. When $q(t)$ changes its sign on $[t_0, \infty)$, $\tau = 0$, and $0 < \lambda < 1$, oscillation criteria for (1.1) were given in [5]. Sun and Saker [6], Sun and Mingarelli [7], and Yang [8] studied the oscillation for a generalized form of (1.1) with $\tau = 0$. When $\tau > 0$, there have been many oscillation criteria for equations of the type (1.1). For example, see [9–14] and references cited therein. We see that all these oscillation criteria depend on time delay.

To the best of our knowledge, little has been known about the oscillatory behavior of (1.1) in the case of oscillatory potentials when $\tau < 0$. Particularly, little has been known about the delay-independent criteria for oscillation of (1.1). Unlike most of papers devoted on delay-dependent oscillation criteria for functional differential equations, the main purpose of this paper is to establish two delay-independent oscillation criteria for (1.1) in both the case $\tau > 0$ and the case $\tau < 0$, where the potential $e(t)$ is not imposed on the Kartsatos condition, and the potential $q(t)$ may change its sign. Finally, two interesting examples are worked out to illustrate the main results.

2. Main Results

Theorem 2.1. *Assume that* $0 < \lambda < 1$ *and* $\tau > 0$. *If*

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^n} \int_{t_0}^{t} \left[(t-s)^n e(s) + Q(t,s)\right] ds = +\infty, \tag{2.1}$$

$$\liminf_{t\to\infty} \frac{1}{(t-t_0)^n} \int_{t_0}^{t} \left[(t-s)^n e(s) - Q(t,s)\right] ds = -\infty, \tag{2.2}$$

where

$$Q(t,s) = (\lambda-1)\lambda^{\lambda/(1-\lambda)} (n!)^{\lambda/(\lambda-1)} \left[(t-s)^n \overline{q}(s)\right]^{1/(1-\lambda)}, \tag{2.3}$$

$\overline{q}(s) = \max\{-q(s), 0\}$, *then all solutions of* (1.1) *are oscillatory for any* $\tau > 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. When $x(t)$ is eventually negative, the proof follows the same argument. Multiplying (1.1) by $(t-s)^n$ and integrating it from t_0 to t yields

$$\begin{aligned}
\int_{t_0}^{t} (t-s)^n e(s) ds &= \int_{t_0}^{t} (t-s)^n x^{(n)}(s) ds + \int_{t_0}^{t} (t-s)^n q(s) x^{\lambda}(s-\tau) ds \\
&\geq \int_{t_0}^{t} (t-s)^n x^{(n)}(s) ds - \int_{t_0}^{t} (t-s)^n \overline{q}(s) x^{\lambda}(s-\tau) ds \\
&= -\sum_{i=0}^{n-1} c_i (t-t_0)^{n-i} x^{(n-i-1)}(t_0) \\
&\quad + c_n \int_{t_0+\tau}^{t+\tau} x(s-\tau) ds - \int_{t_0}^{t} (t-s)^n \overline{q}(s) x^{\lambda}(s-\tau) ds,
\end{aligned} \tag{2.4}$$

where $c_0 = 0$, $c_i = n(n-1)\cdots(n-i+1)$, and $c_n = n!$. Since $\tau > 0$,

$$\int_{t_0}^{t} x(s)ds = \int_{t_0+\tau}^{t+\tau} x(s-\tau)ds, \tag{2.5}$$

we have

$$\int_{t_0+\tau}^{t+\tau} x(s-\tau)ds \geq \int_{t_0+\tau}^{t} x(s-\tau)ds. \tag{2.6}$$

This together with (2.4) and (2.5) yield

$$\begin{aligned}\int_{t_0}^{t} (t-s)^n e(s)ds \geq{}& -\sum_{i=0}^{n-1} c_i (t-t_0)^{n-i} x^{(n-i-1)}(t_0) - c_n \int_{t_0}^{t_0+\tau} x(s-\tau)ds \\ &+ \int_{t_0}^{t} \left[c_n x(s-\tau) - (t-s)^n \overline{q}(s) x^{\lambda}(s-\tau)\right]ds.\end{aligned} \tag{2.7}$$

For given t and s $(t > s)$, set

$$F(x) = c_n x - (t-s)^n \overline{q}(s) x^{\lambda}, \quad x > 0, \ 0 < \lambda < 1. \tag{2.8}$$

It is not difficult to see that $F(x)$ obtains its minimum at $x = [c_n/(t-s)^n\ \overline{q}(s)]^{1/(1-\lambda)}$ and

$$F_{\min} = (\lambda - 1)\lambda^{\lambda/(1-\lambda)}(n!)^{\lambda/(\lambda-1)}\left[(t-s)^n \overline{q}(s)\right]^{1/(1-\lambda)} = Q(t,s). \tag{2.9}$$

It implies that

$$c_n x(s-\tau) - (t-s)^n \overline{q}(s) x^{\lambda}(s-\tau) \geq Q(t,s). \tag{2.10}$$

Therefore, for any $\tau > 0$, multiplying (2.7) by $(t-t_0)^{-n}$, using (2.10), and taking lim inf on both sides of (2.7), we get a contradiction with (2.1). This completes the proof of Theorem 2.1. □

Theorem 2.2. *Assume that $0 < \lambda < 1$ and $\tau < 0$. If (2.1) and (2.2) hold, then all solutions of (1.1) satisfying $x(t) = O(t^n)$ are oscillatory for any $\tau < 0$.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) satisfying $x(t) = O(t^n)$. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$, and there exists a positive constant $M > 0$ such that $x(t) \leq Mt^n$. Similar to the corresponding computation in Theorem 2.1 and noting that $\tau < 0$, we have

$$\begin{aligned}\int_{t_0}^{t} (t-s)^n e(s)ds \geq{}& -\sum_{i=0}^{n-1} c_i (t-t_0)^{n-i} x^{(n-i-1)}(t_0) + c_n \int_{t_0+\tau}^{t_0} x(s-\tau)ds \\ &- c_n \int_{t+\tau}^{t} x(s-\tau)ds + \int_{t_0}^{t} \left[c_n x(s-\tau) - (t-s)^n \overline{q}(s) x^{\lambda}(s-\tau)\right]ds.\end{aligned} \tag{2.11}$$

Since $x(t) \le Mt^n$, we get

$$\int_{t+\tau}^{t} x(s-\tau)ds \le \frac{M}{n+1}\Big[(t-\tau)^{n+1} - t^{n+1}\Big]. \tag{2.12}$$

Then, for any $\tau < 0$, multiplying (2.11) by $(t-t_0)^{-n}$, using (2.10) and (2.12), and taking lim inf on both sides of (2.11), we get a contradiction with (2.1). This completes the proof of Theorem 2.2. □

The main results in this paper can also be extended to the case of time-varying delay. That is, we can consider the following equation:

$$x^{(n)}(t) + q(t)|x(\sigma(t))|^{\lambda-1}x(\sigma(t)) = e(t), \tag{2.13}$$

where $\sigma(t)$ is continuously differentiable on $[t_0,\infty)$, $\lim_{t\to\infty}\sigma(t) = \infty$, and $\sigma'(t) > 0$ for t sufficiently large. Without loss of generality, say $\sigma'(t) > 0$ for $t \ge t_0$. Similar to the analysis as before, we have the following delay-independent and derivative-dependent oscillation criteria for (2.13).

Theorem 2.3. *Assume that* $0 < \lambda < 1$ *and* $\sigma(t) \le t$. *If*

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{(t-t_0)^n}\int_{t_0}^{t}\Big[(t-s)^n e(s) + \widetilde{Q}(t,s)\Big]ds = +\infty, \\ &\liminf_{t\to\infty}\frac{1}{(t-t_0)^n}\int_{t_0}^{t}\Big[(t-s)^n e(s) - \widetilde{Q}(t,s)\Big]ds = -\infty, \end{aligned} \tag{2.14}$$

where

$$\widetilde{Q}(t,s) = (\lambda-1)\lambda^{\lambda/(1-\lambda)}\left(n!\sigma'(s)\right)^{\lambda/(\lambda-1)}\left[(t-s)^n\overline{q}(s)\right]^{1/(1-\lambda)}, \tag{2.15}$$

$\overline{q}(s) = \max\{-q(s),0\}$, *then all solutions of* (2.13) *are oscillatory.*

Theorem 2.4. *Assume that* $0 < \lambda < 1$ *and* $\sigma(t) \ge t$. *If* (2.1) *and* (2.2) *hold, and there exists a continuous function* $\phi(t) \ge 0$ *on* $[t_0,\infty)$ *such that* $\int_{\sigma^{-1}(t)}^{t}\phi(s)ds = O(t^n)$, *where* σ^{-1} *is the inverse of* $\sigma(t)$, *then all solutions of* (2.13) *satisfying* $x(t) = O(\phi(t))$ *are oscillatory.*

3. Examples

In this section, we work out two examples to illustrate the main results.

Example 3.1. Consider the following equation:

$$x^{(n)}(t) + t^{\alpha}\sin t|x(t-\tau)|^{\lambda-1}x(t-\tau) = t^{\beta}\cos t, \quad t \ge 0, \tag{3.1}$$

where $\tau \neq 0$, $\alpha \geq 0$, $\beta > 0$, and $0 < \lambda < 1$ are constants. Note that

$$Q(t,s) \geq \Lambda (t-s)^{n/(1-\lambda)} s^{\alpha/(1-\lambda)}, \tag{3.2}$$

where $\Lambda = (\lambda - 1)\lambda^{\lambda/(1-\lambda)}(n!)^{\lambda/(\lambda-1)} < 0$. We have

$$\begin{aligned} \int_0^t Q(t,s)ds &\geq \Lambda \int_0^t (t-s)^{n/(1-\lambda)} s^{\alpha/(1-\lambda)} ds \\ &= \Lambda B\left(\frac{n}{1-\lambda}, \frac{\alpha}{1-\lambda}\right) t^{((n+\alpha)/(1-\lambda))+1}, \end{aligned} \tag{3.3}$$

where the Beta function $B(n/(1-\lambda), \alpha/(1-\lambda))$ is a positive constant. On the other hand,

$$\int_0^t (t-s)^n s^\beta \cos s ds = t^{n+1+\beta} \int_0^1 (1-u)^n u^\beta \cos tu du = t^{n+1+\beta} I_{n,\beta}(t), \tag{3.4}$$

where $I_{n,\beta}(t)$ has the asymptotic formula

$$I_{n,\beta}(t) = -\Gamma(n+1) t^{-n-1} \cos\left(t - \frac{(n+1)\pi}{2}\right) + o\left(t^{-n-1}\right), \tag{3.5}$$

as $t \to \infty$ [15, pages 49 and 50]. By Theorems 2.1 and 2.2, we have that if

$$\beta > \frac{n+\alpha}{1-\lambda} + 1, \tag{3.6}$$

then all solutions of (3.1) are oscillatory for any $\tau > 0$, and all solutions of (3.1) satisfying $x(t) = O(t^n)$ are oscillatory for any $\tau < 0$.

Example 3.2. Consider the following equation:

$$x^{(n)}(t) + t^\alpha \sin t \left| x\left(\sqrt{t}\right) \right|^{\lambda-1} x\left(\sqrt{t}\right) = t^\beta \cos t, \quad t \geq 0, \tag{3.7}$$

where α, β, and λ are defined as in Example 3.1. Similar to the computation in Example 3.1, we have

$$\begin{aligned} \int_0^t Q(t,s)ds &\geq \Lambda \int_0^t (t-s)^{n/(1-\lambda)} s^{(\alpha+\lambda)/(1-\lambda)} ds \\ &= \Lambda B\left(\frac{n}{1-\lambda}, \frac{\alpha+\lambda}{1-\lambda}\right) t^{(n+\alpha+\lambda)/(1-\lambda)+1}, \end{aligned} \tag{3.8}$$

where $\Lambda = (\lambda - 1)\lambda^{\lambda/(1-\lambda)}(n!/2)^{\lambda/(\lambda-1)}$. Following the same argument in Example 3.1, we have that all solutions of (3.7) are oscillatory if $\beta > (n + \alpha + \lambda)/(1 - \lambda) + 1$.

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